MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 10, Nº 1, 1981, pp. 89—93

DETERMINING SETS FOR FINITELY DEFINED OPERATORS

by

ION RAŞA

(Cluj-Napoca)

1. In this paper we characterize determining sets and Korovkin sets for finitely defined operators, in terms of an appropriate concept of ,,quasi peak point'' (see BERENS—LORENTZ [1]). The sets of this type have been investigated by SHASHKIN [8], MICCHELLI [5], [6], CAVARETTA [3], RUSK [7].

Let Q be a compact metric space. $\mathcal{T}_+$ will be the cone of positive linear operators on $C(Q)$, and $\mathcal{L}_+$ will be the cone of positive linear functionals on $C(Q)$. Let n be a positive integer. We define $\mathcal{D}_n$ to be the set of functionals μ in $\mathcal{L}_+$ for which the support of the representing Radon measure contains a number of points no greater than n.

For q in Q let $\hat{q}$ be the evaluation functional.

If T is a bounded linear operator on $C(Q)$, let T^* denote the adjoint of T. $\mathcal{T}_n$ will be the set of finitely defined (of order n) positive linear operators, i.e. T is in $\mathcal{T}_n$ if for every q in Q, $T^*\hat{q}$ is in $\mathcal{D}_n$.

A subspace X of $C(Q)$ is said to be a Korovkin set for an operator T in $\mathcal{T}_+$ if for any sequence of operators (T_n) in $\mathcal{T}_+$ the convergence of T_nf to Tf in the uniform norm for all f in X implies the convergence of T_nf to Tf for all f in $C(Q)$.

We say that the subspace X is a determining set for T if for any S in $\mathcal{T}_+$ the equality $Sf = Tf$ for all f in X implies $S = T$.

The corresponding concepts of Korovkin set and determining set for positive linear functionals are defined in the obvious way.

We use the following result (see [4], [6], [7]).

THEOREM 1. *A subspace X of $C(Q)$ is a Korovkin set for an operator T in $\mathcal{T}_+$ if and only if X is a determining set for $T^*\hat{q}$ for all q in Q.*

A point p in Q is said to be a quasi peak point for the subspace X if for any $0 < \varepsilon < 1$ and any neighborhood U of p there exists g in X such that

(i) $g(q) \geq 0$ for all q in Q;
(ii) $g(p) < \varepsilon$;
(iii) $g(q) \geq 1$ for all q in $Q \setminus U$.

For f in $C(Q)$ and q in Q let us denote

$$\bar{f}_X(q) = \inf \{g(q) : f \leq g \in X\};$$

$$\underline{f}_X(q) = \sup \{g(q) : f \geq g \in X\}.$$

Let ∂X be the Choquet boundary of X, i.e. the set of all q in Q such that X is determining for $\hat{q}$.

For the following result the reader is referred to [1], [6].

THEOREM 2. *Let X be a subspace of $C(Q)$. The following are equivalent:*
(a) *X is a Korovkin set for the identity operator;*
(b) *$\partial X = Q$;*
(c) *every q in Q is a quasi peak point for X;*
(d) *$\underline{f}_X = \bar{f}_X$ for all f in $C(Q)$.*

2. Let n be a positive integer and suppose card $Q \geq n + 1$. We say that X is a W_n-subspace of $C(Q)$ if for any system (q_i, V_i), $i = 1, \ldots, n$, where q_i are distinct points in Q and V_i is an open neighborhood of q_i, and for any $\varepsilon > 0$ there exists g in X such that:

(1) $g(q) \geq 0$ for all q in Q;
(2) $g(q_i) < \varepsilon$, $i = 1, \ldots, n$;
(3) $g(q) \geq 1$ for all q in $Q \setminus \bigcup_1^n V_i$.

Let X be a subspace of $C(Q)$, f in $C(Q)$ and μ in $\mathfrak{L}_+$. Let us denote $\mu_X(f) = \inf \{\mu(g) : f \leq g \in X\}$.

THEOREM 3. *The following are equivalent:*
(a) *X is a Korovkin set for all μ in $\mathfrak{D}_n$;*
(b) *X is a determining set for all μ in $\mathfrak{D}_n$;*
(c) *$\mu_X(f) = \mu(f)$ for all μ in $\mathfrak{D}_n$ and all f in $C(Q)$;*
(d) *X is a W_n-subspace;*
(e) *X is a Korovkin set for all T in $\mathfrak{F}_n$;*
(f) *X is a determining set for all T in $\mathfrak{F}_n$.*

Proof. (a) ⇒ (b) is obvious.

(b) ⇒ (c). Let X be a determining set for all μ in $\mathfrak{D}_n$. Let μ be in $\mathfrak{D}_n$ and f in $C(Q)$. Then X contains a strictly positive function (see FERGUSON, RUSK [4]). Lemma 1.3 [2] is now appliquable, and

$$\mu_X(f) = \sup \{\nu(f) : \nu \text{ is in } \mathfrak{L}_+,\ \nu|_X = \mu|_X\} = \sup \{\nu(f) : \nu = \mu\} = \mu(f).$$

(c) ⇒ (d). Let $q_1, \ldots, q_n$ be distinct points in Q, $V_1, \ldots, V_n$ open neighborhoods for these points, and $\varepsilon > 0$. By the Urysohn lemma there exists f in $C(Q)$ such that:

(4) $0 \leq f \leq 1$;
(5) $f(q_i) = 0$, $i = 1, \ldots, n$;
(6) $f(q) = 1$, for all q in $Q \setminus \bigcup_1^n V_i$.

Define $\mu = \sum_1^n \hat{q}_i$. Thus μ is in $\mathfrak{D}_n$ and by (c), $\mu_X(f) = \mu(f)$. Hence there exists g in X, $g \geq f$, $\mu(g) - \mu(f) < \varepsilon$. Therefore $g \geq 0$, $g(q) \geq 1$ for all q in $Q \setminus \bigcup_1^n V_i$. From $\mu(f) = \sum_1^n f(q_i) = 0$ we obtain $\mu(g) < \varepsilon$, hence $g(q_i) < \varepsilon$, $i = 1, \ldots, n$. Therefore X is a W_n-subspace.

(d) ⇒ (b). Let μ be in $\mathfrak{D}_n$ and ν be in $\mathfrak{L}_+$ such that $\mu(f) = \nu(f)$ for all f in X. If $\mu = \sum_1^n a_i \hat{q}_i$ where a_i are nonnegative real numbers and q_i are distinct points in Q, we show that supp $\nu \subset \{q_1, \ldots, q_n\}$.

Let y be in supp $\nu \setminus \{q_1, \ldots, q_n\}$, and let $V_y, V_1, \ldots V_n$ be disjoint neighborhoods for $y, q_1, \ldots q_n$. We denote the representing Radon measure of ν also by ν. Since $y \in$ supp ν we have $\nu(V_y) = c > 0$.

By (d), for any $\varepsilon > 0$ there exists g in X such that $g \geq 0$, $g(q_i) < \varepsilon$, $i = 1, \ldots, n$, and $g(q) \geq 1$ for all q in $Q \setminus \bigcup_1^n V_i$.

If q is in V_y then $g(q) \geq 1$. Therefore

$$0 < c \leq \int_{V_y} g\, d\nu \leq \int_Q g\, d\nu = \nu(g) = \mu(g) = \sum_1^n a_i g(q_i) \leq \varepsilon \sum_1^n a_i.$$

If $\varepsilon \to 0$, we obtain $0 < c \leq 0$, a contradiction. Hence $\nu = \sum_1^n b_i \hat{q}_i$, $b_i \geq 0$.

Let q_{n+1} be in $Q \setminus \{q_1, \ldots, q_n\}$, and let V_i be a neighborhood for q_i, $i = 2, \ldots n + 1$ such that q_1 is not in $\bigcup_2^{n+1} V_i$. For any $\varepsilon > 0$ there exists g in X, $g \geq 0$, $g(q_i) < \varepsilon$, $i = 2, \ldots, n + 1$ $g(q_1) \geq 1$. Therefore

$$0 = \mu(g) - \nu(g) = (a_1 - b_1)g(q_1) + \sum_2^n (a_i - b_i)\, g(q_i);$$

$$|a_1 - b_1| \leq |a_1 - b_1|g(q_1) = \left|\sum_2^n (a_i - b_i)\, g(q_i)\right| \leq \varepsilon \sum_2^n |a_i - b_i|.$$

If $\varepsilon \to 0$ we obtain $a_1 = b_1$. A similar argument shows that $a_i = b_i$, $i = 2, \ldots, n$, and hence $\nu = \mu$.

(c) $\Rightarrow$ (a). Let μ be in $\mathfrak{D}_n$, and let (μ_k) be a sequence in $\mathfrak{L}_+$ such that $\lim \mu_k(g) = \mu(g)$ in the uniform norm, for all g in X. Let f be in $C(Q)$. By (c), $\mu_X(f) = \mu(f)$. For any $\varepsilon > 0$ there exists g in X such that

(7) $g \geqslant f$;

(8) $\mu(g) - \mu(f) \leqslant \varepsilon$.

From $\mu_X(-f) = \mu(-f)$ it follows that there exists h in X such that

(9) $h \geqslant -f$;

(10) $\mu(h) - \mu(-f) \leqslant \varepsilon$.

Hence $-h \leqslant f \leqslant g$, i.e. $-\mu_k(h) \leqslant \mu_k(f) \leqslant \mu_k(g)$. From (8) and (10) we obtain now

$$\mu(f) - \varepsilon \leqslant -\mu(h) = \lim(-\mu_k(h)) \leqslant \liminf \mu_k(f) \leqslant \limsup \mu_k(f) \leqslant$$
$$\leqslant \lim \mu_k(g) = \mu(g) \leqslant \mu(f) + \varepsilon.$$

Thus $\lim \mu_k(f) = \mu(f)$ for all f in $C(Q)$.

(e) $\Rightarrow$ (f) is obvious.

(f) $\Rightarrow$ (b). Let μ be in $\mathfrak{D}_n$ and let μ_1 be in $\mathfrak{L}_+$ such that $\mu_1(g) = \mu(g)$ for all g in X. We define the operators T and T_1 on $C(Q)$ by $T(f) = \mu(f) \cdot 1$, $T_1(f) = \mu_1(f) \cdot 1$, where 1 is the constant function. Then T_1 is in $\mathcal{T}_+$ and $T^*\hat{q} = \mu$ for all q in Q, hence T is in $\mathcal{T}_n$. Moreover, $T_1(g) = T(g)$ for all g in X. By (f), $T_1 = T$, and then $\mu_1 = \mu$.

(b) $\Rightarrow$ (e) is a consequence of theorem 1.

Remark. If X is an n^{th} order Korovkin space (see CAVARETTA [3]), then X is a W_n—subspace. Hence from theorem 3 ((d) $\Rightarrow$ (e)) we obtain Cavaretta's theorem 2 [3].

3. Let Q be a compact interval or a circle.

THEOREM 4. *If X is a W_n-subspace of $C(Q)$ and if $\dim X = 2n + 1$, then X is a Cebîşev subspace.*

Proof. We employ the same argument as MICCHELLI [5, theorem 4]. Let X be the linear span of $g_0, g_1, \ldots g_{2n}$.

Suppose X is not a Cebîşev subspace. Then there exist $2n + 1$ distinct points $q_0, \ldots, q_{2n}$ in Q, and $2n + 1$ real numbers $a_0, \ldots, a_{2n}$, not all equal to zero, such that

$$(11) \qquad \sum_{j=0}^{2n} a_j\, g_i\,(q_j) = 0, \qquad i = 0, 1, \ldots, 2n$$

Put $M = \{j : a_j > 0\}$. Multiplying, if necessary, (11) by -1, we may assume card $M \leqslant n$. Define the functionals μ, ν by

$$\mu = \sum_{j \in M} a_j \hat{q}_j \quad (\mu = 0 \text{ if } M = \emptyset), \quad \nu = -\sum_{j \notin M} a_j \hat{q}_j.$$

From (11) we obtain $\mu(g) = \nu(g)$ for all g in X. Moreover, μ is in $\mathfrak{D}_n$, and ν is in $\mathfrak{L}_+$. By theorem 3, X is determining for μ, hence $\mu = \nu$. This contradicts the assumption that $q_0, \ldots, q_{2n}$ are distinct and $a_0, \ldots, a_{2n}$ are not all equal to zero.

REFERENCES

[1] Berens, H., Lorentz, G. G., *Convergence of positive operators.* J. Approximation Theory, **17**, 307–314 (1976)

[2] Boboc, N., Bucur, G., *Conuri convexe de funcţii continue pe spaţii compacte.* Ed. Acad. R.S.R., Bucureşti, 1976.

[3] Cavaretta, Jr., A. S., *A Korokvin theorem for finitely defined operators.* Approximation Theory, ed. G. G. Lorentz. Academic Press, New York, 299–305, 1973.

[4] Ferguson, L. B. O., Rusk, M. D., *Korovkin sets for an operator on a space of continuous functuions.* Pacific J. Math. **65**, 337–345 (1976).

[5] Micchelli, C. A., *Chebyshev subspaces and convergence of positive linear operators.* Proc. Amer. Math. Soc., **40**, 448–452 (1973).

[6] Micchelli, C. A., *Convergence of positive linear operators on C(X).* J. Approximation Theory **13**, 305–315 (1975).

[7] Rusk, M. D., *Determining sets and Korovkin sets on the circle.* J. Approximation Theory, **20**, 278–283 (1977).

[8] Shashkin, Yu. A., *Finitely defined linear operators in spaces of continuous functions* (Russian). Uspehi Mat. Nauk, **20**, no. 6(126), 175–180 (1965).

Received 18. IX. 1980