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THE REPRESENTATION OF n-CONVEX SEQUENCES

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1. The mathematical literature is quite rich in papers which treat problems of the following type: for two sets of sequences, K' and K'' construct a transformation A with the property that $A(K') \subseteq K''$. Usualy, such a transformation is given by a triangular matrix:

$$||p_{m,i}||,\ i = 1, \ldots, m \text{ for } m = 1, 2, \ldots$$

i.e., to the sequence $a = (a_m)_{m=1}^{\infty}$ is attached $A(a) = (A_m(a))_{m=1}^{\infty}$ where:

$$A_m(a) = \sum_{k=1}^{m} p_{m,k} \cdot a_k.$$

Many references to papers concerned to transformations that preserve the n-convexity may be found in [3]. A characterization of such transformations is contained in [2], while [1] presents a characterization of the transformations which map the set of p-monotone sequences in that of q-monotone sequences.

Our aim is to construct a transformation of the set of n-positive sequences, $\mathbf{R}_n^+ = \{(a_m)_{m=1}^{\infty} : a_m \geq 0 \text{ for } m > n\}$, in the set K_n of n-convex sequences. In fact, the transformation is a bijection, so that it gives a representation of n-convex sequences by means of n-positive sequences.

2. Let us to specify some notations and definitions used in what follows.

For a real sequence $(a_m)_{m=1}^{\infty}$, the n-th order difference is defined by:

(1) $$\Delta^0 a_m = a_m,\quad \Delta^n a_m = \Delta^{n-1} a_{m+1} - \Delta^{n-1} a_m.$$

DEFINITION 1. *A sequence $(a_m)_{m=1}^{\infty}$ is said to be convex of order n* (or n-convex) *if $\Delta^n a_m \geq 0$ for all m.*

The set of all n-convex sequences is denoted by K_n.

DEFINITION 2. *A sequence* $(c_m)_{m=1}^{\infty}$ *is said to be n-positive, if* $c_m \geqslant 0$ *for* $m > n$.

3. Before giving the main result, which we have already announced, let us state some auxiliary lemmas, interesting by themselves. Although they are simple enough, we do not find them in the specialized literature.

LEMMA 1. *If*

$$a_m = \sum_{i=1}^{m} b_i \text{ for all } m, \text{ then}$$

$$\Delta^n a_m = \Delta^{n-1} b_{m+1} \text{ for any } m \text{ and any } n \geqslant 1. \tag{2}$$

The *proof* is easy to do by induction. As a direct consequence we have:

LEMMA 2. *The sequence* $(a_m)_{m=1}^{\infty}$ *is n-convex, if and only if there is a sequence* $(b_m)_{m=1}^{\infty}$ *with the property that* $(b_m)_{m=2}^{\infty}$ *is convex of order* $n-1$ *and such that*

$$a_m = \sum_{i=1}^{m} b_i \quad \text{for all } m \geqslant 1. \tag{3}$$

Because 0-convexity means positivity, we will prove by induction

LEMMA 3. *The sequence* $(a_m)_{m=1}^{\infty}$ *is n-convex if and only if there is a n-positive sequence* $(c_m)_{m=1}^{\infty}$ *such that:*

$$a_m = \sum_{i=1}^{m} d_{i,m} \cdot c_i, \tag{4}$$

where the coefficients $d_{i,m}$ *do not depend on the two sequences.*

Proof. By the lemma 2, a sequence $(a_m^1)_{m=1}^{\infty}$ is 1-convex, if and only if there is a 1-positive sequence $(c_m)_{m=1}^{\infty}$, *such that:*

$$a_m^1 = \sum_{i=1}^{m} c_i \text{ for } m \geqslant 1. \tag{5}$$

Then, by the same lemma, the sequence $(a_m^2)_{m=1}^{\infty}$ is 2-convex if and only if:

$$a_m^2 = \sum_{i=1}^{m} a_i^1, \tag{6}$$

where $(a_m^1)_{m=2}^{\infty}$ is 1-convex. So, there is a 1-positive sequence $(c_m^1)_{m=1}^{\infty}$ such that:

$$a_{m+1}^1 = \sum_{i=1}^{m} c_i^1 \quad \text{for } m \geqslant 1. \tag{7}$$

Let us to denote $a_1^1 = c_1$ and $c_i^1 = c_{i+1}$ for $i \geqslant 1$. Then, the sequence $(c_m)_{m=1}^{\infty}$ is 2-positive and (7) becomes:

$$a_m^1 = \sum_{i=2}^{m} c_i \quad \text{for } m \geqslant 2 \tag{7'}$$

and, by (6), we have for $m \geqslant 2$:

$$a_m^2 = a_1^1 + \sum_{i=2}^{m} a_i^1 = c_1 + \sum_{i=2}^{m} \sum_{j=2}^{i} c_j = c_1 + \sum_{j=2}^{m} \sum_{i=j}^{m} c_j,$$

or:

$$a_m^2 = \begin{cases} c_1 & \text{for } m = 1, \\ c_1 + \sum_{j=2}^{m} (m-j+1) c_j & \text{for } m \geqslant 2. \end{cases} \tag{8}$$

Now suppose that a sequence is n-convex, if and only if there is a n-positive sequence $(c_m)_{m=1}^{\infty}$ such that:

$$a_m^n = \begin{cases} \sum_{i=1}^{m} p_{i,m}^n c_i & \text{for } m < n, \\ \sum_{i=1}^{n-1} q_{i,m}^n c_i + \sum_{i=n}^{m} r_{i,m}^n c_i & \text{for } m \geqslant n, \end{cases} \tag{9}$$

where the coefficients $p_{i,m}^n$, $q_{i,m}^n$ and $r_{i,m}^n$ are independent on the two sequences.

By the lemma 2, the sequence $(a_m^{n+1})_{m=1}^{\infty}$ is convex of order $n+1$ if and only if there is a sequence $(a_m^n)_{m=1}^{\infty}$ such that $(a_m^n)_{m=2}^{\infty}$ is n-convex and:

$$a_m^{n+1} = \sum_{i=1}^{m} a_i^n \quad \text{for any } m \geqslant 1. \tag{10}$$

But then, as in (9), we must have a n-positive sequence $(c'_m)_{m=1}^{\infty}$ such that, for $m \geqslant 1$:

$$a_{m+1}^n = \begin{cases} \sum_{i=1}^{m} p_{i,m}^n c'_i & \text{for } m < n, \\ \sum_{i=1}^{n-1} q_{i,m}^n c'_i + \sum_{i=n}^{m} r_{i,m}^n c'_i & \text{for } m \geqslant n. \end{cases} \tag{11}$$

If we denote:

$$a_1^n = c_1 \quad \text{and} \quad c'_i = c_{i+1} \quad \text{for } i \geqslant 1$$

the sequence $(c_m)_{m=1}^{\infty}$ is $n+1$-positive. Moreover, if we replace $i+1$

by i and, after that, $m+1$ by m, from (11) we get for $m \geqslant 2$:

$$(12)\qquad a_m^n = \begin{cases} \sum\limits_{i=2}^{m} p^n_{i-1,m-1}\, c_i & \text{for } m-1 < n, \\ \sum\limits_{i=2}^{n} q^n_{i-1,m-1}\, c_i + \sum\limits_{i=n+1}^{m} r^n_{i-1,m-1}\, c_i & \text{for } m-1 \geqslant n. \end{cases}$$

From (10) and (12) we have:

a) for $m = 1$:

$$a_1^{n+1} = a_1^n = c_1;$$

b) for $1 < m < n+1$:

$$a_m^{n+1} = a_1^n + \sum_{i=2}^{m} a_i^n = c_1 + \sum_{i=2}^{m}\sum_{j=2}^{i} p^n_{j-1,i-1}\, c_j =$$

$$= c_1 + \sum_{j=2}^{m}\sum_{i=j}^{m} p^n_{j-1,i-1}\, c_j = \sum_{j=1}^{m} p^{n+1}_{j,m}\, c_j;$$

c) for $m \geqslant n+1$:

$$a_m^{n+1} = a_1^n + \sum_{i=2}^{n} a_i^n + \sum_{i=n+1}^{m} a_i^n = c_1 + \sum_{i=2}^{n}\sum_{j=2}^{i} p^n_{j-1,i-1}\, c_j +$$

$$+ \sum_{i=n+1}^{m}\left[\sum_{j=2}^{n} q^n_{j-1,i-1}\, c_j + \sum_{j=n+1}^{i} r^n_{j-1,i-1}\, c_j\right] = c_1 + \sum_{j=2}^{n}\sum_{i=j}^{n} p^n_{j-1,i-1}\, c_j +$$

$$+ \sum_{j=2}^{n}\sum_{i=n+1}^{m} q^n_{j-1,i-1}\, c_j + \sum_{j=n+1}^{m}\sum_{i=j}^{m} r^n_{j-1,i-1}\, c_j = \sum_{j=1}^{n} q^{n+1}_{j,m}\, c_j + \sum_{j=n+1}^{m} r^{n+1}_{j,m}\, c_j.$$

This complets the induction and, moreover, gives us the following recurrence relations:

$$(13)\qquad p^{n+1}_{1,m} = 1,\ p^{n+1}_{j,m} = \sum_{i=j}^{m} p^n_{j-1,i-1} \quad \text{for } 2 \leqslant j \leqslant m < n+1;$$

$$(14)\qquad q^{n+1}_{1,m} = 1,\ q^{n+1}_{j,m} = \sum_{i=j}^{n} p^n_{j-1,i-1} + \sum_{i=n+1}^{m} q^n_{j-1,i-1} \quad \text{for } 2 \leqslant j \leqslant n;$$

$$(15)\qquad r^{n+1}_{j,m} = \sum_{i=j}^{m} r^n_{j-1,i-1} \quad \text{for } j \geqslant n+1.$$

Using these relations, we may prove the following:

THEOREM 1. *A sequence $(a_m)_{m=1}^{\infty}$ is n-convex, if and only if there is a n-positive sequence $(c_m)_{m=1}^{\infty}$, such that:*

$$(16)\qquad a_m = \begin{cases} \sum\limits_{i=1}^{m} \binom{m-1}{i-1} c_i & \text{for } m < n, \\ \sum\limits_{i=1}^{n-1} \binom{n-1}{i-1} c_i + \sum\limits_{i=n}^{m} \binom{m+n-i-1}{n-1} c_i & \text{for } m \geqslant n. \end{cases}$$

Proof. Let:

$$(17)\qquad s_0(m) = m,\quad s_j(m) = \sum_{i=1}^{m} s_{j-1}(i) \quad \text{for } j \geqslant 1.$$

From (13) we have successively:

$$p^n_{1,m} = p_{1,m} = 1,$$

$$p^n_{2,m} = \sum_{i=2}^{m} p^{n-1}_{1,i-1} = \sum_{i=2}^{m} p_{1,i-1} = p_{2,m} = m-1 = s_0(m-1),$$

$$p^n_{3,m} = \sum_{i=3}^{m} p_{2,i-1} = p_{3,m} = \sum_{i=3}^{m} s_0(i-2) = s_1(m-2).$$

Now suppose that for any n and m:

$$(18)\qquad p^n_{j,m} = p_{j,m} = s_{j-2}(m-j+1).$$

Again by (13) we have then:

$$p^n_{j+1,m} = \sum_{i=j+1}^{m} p^{n-1}_{j,i-1} = \sum_{i=j+1}^{m} s_{j-2}(i-1-j+1) = p_{j+1,m} =$$

$$= \sum_{i=1}^{m-j} s_{j-2}(i) = s_{j-1}(m-j),$$

that is (18) is true for any $j \geqslant 2$.

Similarly, from (14) we have:

$$q^n_{1,m} = q_{1,m} = 1 = p_{1,m},$$

$$q^n_{2,m} = \sum_{i=2}^{n} p^n_{1,i-1} + \sum_{i=n+1}^{m} q^n_{1,i-1} = \sum_{i=2}^{m} p_{1,i-1} = p_{2,m}.$$

Supposing:

$$(19)\qquad q^n_{j,m} = p_{j,m},$$

from (14) and (13) we have:

$$q^n_{j+1,m} = \sum_{i=j+1}^{n-1} p^{n-1}_{j,i-1} + \sum_{i=n}^{m} q^{n-1}_{j,i-1} = \sum_{i=j+1}^{n-1} p_{j,i-1} + \sum_{i=n}^{m} p_{j,i-1} = p_{j+1,m},$$

that is (19) holds.

As we saw in (8):

$$r^2_{j,m} = m - j + 1 = s_0(m - j + 1) \quad \text{for} \quad 2 \leqslant j \leqslant m.$$

From (15) we have for $j \geqslant 3$:

$$r^3_{j,m} = \sum_{i=j}^{m} r^2_{j-1,\, i-1} = \sum_{i=j}^{m} s_0(i - j + 1) = \sum_{i=1}^{m-j+1} s_0(i) = s_1(m - j + 1).$$

Let us suppose that:

$$r^n_{j,m} = s_{n-2}(m - j + 1). \tag{20}$$

Then

$$r^{n+1}_{j,m} = \sum_{i=j}^{m} r^n_{j-1,\, i-1} = \sum_{i=j}^{m} s_{n-2}(i - j + 1) = \sum_{i=1}^{m-j+1} s_{n-2}(i),$$

that is, by induction for n, the assumption (20) is proved.

To finish the proof of the theorem, it is enough to determine the coefficients $s_k\ (m)$. We have:

$$s_1(m) = \sum_{i=1}^{m} s_0(i) = \sum_{i=1}^{m} i = \frac{m(m+1)}{2} = \binom{m+1}{2}.$$

Let us suppose that:

$$s_k(m) = \binom{m+k}{k+1}. \tag{21}$$

Then:

$$s_{k+1}(m) = \sum_{i=1}^{m} s_k(i) = \sum_{i=1}^{m} \binom{i+k}{k+1} = \binom{m+k+1}{k+2}.$$

From (9), (18), (19) and (21) we have (16).

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