MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
TOME 10, N° 1, 1981, pp. 11–16

ON SOME TYPES OF OPTIMIZATION PROBLEMS IN COMPLEX SPACE

by

DOREL I. DUCA
(Cluj-Napoca)

1. Introduction

In the papers dealing with the mathematical programming problem in complex space this is formulated as an optimization problem of the form:

$$\text{Minimize } \operatorname{Re} f(z) \text{ subject to } g(z) \in S,$$

where $f: C^n \to C$, $g: C^n \to C^m$ and $S \subseteq C^m$, $S \neq \emptyset$.

Given the object function f and the feasible set $Y = \{z \in C^n / g(z) \in S\}$ in the present paper some new optimization problems in complex space are formulated, and then relations among the solutions of these problems are established.

2. Notations

Denote by $C^n(\mathbf{R}^n)$ n-dimensional complex(real) vector space. If $z = (z_j) \in C^n$ is a vector, then z^T, $\bar{z}$ and z^H denote its transpose, complex conjugate and conjugate transpose, respectively. If $b \in C$ is a complex number, then $\operatorname{Re} b$, $\operatorname{Im} b$, $\arg b$ and $|b|$ denote the real part, imaginar part, argument, and modulus of b, respectively.

3. Formulation of problems

Let Y be a nonempty set in C^n and let $f: Y \to C$.

In the papers dealing with the mathematical programming problem in complex space this is formulated as an optimization problem of the form:

(PR) $$\text{Minimize } \operatorname{Re} f(z) \text{ subject to } z \in Y.$$

If $Y = \{x \in \mathbf{R}^n / g(x) \leqslant 0, h(x) = 0\}$, where $g : \mathbf{R}^n \to \mathbf{R}^m$, $h : \mathbf{R}^n \to \mathbf{R}^p$, and $f : Y \to \mathbf{R}$, then Problem (PR) is that of real mathematical programming with m inequalities and p equalities.

Of course, given the feasible set $Y \subseteq C^n$, $Y \neq \emptyset$ and the function $f : Y \to C$, we can also formulate an optimization problem of the form:

(PI) *Minimize* $\operatorname{Im} f(z)$ *subject to* $z \in Y$.

Since $\operatorname{Im} f(z) = \operatorname{Re}\,[-if(z)]$ for all $z \in Y$, it follows that Problem (PI) is of the same type as Problem (PR).

In general, the solutions of the problems (PR) and (PI) are not identical (see the examples below).

Since, in Problem (PR) the imaginary part of the function f is not taken into account, while in Problem (PI) it is the real part of f that is not taken into consideration, it seems to be of interest to study problems in which both the real and the imaginary part of f are simultaneously considered. The solutions of these problems will make a compromise between the solutions of Problems (PR) and (PI), minimizing simultaneously both the function $\operatorname{Re} f$ and the function $\operatorname{Im} f$ on Y.

A problem in which one takes into account simultaneously both the real and the imaginary part of f is the following:

(P_λ) *Minimize* $\operatorname{Re} \bar{\lambda} f(z)$ *subject to* $z \in Y$,

where $\lambda = \lambda_1 + i\lambda_2 \in C$ $(\lambda_1, \lambda_2 \in \mathbf{R})$, is a fixed complex number.

If $\lambda = 1$, then Problem (P_λ) turns into Problem (PR), while if $\lambda = i$, then Problem (P_λ) reduces to Problem (PI).

Of course, Problem (P_λ) is of the type of (PR).

Another example in which both the real and the imaginary part of f are simultaneously taken into consideration is the following:

(PM) *Minimize* $|f(z)|$ *subject to* $z \in Y$.

The problem

(Pminmax) *Minimize* $[\max\{\operatorname{Re} f(z), \operatorname{Im} f(z)\}]$ *subject to* $z \in Y$,

is again a problem in which one has to minimize on Y simultaneously both the real and the imaginary part of f.

Naturally, the problems (P_λ), (PM), (Pminmax) can be considered as particular cases of the problem:

(PF) *Minimize* $\operatorname{Re} F(f(z))$ *subject to* $z \in Y$,

where $F : C \to C$ is a given function.

Obviously, Problem (PF) is of the type of (PR).

Another point of view in dealing with the problem of minimizing simultaneously both the real and the imaginary part of f on Y, might be the following: a point $z^0 \in Y$ should be a solution if

$$\left.\begin{array}{c} z \in Y \\ \operatorname{Re} f(z) < \operatorname{Re} f(z^0) \end{array}\right\} \Rightarrow \operatorname{Im} f(z) > \operatorname{Im} f(z^0),$$

and

$$\left.\begin{array}{c} z \in Y \\ \operatorname{Im} f(z) < \operatorname{Im} f(z^0) \end{array}\right\} \Rightarrow \operatorname{Re} f(z) > \operatorname{Re} f(z^0).$$

We give in this sense the following

DEFINITION. *Let Y be a nonempty set in C^n and let $f : Y \to C$. A point $z^0 \in Y$ is called a v-minimum point with respect to f on Y if there exists no point $z \in Y$ such that*

$$\left\{\begin{array}{l} \operatorname{Re} f(z) \leqslant \operatorname{Re} f(z^0) \\ \operatorname{Im} f(z) \leqslant \operatorname{Im} f(z^0) \\ f(z) \neq f(z^0). \end{array}\right.$$

The problem of determining all v-minimum points $z \in Y$ with respect to f on Y will be denoted by

(PV) *v-Minimize* $f(z)$ *subject to* $z \in Y$.

A v-minimum point with respect to f on Y will also be called, a v-minimum point for Problem (PV).

Examples. 1. Let $f(z) = z\bar{z} - z$ for all $z \in C$, let $Y = \{z \in C / |\arg z| \leqslant \pi/4, \operatorname{Re}(1 - z) \geqslant 0\}$, let $F(w) = w^2 - 2w - w\bar{w}$ for all $w \in C$, and let $\lambda = 1 + i$. Problem (PR) has the solution $z^1 = 1/2$, Problem (PI) has the solution $z^2 = 1 + i$, Problem (P_λ) has the solution $z^3 = (1 + i)/2$, Problem (PM) has the solutions $z^4 = 0$ and $z^5 = 1$, Problem (Pminmax) has the solution $z^6 = (1 + (\sqrt{2} - 1))/2$, Problem (PF) has the solutions $z = (1 + ti)/2$ for all $t \in [-1, 1]$, and Problem (PV) has the v-minimum points $z = (1 + it)/2$ for all $t \in [0, 1]$ and $z = s + is$ for all $s \in [1/2, 1]$.

2. Let $f(z) = (1 - i)(z^2 + \bar{z}^2) + 2(1 + i)z\bar{z} - 4z$ for all $z \in C$, let $Y = \{z \in C / |\arg z| \leqslant \pi/4, \operatorname{Re}(1 - z) \geqslant 0\}$, let $F(w) = (1 + i)w$ for all $w \in C$, and let $\lambda = 1 + i$. Problem (PR) has the solutions $z = (1 + it)/2$ for all $t \in [-1, 1]$, Problem (PI) has the solutions $z = (s + i)/2$ for al $s \in [1, 2]$, Problem (P_λ) has the solution $z^1 = (1 + i)/2$, Problem (PM has the solutions $z^2 = 0$, $z^3 = 1$ and $z^4 = 1 + i$, Problem (Pminmax) ha the solution $z^5 = z^1$ and Problem (PV) has the solution $z^6 = z^1$.

4. Results

We shall new establish relations among the solutions of Problems (PR), (PI), (P_λ), (PM), (Pminmax), (PF) and (PV).

THEOREM 1. *Let Y be a nonempty set in C^n and let $f : Y \to C$. Let $F : C \to C$ be a function satisfying the following condition:*

$$(1)\qquad \left.\begin{array}{c} w, u \in C \\ \operatorname{Re} w \leqslant \operatorname{Re} u \\ \operatorname{Im} w \leqslant \operatorname{Im} u \\ w \neq u \end{array}\right\} \Rightarrow \operatorname{Re} F(w) < \operatorname{Re} F(u).$$

If $z^0 \in C$ is a solution of Problem (PF), *then z^0 is a v-minimum point for Problem* (PV).

Proof. Obviously, we have $z^0 \in Y$. Assume that z^0 is not a v-minimum point for Problem (PV). Then there exists $z^1 \in Y$ such that

$$(2)\quad \begin{aligned} \operatorname{Re} f(z^1) &\leqslant \operatorname{Re} f(z^0) \\ \operatorname{Im} f(z^1) &\leqslant \operatorname{Re} f(z^0) \\ f(z^1) &\neq f(z^0). \end{aligned}$$

By (2) and (1) it follows that $\operatorname{Re} F(f(z^1)) < \operatorname{Re} F(f(z^0))$, i.e. z^0 is not a solution of Problem (PF). But this is a contradiction, therefore z^0 is a v-minimum point for Problem (PV).

COROLLARY 1. *Let Y be a nonempty set in C^n, let $f: Y \to C$ and let $\lambda = \lambda_1 + i\lambda_2 \in C$ with $\lambda_1, \lambda_2 > 0$ be fixed.*

If $z^0 \in C^n$ is a solution of Problem (P_λ), *then z^0 is a v-minimum point for Problem* (PV).

Proof. Apply Theorem 1 for the function $F: C \to C$ defined by the formula $F(w) = \bar{\lambda} w$ for all $w \in C$.

COROLLARY 2. *Let Y be a nonempty set in C^n, and let $f: Y \to C$ be a function with $\operatorname{Re} f(z) \geqslant 0$, $\operatorname{Im} f(z) \geqslant 0$ for all $z \in Y$.*

If $z^0 \in C^n$ is a solution of Problem (PM), *then z^0 is a v-minimum point for Problem* (PV).

Proof. Apply Theorem 1 for the function $F: C \to C$ defined by the formula $F(w) = |w|$ for all $w \in C$.

THEOREM 2. *Let Y be a nonempty set in C^n and let $f: Y \to C$. Let $F: C \to C$ be a function satisfying the following condition:*

$$(3)\quad \left.\begin{aligned} w, u &\in C \\ \operatorname{Re} w &\leqslant \operatorname{Re} u \\ \operatorname{Im} w &\leqslant \operatorname{Im} u \end{aligned}\right\} \Rightarrow \operatorname{Re} F(w) \leqslant \operatorname{Re} F(u).$$

If Problem (PF) *has a unique solution $z^0 \in C^n$, then z^0 is a v-minimum point for Problem* (PV).

Proof. Obviously, we have $z^0 \in Y$. Assume that z^0 is not a v-minimum point for Problem (PV), i.e. there exists $z^1 \in Y$ with $z^1 \neq z^0$ such that (2) holds. By (3) and (2) it follows that $\operatorname{Re} F(f(z^1)) \leqslant \operatorname{Re} F(f(z^0))$, which contradicts the fact that z^0 is the unique solution of Problem (PF).

COROLLARY 3. *Let Y be a nonempty set in C^n and let $f: Y \to C$. Let $\lambda = \lambda_1 + i\lambda_2 \in C$ with $\lambda_1, \lambda_2 \geqslant 0$ be fixed. If Problem* (P_λ) *has a unique solution $z^0 \in C^n$, then z^0 is a v-minimum point for Problem* (PV).

Proof. Apply Theorem 2 for the function $F: C \to C$ defined by $F(w) = \bar{\lambda} w$ for all $w \in C$.

COROLLARY 4. *Let Y be a nonempty set in C^n and let $f: Y \to C$.* If Problem (PR) *has a unique solution $z^0 \in C^n$, then z^0 is a v-minimum point for Problem* (PV).

Proof. Apply Corollray 3 with $\lambda = 1$.

COROLLARY 5. *Let Y and f be as in Corollary 4. If Problem* (PI) *has a unique solution $z^0 \in C^n$, then z^0 is a v-minimum point for* (PV).

Proof. Apply Corollary 3 with $\lambda = i$.

COROLLARY 6. *Let Y be a nonempty set in C^n and let $f: Y \to C$. If Problem* (Pminmax) *has a unique solution $z^0 \in C^n$, then z^0 is a v-minimum point for Problem* (PV).

Proof. Apply Theorem 2 for the function $F: C \to C$ defined by the formula $F(w) = \max\{\operatorname{Re} w, \operatorname{Im} w\}$ for all $w \in C$.

THEOREM 3. *Let $Y \subseteq C^n$, $Y \neq \emptyset$ and let $f: Y \to C$. If $z^0 \in C^n$ is a common solution of Problem* (PR) *and* (PI), *then z^0 is a v-minimum point for Problem* (PV).

Proof. Obviously, we have $z^0 \in Y$. Assume that z^0 is not a v-minimum point for Problem (PV), i.e. there exists $z^1 \in Y$ such that (2) holds.

If $\operatorname{Re} f(z^1) < \operatorname{Re} f(z^0)$ and $\operatorname{Im} f(z^1) \leqslant \operatorname{Im} f(z^0)$, it follows that z^0 is not a solution of Problem (PR), which is a contradiction.

If $\operatorname{Re} f(z^1) \leqslant \operatorname{Re} f(z^0)$ and $\operatorname{Im} f(z^1) < \operatorname{Im} f(z^0)$, it follows that z^0 is not a solution of Problem (PI), which is a contradiction, too.

Therefore, z^0 is a v-minimum point for Problem (PV).

THEOREM 4. *Let Y be a nonempty set in C^n and let $f: Y \to C$. Let $F: C \to C$ be a function satisfying condition* (3). *If $z^0 \in C^n$ is a common solution of Problems* (PR) *and* (PI), *then z^0 is a solution of Problem* (PF).

Proof. Let z^0 be a common solution of Problems (PR) and (PI). Then $\operatorname{Re} f(z^0) \leqslant \operatorname{Re} f(z)$ and $\operatorname{Im} f(z^0) \leqslant \operatorname{Im} f(z)$ for all $z \in Y$. Since the function F satisfies Condition (3), it follows that $\operatorname{Re} F(f(z^0)) \leqslant \operatorname{Re} F(f(z))$ for all $z \in Y$, i.e. z^0 is a solution of Problem (PF).

COROLLARY 7. *Let $Y \subseteq C^n$, $Y \neq \emptyset$ and let $f: Y \to C$. If $z^0 \in C^n$ is a common solution of Problem* (PR) *and* (PI), *then z^0 is a solution of Problem* (Pminmax).

Proof. Apply Theorem 4 for the function $F: C \to C$ defined by $F(w) = \max\{\operatorname{Re} w, \operatorname{Im} w\}$ for all $w \in C$.

COROLLARY 8. *Let $Y \subseteq C^n$, $Y \neq \emptyset$, and let $f: Y \to C$. If $z^0 \in C^n$ is a common solution of Problems* (PR) *and* (PI), *then z^0 is a solution of Problem* (P_λ) *for any $\lambda = \lambda_1 + i\lambda_2 \in C$ with $\lambda_1, \lambda_2 \geqslant 0$.*

Proof. Apply Theorem 4 for the function $F: C \to C$ defined by $F(w) = \bar{\lambda} w$ for all $w \in C$.

REFERENCES

[1] Abrams, Robert A. and Ben-Israel, A., *Nonlinear Programming in Complex Space: Necessary Conditions.* SIAM J. Control, **9**, 606–620 (1971).

[2] Abrams, Robert A., *Nonlinear Programming in Complex Space: Sufficient Conditions and Duality.* J. Math. Anal. Appl., **38**, 619–632 (1972).

[3] Ben-Israel, A., *Linear Equations and Inequalities on Finite Dimensional, Real or Complex Space: a Unified Theory.* J. Math. Anal. Appl., **27**, 367–389 (1969).

[4] Craven, B. D. and Mond, B., *A Fritz John Theorem in Complex Space.* Bull. Austral. Math. Soc., **8**, 215–220 (1973).

[5] Duca, Dorel, *Constraint Qualifications in Nonlinear Programming in Complex Space.* Studia Univ. Babeş-Bolyai, Mathematica, **23**, 61–65 (1978).

[6] Duca, Dorel I., *On Vectorial Programming Problem in Complex Space.* Studia Univ. Babeş-Bolyai, Mathematica, **24**, 51–56 (1979).

[7] Duca, Dorel I., *The Vectorial Programming Problem in Complex Space* (to apper).

[8] Levinson, Norman, *Linear Programming in Complex Space.* J. Math. Anal. Appl., **14**, 44–62 (1966).

Received 6. VI. 1980

Faculty of Mathematics
„Babeş-Bolyai" University
3400 Cluj-Napoca, Romania