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AN ITERATIVE METHOD FOR THE SOLUTION OF THE EQUATION $x = f(x, \ldots, x)$

by

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1. Introduction. The following theorem was proved in [6] (see also [2] pp. 94–95, N. CIOBANU [1], D. KUREPA [2], M. MARJANOVIČ [10]).

THEOREM 1. PREŠIČ (1965). *Let (X, d) be a complete metric space and $f: X^m \to X$ a mapping. If there exist $a_i \in \mathbf{R}_+$, $i = \overline{1, m}$, $a_1 + \ldots + a_m = a < 1$, such that*

$$d(f(x_0, \ldots, x_{m-1}), f(x_1, \ldots, x_m)) \leqslant \sum_{i=1}^{m} d(x_{i-1}, x_i)$$

for all $x_0, \ldots, x_m \in X$, then

(i) *there exists a unique $x^* \in X$ such that*

$$x^* = f(x^*, \ldots, x^*);$$

(ii) *for all $x_0, \ldots, x_{m-1} \in X$, the sequence $(x_{m+k})_{k \in \mathbf{N}}$, $x_{m+k} = f(x_k, \ldots, x_{k+m-1})$, converges to x^* and*

$$d(x_n, x^*) \leqslant \frac{a^{n+1-m}}{1-a} \max(d(x_0, x_1), \ldots, d(x_{m-1}, x_m)), \quad n > m.$$

In the prezent paper we will give a generalization of the theorem 1. Let $\varphi: \mathbf{R}_+^m \to \mathbf{R}_+$ be a mapping with the following properties

(a) $(r \leqslant s,\ r, s \in \mathbf{R}_+^n) \Rightarrow (\varphi(r) \leqslant \varphi(s))$;

(b) $(r \in \mathbf{R}_+,\ r > 0) \Rightarrow (\varphi(r, \ldots, r) < r)$;

(c) the mapping φ is continuous.

Example 1. Let $a_i \in \mathbf{R}_+$, $i = \overline{1, m}$, $\sum_{i=1}^{m} a_i < 1$. We define

$$\varphi : \mathbf{R}_+^m \to \mathbf{R}_+, \text{ by } \varphi(r) = \sum_{i=1}^{m} a_i r_i .$$

Example 2. Let $a \in \mathbf{R}_+$, $a < \frac{1}{m}$. We define

$$\varphi : \mathbf{R}_+^m \to \mathbf{R}_+, \text{ by } \varphi(r) = a(r_1 + \ldots + r_m).$$

Example 3. For $a \in]0, 1[$, we define $\varphi : \mathbf{R}_+^m \to \mathbf{R}_+$

$$\text{by } \varphi(r) = a \max \{r_1, \ldots, r_m\}.$$

Example 4. Let $a_i \in \mathbf{R}_+$, $\sum_{i=1}^{m} a_i < 1$, and $p \in \mathbf{N}$. We define

$$\varphi : \mathbf{R}_+^m \to \mathbf{R}_+ \text{ by } \varphi(r) = \left(\sum_{i=1}^{m} a_i r_i^p\right)^{\frac{1}{p}}.$$

Example 5. Let $\varphi_i : \mathbf{R}_+ \to \mathbf{R}_+$, $i = \overline{1, m}$ be mappings with the following properties:

(a') $(r \leqslant s,\ r, s \in \mathbf{R}_+) \Rightarrow (\varphi_i(r) \leqslant \varphi_i(s),\ i = \overline{1, m})$;

(b') $(r \in \mathbf{R}_+,\ r > 0) \Rightarrow \left(\sum_{i=1}^{m} \varphi_i(r) < r\right)$;

(c') the mappings φ_i, $i = \overline{1, m}$ are continuous. We define $\varphi : \mathbf{R}_+^m \to$ $\to \mathbf{R}_+$, by $\varphi(r) = \sum_{i=1}^{m} \varphi_i(r_i)$.

Let $r \in \mathbf{R}_+$. For simplicity we put

$$\varphi^2(r, \ldots, r) = \varphi(\varphi(r, \ldots, r), \ldots, \varphi(r, \ldots, r)), \ldots ;$$

$$\varphi^n(r, \ldots, r) = \varphi(\varphi^{n-1}(r, \ldots, r), \ldots, \varphi^{n-1}(r, \ldots, r)), \ldots$$

2. The main result. Let (X, d) be a metric space and $f : X^n \to X$. Let $\tilde{f} : X \to X$ be given by $x \to f(x, \ldots, x)$. The set of the solution of the equation

$$x = f(x, \ldots, x) \tag{1}$$

is equal to the fixed point set of $\tilde{f}$.

The main result of the present paper is the following

THEOREM 2. *Let (X, d) be a complete metric space and $f : X^m \to X$ be such that there exists $\varphi : \mathbf{R}^m \to \mathbf{R}$ with the following properties*

(a) $(r \leqslant s,\ r, s \in \mathbf{R}_+^m) \Rightarrow (\varphi(r) \leqslant \varphi(s))$;

(*b*) $(r \in \mathbf{R}_+,\ r > 0) \Rightarrow (\varphi(r, \ldots, r) < r)$;

(c) φ *is continuous*;

(d) $\sum_{k=0}^{\infty} \varphi^k(r) < +\infty$;

(e) $\varphi(r, 0, \ldots, 0) + \varphi(0, r, 0, \ldots, 0) + \ldots + \varphi(0, \ldots, 0, r) \leqslant$ $\leqslant \varphi(r, \ldots, r),\ \forall r \in \mathbf{R}_+$

(f) *for all* $x_0, x_1, \ldots, x_m \in X$, $d(f(x_0, \ldots, x_{m-1}), f(x_1, \ldots, x_m)) \leqslant$ $\leqslant \varphi(d(x_0, x_1), \ldots, d(x_{m-1}, x_m))$.

Then

(i) $F_f = \{x^*\}$;

(ii) *for any* $\tilde{x}_0 \in X$, *the sequence* $(\tilde{x}_n)_{n \in \mathbf{N}}$ $\tilde{x}_n = f(\tilde{x}_{n-1}, \ldots, \tilde{x}_{n-1})$, *converges to* x^*;

(iii) *for all* $x_0, \ldots, x_{m-1} \in X$, *the seguence* $(x_{m+n})_{n \in \mathbf{N}}$, $x_{m+n} = f(x_n, \ldots$ $\ldots, x_{n+m-1})$, *converges to* x^* *and* $d(x_n, x^*) \leqslant m \sum_{k=0} \varphi^{[\frac{n}{m}]+k}_{(d_0)}$,

wher:

$$d_0 = \max (d(x_0, x_1), \ldots, d(x_{m-1}, x_m)).$$

Proof (i) + (ii) From (e) and (f) we have:

$$d(f(x), f(y)) = d(f(x, \ldots, x), f(y, \ldots, y)) \leqslant d(f(x, \ldots, x), f(x, \ldots, x, y)) + d(f(x, \ldots, x, y), f(x, \ldots, x, y, y)) + \ldots + d(f(x, y, \ldots, y), f(y, \ldots, y)) \leqslant \varphi(0, \ldots, 0, d(x, y)) + \varphi(0, \ldots, 0, d(x, y), 0) + \ldots + \varphi(d(x, y), \ldots, 0) \leqslant \varphi(d(x, y), \ldots, d(x, y)).$$

From the fixed point theorem given in [7] we have (i) + (ii).

(iii) Let $x_0, x_1, \ldots, x_{m-1} \in X$ and $x_n = f(x_{n-m}, \ldots, x_{n-1})$, $n \geqslant m$. If we take $0 < d_0 \geqslant \max (d(x_0, x_1, \ldots, d(x_{m-1}, x_m))$, we have from (a)—(e):

$$d(x_m, x_{m+1}) = d(f(x_0, \ldots, x_{m-1}), f(x_1, \ldots, x_m)) \leqslant \varphi(d_0, \ldots, d_0) < d_0;$$

$$d(x_{m+1}, x_{m+2}) \leqslant \varphi(d(x_1, x_2), \ldots, d(x_m, x_{m+1})) \leqslant$$
$$\leqslant \varphi(d_0, \ldots, d_0, \varphi(d_0)) \leqslant \varphi(d_0, \ldots, d_0) < d_0;$$

. . . .

$$d(x_{2m-1}, x_{2m}) \leqslant \varphi(d(x_{m-1}, x_m), \ldots, d(x_{2m-2}, x_{2m-1})) \leqslant$$
$$\leqslant \varphi(d_0, \varphi(d_0, \ldots, d_0), \ldots, \varphi(d_0, \ldots, d_0)) < d_0\,;$$
$$d(x_{2m}, x_{2m+1}) \leqslant \varphi(d(x_m, x_{m+1}), \ldots, d(x_{2m-1}, x_{2m})) \leqslant$$
$$\leqslant \varphi(\varphi(d_0, \ldots, d_0), \ldots, \varphi(d_0, \ldots, d_0)) =$$
$$= \varphi^2(d_0, \ldots, d_0) < \varphi(d_0, \ldots, d_0)$$

and by induction we have:

$$d(x_n, x_{n+1}) \leqslant \varphi^{[\frac{n}{m}]}(d_0, \ldots, d_0) < \varphi^{[\frac{n}{m}]-1}(d_0, \ldots, d_0)) \quad n \geqslant m$$

and

$$d(x_{m+p}, x_n) \leqslant m \sum_{k=0}^{\infty} \varphi^{[\frac{n}{m}]+k}(d_0), \quad n \geqslant m, \quad p \in \mathbf{N}.$$

Hence, $(x_{m+k})_{k\in\mathbf{N}}$ is a Cauchy sequence. Let

$$x^* = \lim_{n\to\infty} x_n.$$

Let us prove that x^* is a solution of the equation

$$x = f(x, \ldots, x).$$

We have from the hypotheses on f

$$d(x_{m+n}, f(x^*, \ldots, x^*)) = d(f(x_n, \ldots, x_{n+m-1}), f(x^*, \ldots, x^*)) \leqslant$$
$$\leqslant \varphi(d(x_n, x_{n+1}), \ldots, d(x_{m+n-2}, x_{m+n-1}), d(x_{m+n-1}, x^*)) +$$
$$+ \ldots + \varphi(d(x_{m+n-1}, x^*), 0, \ldots, 0).$$

Making $n \to +\infty$, we have

$$d(x^*, f(x^*, \ldots, x^*)) \leqslant \varphi(0, \ldots, 0) + \ldots + \varphi(0, \ldots, 0) = 0,$$

i.e. $x^* = f(x^*, \ldots, x^*)$.

3. Remarks

3.1. For φ as in example 1 we have theorem 1.

3.2. For φ as in example 2 we have a result given in [1].

3.3. We have the following

THEOREM 3. *Let* (X, d) *be a complete metric space* $f: X^m \to X$ *be such that there exists* $\varphi: \mathbf{R}_+^m \to \mathbf{R}_+$ *with the properties* (a) — (d), *and* (f) *from theorem* 2.

Then for all $x_0, \ldots, x_{m-1} \in X$, *the sequence* $(x_{m+k})_{k\in\mathbf{N}}$, $x_{m+k} = f(x_k, \ldots, x_{k+m-1})$, *converges to a solution of the equation* (1).

4. Exemples

Example 6. Let $f: \mathbf{R}^2 \to \mathbf{R}$ be such

$$|f(x_0, x_1) - f(x_1, x_2)| \leqslant \varphi(|x_0 - x_1|, |x_1 - x_2|)$$

for all $x_0, x_1, x_2 \in \mathbf{R}$, where $\varphi: \mathbf{R}_+^2 \to \mathbf{R}$ is as in the theorem 2. Then the equation

$$x = f(x, x)$$

has a unique solution, x^*, and the sequance $(x_n)_{n\in\mathbf{N}}$, $x_n = f(x_{n-1}, x_n)$, converges to x^* for all $x_0, x_1 \in X$.

Example 7. Let $\Omega \subset \mathbf{R}^r$ be a bounded domain, and $C(\bar\Omega)$, the Banach space of all functions defined and continuous on $\bar\Omega$, with $||x|| = \max_{t\in\Omega}|x(t)|$. Let $f: C(\bar\Omega) \times C(\bar\Omega) \to C(\bar\Omega)$, be given by

$$f(x, y,)(t) = \int_\Omega K(t, s, x(s), y(s))ds,$$

where

$$K \in C(\bar\Omega \times \bar\Omega \times \mathbf{R} \times \mathbf{R}).$$

We suppose that

$$|K(t, s, u, v) - K(x, y, v, w)| \leqslant \Psi(|u - v|, |v - w|)$$

for all $x, y \in \bar\Omega$, $u, v, w \in \mathbf{R}$, where $\Psi: \mathbf{R}_+^2 \to \mathbf{R}^+$, is such that, $\varphi = m(\Omega)\Psi$ is as in the theorem 2. Then the equation

$$x(t) = \int_\Omega K(t, s, x(s), x(s))ds, \qquad t \in \bar\Omega$$

has in $C(\bar\Omega)$ a unique solution, x^*, and the sequence $(x_n)_{n\in\mathbf{N}}$,

$$x_n(t) = \int_\Omega K(t, s, x_{n-2}(s), x_{n-1}(s))ds, \quad t \in \bar\Omega$$

converges to x^* for all $x_0, x_1 \in C(\bar\Omega)$.

5. Generalization. Let (X, d, ρ) be a two-metric space (see [7], [8], [9]) and $f: X^m \to X$ a mapping. For such type of mappings we have

THEOREM 4. *We suppose that*:

(1) $d(x, y) \leqslant \rho(x, y)$, $\forall\, x, y \in X$;

(2) (X, d) *is a complete metric space*;

(3) $f: (X^m, d) \to (X, d)$ *is continuous*;

(4) *there exists* $\varphi : \mathbf{R}_+^m \to \mathbf{R}_+$ *with the properties* (a) — (f) *in* (X, ρ), *from the theorem* 2.

Then

(i) $F_f = \{x^*\}$;

(ii) *for any* $x_0 \in X$, *the sequence* $(x_n)_{n \in \mathbf{N}}$, $x_n = f(x_{n-1}, \ldots, x_{n-1})$, *converges in* (X, d) *to* x^*;

(iii) *for all* $x_0, \ldots, x_{m-1} \in X$, *the sequence* $(x_{m+n})_{n \in \mathbf{N}}$, $x_{m+n} = f(x_n, \ldots, x_{n+m-1})$, *converges in* (X, d) *to* x^*.

REFERENCES

[1] N. Ciobanu, *Aplicații ale teoremei de punct fix a lui Banach*. Lucrările comunicate la sesiunea științifică jubiliară, Institutul Politehnic Iași, 28—29 octombrie 1978, 63—67.

[2] Gh. Coman, G. Pavel, I. Rus, I. A. Rus, *Introducere în teoria ecuațiilor operatoriale*. Editura Dacia, Cluj-Napoca, 1976.

[3] D. Kurepa, *Some fixed point theorems*. Math. Balkanica, **2**, 102—108 (1972).

[4] M. Morjanović, S. B. Prešič, *Remark on the convergence of a sequence*. Publ. Fac. d'électr. de l'Univ. à Belgrade, No. **155**, 63—64 (1955).

[5] E. Pascali, *Osservazioni sulle F-contrazioni*. Matematische (Catania), **30**, 123—132 (1975).

[6] S. B. Prešič, *Sur une classe d'inéquations aux différence finies et sur la convergence de certaines suites*. Publ. Inst. Math., **5**, 75—78 (1965).

[7] I. A. Rus, *Some metrical fixed point theorems*. Studia Univ. Babeș-Bolyai, Mathematica, **1**, 1979, 73—77.

[8] I. A. Rus, *Metrical fixed point theorems*. Univ. of Cluj-Napoca, Department of Mathematics, Cluj-Napoca, 1979.

[9] I. A. Rus, *Principii și aplicații ale teoriei punctului fix*. Editura Dacia, Cluj-Napoca, 1979.

[10] M. R. Taskovic, *Banach's mappings and some generalizations*. Publ. de l'Inst. de Math., **16** (**30**) (1973).

[11] C. S. Wong, *Maps of contractive type*. *Fixed point theory and its applications* (ed. S. Swaminathan). Academic Press, New York, 1976, 197—207.

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