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ON THE MULTI-VALUED METRIC PROJECTION IN NORMED VECTOR SPACES

by

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In this paper we'll define countable and finite multi-valued metric projections and we shall examine their existence in normed, locally convex and metrizable topological vector spaces.

Let X be a real or complex normed vector space and M an arbitrary subset of X. We denote by P_M the metric projection with respect to M i.e.,

$$P_M(x) = \{m_0 \in M \mid \|x - m_0\| = \inf_{m \in M} \|x - m\|\},$$

and by $\mathfrak{U}_M$ the set

$$\mathfrak{U}_M = \{x \in X \mid \operatorname{card} P_M(x) \leqslant 1\}.$$

Let M be a non-void proper subset of X. If for every $x \in X \setminus M$ we have

$$2 \leqslant \operatorname{card} P_M(x) < \infty,$$

then we shall say that P_M is a *finitely multi-valued metric projection.*

S. B. STEČKIN [5] proved that, $\mathfrak{U}_M = X$ for every subset M of X, if and only if X is a strictly convex normed vector space.

It is obvious now, that if X is a strictly convex normed vector space, then for every subset M of X, the corresponding P_M isn't a finitely multi-valued metric projection.

Naturally, the question is what happens in more general spaces, i.e. in normed spaces which aren't strictly convex.

THEOREM 1. *Let M be an arbitrary non-void, proper subset of a normed space X. Then P_M isn't a finitely multi-valued metric projection.*

Proof. a) If $\overline{M} = X$, then for every $x \in X \setminus M$ we have $P_M(x) = \emptyset$ hence P_M isn't a finitely multi-valued metric projection.

b) If $\overline{M} \neq X$, then let $x_0 \in X \setminus \overline{M}$. Since X is a regular topological space, there exists a neighbourhood of x_0 contained in $X \setminus \overline{M}$.

Let be $r = \inf_{m \in M} \|x_0 - m\| > 0$. We suppose that P_M is a finitely multi-valued metric projection.

Let $m_1, m_2, \ldots, m_k$, $k \geqslant 2$ be the finite set of elements of best approximation of x_0 by elements from M.

Let us denote by $B(x_0, r)$, respectively $\overline{B}(x_0, r)$ the opened (respectively the closed) balls with center in x_0 and radius r.

Then we have: $\|x_0 - m_i\| = r$, $i = 1, 2, \ldots, k$. Let be

$$y_0 = \lambda x_0 + (1 - \lambda) m_1,\ 0 < \lambda < 1, \text{ and } 0 < 3\lambda r < \min_{i=\overline{2,k}} \|m_1 - m_i\|.$$

We shall prove that

1° $\overline{B}(y_0, \lambda r) \subseteq \overline{B}(x_0, r)$

2° $\overline{B}(y_0, \lambda r) \cap M = m_1$.

1° If $x \in \overline{B}(y_0, \lambda r)$ then:

$$\|x - x_0\| \leqslant \|x - y_0\| + \|y_0 - x_0\| \leqslant \lambda r + \|(1 - \lambda)(m_1 - x_0)\| =$$
$$= \lambda r + (1 - \lambda) r = r, \text{ hence } x \in \overline{B}(x_0, r).$$

2° Since $\overline{B}(y_0, \lambda r) \subseteq \overline{B}(x_0, r)$ and $\overline{B}(x_0, r) \cap M = \bigcup_{i=1}^{k} m_i$, it follows that $B(y_0, \lambda r) \cap M \subseteq \bigcup_{i=1}^{k} m_i$.

We shall prove now that

2a) $m_1 \in \overline{B}(y_0, \lambda r)$;

2b) $m_i \notin \overline{B}(y_0, \lambda r)$, $i = 2, 3, \ldots, k$.

2a) $\|y_0 - m_1\| = \|\lambda x_0 + (1 - \lambda) m_1 - m_1\| = \lambda \|x_0 - m_1\| = \lambda r$.

2b) By $\|m_1 - m_i\| \leqslant \|m_1 - y_0\| + \|y_0 - m_i\|$, it follows that

$$\|y_0 - m_i\| \geqslant \|m_1 - m_i\| - \|m_1 - y_0\| \geqslant 3\lambda r - \lambda r = 2\lambda r.$$

Then $m_i \notin \overline{B}(y_0, \lambda r)$, $i = 2, 3, \ldots, k$.

From 2a) and 2b) it follows that $\overline{B}(y_0, \lambda r) \cap M = m_1$; then $y_0 \in \mathcal{U}_M$ and P_M is not finitely multi-valued, which contradicts the hypothesis.

Let X be now a semi-normed vector space whose topology is given by the semi-norm $p(x)$. We define the metric projection on M, with respect to the semi-norm $p(x)$ by

$$P_M(x) = \{m_0 \in M \mid p(x - m_0) = \inf_{m \in M} p(x - m)\}.$$

In the following proposition we shall give a characterization of the semi-normed vector spaces which aren't normed, in the terms of finitely multi-valued metric projections.

PROPOSITION 2. *In every semi-normed vector space X, which isn't normed there exist the sets $M_2, M_3, \ldots, M_n, \ldots$ as well as the sets A and B such that*

(i) $$\operatorname{card} P_{M_n}(x) = n,$$

for every $x \in X \setminus M_n$, and every $n \in \mathbf{N}$, and

(ii) $$\operatorname{card} P_A(x) = \aleph_0,$$

(iii) $$\operatorname{card} P_B(x) = \aleph,$$

for every $x \in X \setminus A$ respectively for every $x \in X \setminus B$.

Proof. Since B is a semi-normed but non-normed vector space, there exists an element $x_0 \neq 0$ with $p(x_0) = 0$. We shall prove that the sets

$$M_2 = \{x_0, x_0/2\},$$
$$M_3 = \{x_0, x_0/2, x_0/3\},$$
$$\ldots\ldots\ldots\ldots$$
$$M_n = \{x_0, x_0/2, x_0/3, \ldots, x_0/n\},$$
$$A = \{x_0, x_0/2, x_0/3, \ldots, x_0/n, \ldots\},$$
$$B = \{\lambda x_0\}_{\lambda \in (0,1)},$$

have the properties (i)—(iii) respectively.

We shall prove only that

$$\operatorname{card} P_{M_n}(x) = n,$$

for every $x \in X \setminus M_n$.

Let $x \in X \setminus M_n$, then we have

$$p(x - m) \leqslant p(x) + p(-m) = p(x), \text{ for every } m \in M_n, \text{ and}$$
$$p(x) \leqslant p(x - m) + p(m) = p(x - m), \text{ for every } m \in M_n.$$

Then $p(x) = p(x - m)$, for every $m \in M_n$. We have now that

$$\operatorname{card} P_M(x) = \operatorname{card} M_n = n.$$

We can prove similarly the other statements in the propositions.

COROLLARY. *The semi-normed vector space X isn't a normed vector space if and only if there exists a subset M of X such that P_M is a finitely multi-valued metric projection.*

If X is a locally convex space and the topology of X is generated by the family of semi-norms $\{p_i\}_{i \in I}$, then it can be defined the metric projection (see [4]) by

$$P_M(x) = \{m_0 \in M \mid \forall i \in I,\ p_i(x - m_0) = \inf_{m \in M} p_i(x - m)\}.$$

With this definition of the metric projection and with a proof analogous to those of Proposition 2 we have:

PROPOSITION 3. *If X is a locally convex vector space, and if $\{p_i\}_{i\in I}$ is the generating family of semi-norms for the topology of X, then the following statements are equivalent:*

1. *There exists a set $M \subset X$ such that P_M is a finitely multi-valued metric projection.*
2. *X isn't Hausdorff.*

A linear metric space is a vector space whose topology (not necessarily compatible with the linear structure) is given by a metric ρ. In such a space one can easily given an example of a set M with P_M finitely multi-valued. Here the metric projection is defined by

$$P_M(x) = \{m_0 \in M | \rho(x, m_0) = \inf_{m\in M} \rho(x, m)\}.$$

If the linear metric space is a metrizable topological vector space, then the metric ρ is given by a pseudo-norm.

Let be $X = \mathbf{R}^2$. If $x = (x_1, x_2)$ is an arbitrary element of $\mathbf{R}^2$, then let q be the pseudo-norm given by

$$q(x) = \sqrt{|x_1|} + \sqrt{|x_2|}.$$

With the metric $\rho(x, y) = q(x - y)$, $\mathbf{R}^2$ becomes a Hausdorff metrizable topological vector space.

Let M be the set

$$\{x \in \mathbf{R}^2 | x = (a, a), a \in \mathbf{R}\}.$$

Then we have card $P_M(x) = 2$, for every $x \in \mathbf{R}^2 \setminus M$, and hence P_M is a finitely multi-valued metric projection.

THEOREM 4. *If in a metrizable topological vector space X, P_M is a finitely multi-valued metric projection then M is a perfect set of X.*

Proof. Let P_M be a finitely multi-valued metric projection. We suppose that M isn't a closed set. Let $x \in \bar{M} \setminus M$. Since $x \in \bar{M}$, it follows that for every $\varepsilon > 0$, there exists an element $m \in M$ such that $\rho(x, m) = q(x - m) < \varepsilon$, with q the mentioned pseudo-norm. If $m_0 \in P_M(x)$, then

$$q(x - m_0) = \inf_{m\in M} q(x - m) = 0,$$

and it follows that $x = m_0$ and this contradicts the fact that $x \in \bar{M} \setminus M$. Hence M is a closed set.

We suppose now that M isn't dense in itself. Let $m_0 \in M$ be an isolated point. Then there exists an $\varepsilon > 0$ such that the set

$$\{x \in X | q(x - m_0) < \varepsilon\},$$

doesn't contain points of M different from m_0. Then we have $q(m - m_0) \geq \varepsilon$ for every $m \in M$, $m \neq m_0$. Let $\lambda_n \to 0$ be a null sequence and $x_n = (1 - \lambda_n)m_0$. From the continuity of the mapping $(\lambda, x) \to \lambda x$, we obtain

$$q(x_n - m_0) = q(-\lambda_n m_0) = q(\lambda_n m_0) \leq \varepsilon/3$$

for every $n \geq N_0$.

Then for every $m \in M$, $m \neq m_0$, and $n \geq N_0$ we obtain

$$q(x_n - m) = q[(1 - \lambda_n)m_0 - m] = q(m_0 - m - \lambda_n m_0) \geq$$

$$\geq |q(m_0 - m) - q(\lambda_n m_0)| \geq \varepsilon - \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} \geq \frac{\varepsilon}{3} + q(x_n - m_0).$$

Hence we have

$$q(x_n - m) \geq q(x_n - m_0) + \frac{\varepsilon}{3},$$

for every $m \neq m_0$, $n \geq N_0$ and it follows that $P_M(x_n) = m_0$, for every $n \geq N_0$. Then P_M isn't a finitely multi-valued metric projection, which contradicts our assumption. Then M is indeed a perfect set.

In the following theorems we shall consider the normed spaces in which there exists a set M with the property that for every point $x \in X \setminus \bar{M}$ we have card $P_M(x) = \aleph_0$. First, we give some definitions.

Let M be a non-void, proper subset of the normed vector space X. We shall say that the corresponding metric projection P_M is a *countable multi-valued metric projection* if for every $x \in X \setminus M$ we have

$$\text{card } P_M(x) = \aleph_0.$$

If the above relation is verified only for $x \in X \setminus \bar{M} \neq \emptyset$, then we shall say that P_M is a *weakly countable multi-valued metric projection.*

By the Steckin's above mentioned result it follows that such metric projections must be tryed only in the non-strictly convex spaces.

As a simple remark, we have that for a convex set M, the corresponding P_M isn't a finitely, or a weakly countable, multi-valued metric projection.

Indeed, if M is a convex set, $x \in X \setminus \bar{M}$ and if $m_1, m_2 \in P_M(x)$, with $m_1 \neq m_2$, then, for every $\lambda \in (0, 1)$ we have

$$||x - \lambda m_1 - (1 - \lambda)m_2|| = ||\lambda(x - m_1) + (1 - \lambda)(x - m_2)|| \leq$$

$$\leq \lambda||x - m_1|| + (1 - \lambda)||x - m_2|| = d(x, M),$$

where $d(x, M)$ is the distance of x to M.

It follows that $\lambda m_1 + (1 - \lambda)m_2 \in P_M(x)$, for every $\lambda \in (0,1)$. Hence in this case P_M isn't a finitely or a weakly countable multi-valued metric projection. When X is a metrizable topological vector space, and P_M is a countable multi-valued metric projection, we have (in analogy with Theorem 4).

PROPOSITION 5. *Let X be a metrizable topological vector space and M a non-void proper subset of X. If P_M is a countable multi-valued metric projection, then M is a perfect subset of X.*

The *proof* is similar with the proof of Theorem 4.

In the case when X is a Banach space we have

THEOREM 6. *If X is a Banach space, and M a non-void, proper subset of X, then P_M cannot be a countable, multi-valued metric projection.*

Proof. We suppose that there exists a set $M \subset X$, with the property that P_M is a countable multi-valued metric projection.

Then M is, according to the last proposition, a perfect set. If $x \in X \setminus M$, then card $P_M(x) = \aleph_0$. But, generally, the intersection of two perfect sets isn't a perfect set. However, we prove that the intersection

$$P_M(x) = M \cap S(x, d(x, M)),$$

where $S(x, d(x, M))$ is the sphere with center in x and radius $d(x, M)$, is a perfect set.

a) $P_M(x)$ is a closed set as an intersection of two closed sets.

b) $P_M(x)$ is a set that is dense in itself. Indeed, if m_0 is an isolated point of $P_M(x)$, then there exists a ball $B(m_0, \varepsilon)$, with center in m_0 and radius $\varepsilon > 0$, such that

$$P_M(x) \cap B(m_0, \varepsilon) = m_0.$$

Let us consider the point

$$x_\lambda = (1 - \lambda)m_0 + \lambda x \in X,$$

with $0 < \lambda < \varepsilon/(2||x - m_0||) < 1$. Since

$$||x - x_\lambda|| = ||x - (1 - \lambda)m_0 - \lambda x|| = (1 - \lambda)||x - m_0|| = (1 - \lambda)\, d(x, M),$$

it follows that $x \in X \setminus M$. We have

$$||x_\lambda - m_0|| = ||(1 - \lambda)m_0 + \lambda x - m_0|| = \lambda||x - m_0|| < \varepsilon/2.$$

On the other hand let $m \in M$, $m \neq m_0$. Then $m \in P_M(x) \setminus \{m_0\}$ or $m \notin \bar{B}(x, d(x, M))$. We shall prove that

$$||x_\lambda - m|| > ||x_\lambda - m_0||,$$

for both cases.

If $m \in P_M(x) \setminus \{m_0\}$ we have:

$$||x_\lambda - m|| \geq |\,||m_0 - m|| - ||x_\lambda - m_0||\,| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > ||x_\lambda - m_0||.$$

If $m \notin \bar{B}(x, d(x, M))$ we have:

$$||x_\lambda - m|| \geq |\,||x - m|| - ||x - x_\lambda||\,| > (d(x, M) - (1 - \lambda)d(x, M)) =$$
$$= \lambda d(x, M) = ||x_\lambda - m_0||.$$

It follows that $P_M(x) = \{m_0\}$, and $x_\lambda \in X \setminus M$. Then $P_M(x)$ is dense in itself and hence a perfect set. But, if $P_M(x)$ is a perfect set of the complet metric space X, then (see [1]) $P_M(x)$ is an uncountable set. The theorem is proved.

However, if X is a non- Hausdorff locally convex space, $\{p_i\}_{i \in I}$ is the generating set of semi-norms for the topology of X and $x_0 \neq 0$ is a point for which $p_i(x_0) = 0$, for every $i \in I$, then the metric projection with respect to te set A

$$A = \{x_0, x_0/2, \ldots, x_0/n, \ldots\},$$

is a countable multi-valued metric projection. A proof is given in Proposition 2.

We shall show in the sequel that there exist a wide class of normed spaces with *weakly* countable multi-valued metric projections.

Notice first, that, in virtue of a well-known result, in every non strictly convex normed space there exists a closed real hyperplane H such that card $P_H(x) \geq \aleph$, for every $x \in X \setminus H$.

Indeed, if X isn't a strictly convex normed space then there exist $x_1, x_2 \in S(0, 1) = \mathrm{Fr}\,(B(0, 1))$ such that $\lambda x_1 + (1 - \lambda)x_2 \in S(0, 1)$ for every $\lambda \in (0,1)$. (see for instance M. G. KREIN [2]). Then from a well-known separation theorem, (see H. H. SCHAEFER [3]) there exists a close real hyperplane containing the segment $[x_1, x_2]$ and not intersecting the interior of the unit ball. Let H be this hyperplane. We have $H = x_1 + H_0$ where x_1 is one of the previous points and H_0 is a hyperplane parallel to H and passing through the origin. We'll show now that for every $x \in X \setminus H$, card $P_H(x) \geq \aleph$. Let $x \in X \setminus H$. We have $x = \lambda x_1 + h_x$ with $\lambda \neq 1$, $h_x \in H_0$ and the decomposition is unique.

$$P_H(x) = \{x_1 + h_1 \mid ||\lambda x_1 + h_x - x_1 - h_1|| =$$
$$= \inf_{h \in H_0} ||\lambda x_1 + h_x - x_1 - h||\} = x_1 + P_{H_0}((\lambda - 1)x_1 + h_x),$$

and from the quasi-additivity and homogeneity of the multi-valued metric projection (with respect to a subspace) we have:

$$P_H(x) = x_1 + h_x + (\lambda - 1)P_{H_0}(x_1).$$

If $x = 0$, we have $h_x = 0$, $\lambda = 0$ and $P_H(0) = x_1 - P_{H_0}(x_1)$.

Since $P_H(0)$ contains the segment $[x_1, x_2]$ it follows that

$$\aleph \leq \text{card}\, P_H(0) = \text{card}\, P_{H_0}(x_1) = \text{card}\, P_H(x),$$

for $\lambda \neq 1$, i.e. for every $x \in X \setminus H$.

Let A be a convex set and H a closed real hyperplane of a normed space X. Then, H is called a *supporting hyperplane* of A if $A \cap H \neq \emptyset$ and if A is contained in one of the closed semi-spaces determined by H.

The hyperplane H considered in the previous example was the supporting hyperplane of the unit ball of a non-strictly convex space.

THEOREM 7. *Let X be a normed space. If the unit ball $\bar{B}(0, 1)$ of X has at least one supporting hyperplane H with the property that*

$$1 \leqslant \dim [H \cap \bar{B}(0, 1)] \leqslant \aleph_0,$$

then there exists a set $M \subset H$, dense in H such to P_M be a weakly countable multi-valued metric projection.

Proof. Let $S = H \cap \bar{B}(0, 1)$. Then S is a convex set of dimension $\leqslant \aleph_0$ which contains at least a segment. From the previous remarks it results that if $x \in X \setminus H$, then

$$P_H(x) = x_1 + h_x + (\lambda - 1) P_{H_0}(x_1) =$$
$$= x_1 + h_x + (\lambda - 1)(x_1 - P_H(0)) =$$
$$= \lambda x_1 + h_x - (\lambda - 1) P_H(0) = x - (\lambda - 1) P_H(0),$$

with the above notations, and $\lambda \neq 1$, for $x \in X \setminus H$.

Since $P_H(0) = H \cap \bar{B}(0, 1) = S$, we have

$$\operatorname{card} P_H(x) = \operatorname{card} S = \aleph,$$

for every $x \in X \setminus H$.

Consider the Hamel basis $\{e_i\}_{i \in I}$ in H_0 and let M_0 be the linear hull of $\{e_i\}$ with rational coefficients.

It is easy to see that $M = M_0 + x_0$ is dense in H, and for every $x \in X \setminus \bar{M} = X \setminus H$ we have

$$\operatorname{card} P_M(x) = \aleph_0(\dim S) = \aleph_0,$$

since $1 \leqslant \dim S \leqslant \aleph_0$.

R e m a r k. Every non strictly convex normed space of algebraic dimension at most $\aleph_0$ has weakly countable multi-valued metric projections.

However, there exist non strictly convex normed spaces without the property in the precedent theorem. A such space is c_0, the space of real sequences converging to 0. If $x = (x_1, x_2, \ldots, x_n, \ldots) \in c_0$, then $||x|| = \max \{|x_i|, i \in \mathbf{N}\}$.

It is well-known that c_0 is a separable Banach space and his unit ball has no extremal points. The intersection of the closed unit ball with an arbitrary supporting hyperplane of its is a closed convex set of algebraic dimension $\aleph$, however we have:

PROPOSITION 8. *There exists in c_0 a weakly countable multi-valued metric projection.*

Proof. Let H be a supporting hyperplane of the unit ball of c_0 and $x_0 \in H \cap \bar{B}(0, 1)$. Then

$x_0 = (x_1^0, x_2^0, \ldots, x_n^0, \ldots)$ where $x_{i_1}^0, x_{i_2}^0, \ldots, x_{i_n}^0 \in \{-1, 1\}$ and x_i with $i \neq i_1, i_2, \ldots, i_n$ will be in an interval $[-r, r]$ with $r \in (0, 1)$.

For a complete proof we need the following lemma (see I. SINGER [4]).

LEMMA 9. *Let X be a normed vector space, $x \in X$ and $r > 0$. For any $f \in X^*$ with $||f|| = 1$, the hyperplane $H \subset X$ defined by*

$$(*) \qquad H = \{y \in X | f(x - y) = r\} = \{y \in X | f(y) = f(x) - r\},$$

supports the ball $\bar{B}(x, r)$, and conversely, for any supporting hyperplane H of the ball $\bar{B}(x, r)$ there exists a unique $f \in E^*$ with $||f|| = 1$ such that the equality $(*)$ holds.

We use only the second part of the lemma. Particularly, we have that each supporting hyperplane of the unit ball can be written in the form

$$(**) \qquad H = \{x \in X | f(x) = -1\},$$

where $f \in X^*$ and $||f|| = 1$.

For c_0, the theorem of representation of continuous linear functionals give

$$f(x) = f_u(x) = \sum_{i=1}^{\infty} u_i x_i$$

where $u = (u_1, u_2, \ldots) \in l^1$, $x = (x_1, x_2, \ldots)$ and

$$||f|| = ||u||_{l^1}.$$

From $(**)$ we obtain

$$\sum_{i=1}^{\infty} u_i x_i = -1 \text{ and } ||f|| = ||u||_{l^1} = \sum_{i=1}^{\infty} |u_i| = 1.$$

From the fact that $x_0 \in H$ we have

$$1 = \left| \sum_{i=1}^{\infty} u_i x_i^0 \right| \leqslant \sum_{i=1}^{\infty} |u_i| \, |x_i^0| \leqslant \sum_{i=1}^{\infty} |u_i| = 1.$$

It follows that in order to have equalities in the above sequences of inequalities it is necessary and sufficient that

$$\begin{cases} \operatorname{sign} u_i = -\operatorname{sign} x_i^0, \text{ for every } i \in \mathbf{N}; \\ u_i = 0, \text{ for any } i \neq i_1, i_2, \ldots, i_n; \\ \sum_{k=1}^{n} |u_{i_k}| = 1. \end{cases}$$

From this conditions it follows that the supporting hyperplane which passes through x_0 is

$$H = \left\{ x \in c_0 \,\middle|\, \sum_{k=1}^{n} u_{i_k} x_{i_k} = -1, \sum_{k=1}^{n} |u_{i_k}| = 1 \right\},$$

where u_{i_k} are uniquely determined for a given supporting hyperplane H. It is easy to see now that

$$H \cap \bar{B}(0, 1) = \{x \in c_0 | x_{i_k} = -\text{sign}\, u_{i_k},\ k = \overline{1, n},\ |x_i| \leqslant 1,\ i \in \mathbf{N}\} =$$
$$= \{x \in c_0 | x_{i_k} = x_{i_k}^0,\ k = \overline{1, n},\ |x_i| \leqslant 1,\ i \in \mathbf{N}\}.$$

Then $H \cap \bar{B}(0,1)$ is a closed set of algebraic dimension $\aleph$.

Let us denote by M the reunion of the exterior of $\bar{B}(0, 1)$ and the set N in $S(0, 1)$ given by

$$N = \{x \in c_0 | \ \|x\| = 1,\ x = (x_i),\ x_i \text{ rational, and } x_i \neq 0 \text{ for a finite set of indices}\}.$$

We prove now that for this set M, P_M is a weakly countable multi-valued metric projection.

We have immediatly

$$P_H(0) = N, \text{ and hence card } P_M(0) = \aleph_0.$$

If $x \in B(0,1)$, $x \neq 0$, it follows that there exists an $x_0 \in \text{Fr}\, B(0, 1)$ and an $\lambda \in (0,1)$ such that $x = \lambda x_0$. Let H be a supporting hyperplane of the unit ball which passes through x_0. The existence of a such hyperplane is guaranteed by separation theorems. (see H. H. SCHAEFER [3]). We have

$$P_H(\lambda x_0) = \lambda x_0 + (1 - \lambda) P_H(0) =$$
$$= \{x \in c_0 | x_{i_k} = x_{i_k}^0,\ k = \overline{1, n},\ x_i = \lambda x_i^0 + (1 - \lambda)\alpha_i,$$
$$|\alpha_i| \leqslant 1,\ i \neq i_1, i_2, \ldots, i_n, \text{ and } \alpha_i \to 0 \text{ when } i \to \infty\},$$

where $x_{i_k}^0$ and x_i^0 were given in the first part of the theorem. It is easy to see that α_i can be chosen such to x_i be all rational. Moreover, since

$$\|x_i^0\| < (1 - \lambda)/\lambda$$

for any $i \geqslant N_0 \geqslant 2i_n$, all the α_i's can be chosen $-\lambda x_i^0/(1 - \lambda)$, for $i \geqslant N_0$.

Since

$$|x_i| = |\lambda x_i^0 + (1 - \lambda)\alpha_i| \leqslant \lambda |x_i^0| + (1 - \lambda)|\alpha_i| \leqslant 1,$$

for every $i \neq i_1, i_2, \ldots, i_n$, and $x_{i_k} \in \{-1, 1\}$, $k = \overline{1, n}$ and since the choice of α_i with the previous properties can be made for a countable number of points from

$$P_H(\lambda x_0) \cap \bar{B}(0,1) \cap M,$$

it follows that we have

$$\aleph_0 \geqslant \text{card } P_M(x) = \text{card } P_M(\lambda x_0) \geqslant \text{card } (P_H(\lambda x_0) \cap \bar{B}(0, 1) \cap M) \geqslant \aleph_0.$$

Hence card $P_M(x) = \aleph_0$, for every $x \in X \setminus \bar{M} = B(0, 1)$ and P_M is a weakly countable multi-valued metric projection. Q.E.D.

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