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SOME REMARKS ON MINIMAL POINTS IN NORMED LINEAR SPACES

by

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Let $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ be two real normed linear spaces such that $Y \subset X$ and let $\lambda \geqslant 1$. We assign to each nonempty set $M \subset Y$ a set $M^{\lambda}_{Y,X} \subset X$ in the following way : $x \in M^{\lambda}_{Y,X}$ if there exists no $y \in Y$, $y \neq x$ such that

$$||y - m||_Y \leqslant \lambda ||x - m||_X \quad \text{for each } m \in M.$$

For $y_0 \in Y$ and $r \geqslant 0$ we denote

$$B_Y(y_0, r) = \{y \in Y : ||y - y_0||_Y \leqslant r\}.$$

Then obviously $x \in M^{\lambda}_{Y,X}$ if and only if the set

$$\bigcap_{m \in M} B_Y(m, \lambda \, || \, x - m||_X)$$

is either empty or $\{x\}$.

When $Y = X$, $||x||_Y = |x||_X$ for each $x \in X$ and $\lambda = 1$, then for $M \subset X$, the set $M^1_{X,X}$ is the *set of minimal points with respect to* M studied by B. BEAUZAMY and B. MAUREY in [1], [2] and denoted there by $\min M$. When $Y \subset X$, $||y||_Y = ||y||_X$ for each $y \in Y$ and $\lambda = 1$, then for $M \subset Y$, the set $M^1_{Y,X}$ was introduced and studied in [3] and denoted there by $M_{Y,X}$. When $||y||_Y = ||y||_X$ for each $y \in Y$ and $\lambda \geqslant 1$, some results of this paper have been announced, without proofs, in [4].

In the next remark we extend for $M^{\lambda}_{Y,X}$ some elementary properties of $\min M$ given in [2] (see also [3] for the case $||y||_Y = ||y||_X$ for each $y \in Y$ and $\lambda = 1$), the proofs being similar and simple.

Remark 1. Let M be a nonempty subset of Y and $\lambda \geqslant 1$. Then:

a) For each real α we have $(\alpha M)^{\lambda}_{Y,X} = \alpha M^{\lambda}_{Y,X}$.

b) For each $y \in Y$ we have $(M + y)^{\lambda}_{Y,X} = M^{\lambda}_{Y,X} + y$.

c) If $M \subset L \subset Y$ then $M \subset M^{\lambda}_{Y,X} \subset L^{\lambda}_{Y,X}$. If M is a dense subset of L (in the $||.||_Y$ topology) then $M^{\lambda}_{Y,X} = L^{\lambda}_{Y,X}$.

d) If M is a bounded set in both $||.||_Y$ and $||.||_X$, then $M^{\lambda}_{Y,X}$ is a bounded set of X.

In Remark 2 of [3] we gave some simple connections between $M^{1}_{Y,Y}$ and $M^{1}_{Y,X}$, as well as $M^{1}_{X,X}$ and $M^{1}_{Y,X}$, when $||y||_Y = ||y||_X$ for each $y \in Y$. In the next remark we extend these results, the proofs being straightforward.

Remark 2. Let $M \subset Y$ be a nonempty set and $\lambda \geqslant 1$.

a) If $||y||_Y \geqslant ||y||_X$ for each $y \in Y$, then

$$M^{\lambda}_{Y,Y} \subset M^{\lambda}_{Y,X} \cap Y \tag{1}$$

$$M^{\lambda}_{X,X} \subset M^{\lambda}_{Y,X} \tag{2}$$

b) If $||y||_Y \leqslant ||y||_X$ for each $y \in Y$, then

$$M^{\lambda}_{Y,X} \cap Y \subset M^{\lambda}_{Y,Y} \tag{3}$$

In particular, if $||y||_Y = ||y||_X$ for each $y \in Y$, then

$$M^{\lambda}_{Y,Y} = M^{\lambda}_{Y,X} \cap Y \tag{4}$$

c) If $1 \leqslant \lambda \leqslant \mu$, then

$$M^{\mu}_{Y,X} \subset M^{\lambda}_{Y,X} \tag{5}$$

Examples showing that even when $||y||_Y = ||y||_X$ for each $y \in Y$, the inclusions $M^{1}_{Y,Y} \subset M^{1}_{Y,X}$ and $M^{1}_{X,X} \subset M^{1}_{Y,X}$ are strict in general have been given in [3]. Other examples showing that the inclusions in (1), (3) and (5) could be strict will be given after Remark 3 below. Clearly, when $M = \{m\}$, $m \in Y$ then $M^{\lambda}_{Y,Y} = M^{\lambda}_{X,X} = M^{\lambda}_{Y,X} = M$. When $M = \{m_1, m_2\}$, $m_1, m_2 \in Y$, $m_1 \neq m_2$, then these equalities are no longer true in general, as the results below will show (see formula (6) and Example 1), but when $||y||_Y = ||y||_X$ for each $y \in Y$ and $\lambda > 1$ then these equalities hold (see Remark 3).

When X is a normed linear space, we shall denote by ex $B_X(0, 1)$ the set of all extreme points of $B_X(0,1)$. For $x_1, x_2 \in X$, we denote $[x_1, x_2] = \{\alpha x_1 + (1 - \alpha)x_2 : 0 \leqslant \alpha \leqslant 1\}$.

In [3], Theorem 1, we have proved that when $(X, ||.||)$ is a normed linear space, Y a linear subspace of X (i.e., when $||y||_Y = ||y||_X = ||y||$ for each $y \in Y$) and $M = \{m_1, m_2\}$, $m_1, m_2 \in Y$, $m_1 \neq m_2$, then

$$M^{1}_{Y,Y} = M^{1}_{Y,X} = M \text{ or } [m_1, m_2] \tag{6}$$

and we have $M^{1}_{Y,X} = [m_1, m_2]$ if and only if $(m_1 - m_2)/||m_1 - m_2|| \in$ ex $B_Y(0,1)$.

Remark 3. When $||y||_Y = ||y||_X$ for each $y \in Y$ and $M = \{m_1, m_2\}$, $m_1, m_2 \in Y$, $m_1 \neq m_2$, then for each $\lambda > 1$ we have

$$M^{\lambda}_{Y,Y} = M^{\lambda}_{X,X} = M^{\lambda}_{Y,X} = M \tag{7}$$

Indeed, for $\lambda > 1$ we have, by (5) and (6) that $M^{\lambda}_{Y,X} \subset [m_1, m_2]$. Let $x = \alpha m_1 + (1 - \alpha)m_2$, $0 < \alpha < 1$ and let $y \in Y$, $y = \beta m_1 + (1 - \beta)m_2$, $\beta = \alpha\lambda$. We have $x \neq y$ and $||y - m_i||_X \leqslant \lambda||x - m_i||_X$, that is $x \notin M^{\lambda}_{Y,X}$. By Remark 1c) it follows $M = M^{\lambda}_{Y,X}$. Hence (7) is proved.

Now we shall give examples showing that the inclusions in (1), (3), (5) are strict in general.

Example 1. In order to show that the inclusion in (1) could be strict, let X be the linear space of all real sequences $x = (\xi_1, \ldots, \xi_n, \ldots)$ with $\sum_{i=1}^{\infty} |\xi_i| < \infty$ endowed with the norm $||x||_X = ||x||_{c_0}$ and let $Y = X$ endowed with the norm $||y||_Y = ||y||_{l^1}$. Then for each $y \in Y$ we have $||y||_X \leqslant ||y||_Y$. Let k be an integer greater than 3λ and let $m = (\eta_1, \ldots, \eta_n, \ldots) \in Y$ where $\eta_n = 1$ for $n \leqslant k$ and $\eta_n = 0$ for $n > k$. Let $M = \{0, m\} \subset Y$. We show that $2m \in M^{\lambda}_{Y,X} \cap Y$. Let $y \in Y$ such that $||y||_Y \leqslant \lambda ||2m||_X = 2\lambda$ and $||y - m||_Y \leqslant \lambda|| 2m - m||_X = \lambda$. Then $\lambda \geqslant ||y - m||_Y \geqslant ||m||_Y - ||y||_Y \geqslant k - 2\lambda$ which is impossible since $k > 3\lambda$. Therefore $2m \in M^{\lambda}_{Y,X} \cap Y$. By (6), (for $\lambda = 1$) or (7), (for $\lambda > 1$), $2m \notin M^{\lambda}_{Y,Y}$ and so in this case the inclusion in (1) is strict. This example shows also that when the assumption $||y||_Y = ||y||_X$ for each $y \in Y$ is not fulfilled, then it can happen that $M^{\lambda}_{Y,Y} \neq M^{\lambda}_{Y,X}$, or $M^{\lambda}_{X,X} \neq M^{\lambda}_{Y,X}$, for some sets $M = \{m_1, m_2\}$, $m_1, m_2 \in Y$, $m_1 \neq m_2$.

Example 2. In order to show that the inclusion in (3) could be strict, let X be the same linear space given in Example 1 endowed with the norm $||x||_X = ||x||_{l^1}$ and $Y = X$ endowed with the norm $||y||_Y = ||y||_{c_0}$. For $n = 1, 2, \ldots$, let $m_n = (\eta_{1n}, \ldots, \eta_{jn}, \ldots) \in Y$ where $\eta_{jn} = 1/n$ for $j \leqslant n$ and $\eta_{jn} = 0$ otherwise. Let $M = \{m_n : n = 1, 2, \ldots\}$. Since 0 is in the $||\cdot||_Y$ closure of M, by Remark 1c), it follows $0 \in M^{\lambda}_{Y,Y}$. On the other hand, $0 \notin M^{\lambda}_{Y,X}$ since for $y = (1, 0, 0, \ldots) \in Y$, we have for each n, $||y - m_n||_Y \leqslant \lambda||m_n||_Y = \lambda$ and so the inclusion in (3) is strict in this case.

Example 3. Finally, to show that the inclusion in (5) could be strict, let $(X, ||\cdot||)$ be a normed linear space such that ex $B_X(0,1)$ is nonempty. Let $m \in$ ex $B_X(0,1)$ and $M = \{0, m\}$. By Remark 3 we have $M^{\lambda}_{X,X} = M$ for $\lambda > 1$ and by [3], Theorem 1, $M^{1}_{X,X} = \{\alpha m : 0 \leqslant \alpha \leqslant 1\}$.

By Remark 1c) it follows that we have $M^{\lambda}_{Y,X} \subset Y$ for each $M \subset Y$ if and only if $Y^{\lambda}_{Y,X} = Y$. We shall give now an equivalent condition for $Y^{\lambda}_{Y,X} = Y$ in terms of the existence of some mappings of X onto Y.

Let $\tilde{P}(X, Y, \lambda)$ be the set of all mappings $\tilde{P}: X \to Y$ with the following properties:

$$\tilde{P}(\alpha x) = \alpha \tilde{P}(x) \qquad (x \in X,\ \alpha \in \mathbf{R}) \tag{8}$$

$$\tilde{P}(x + y) = \tilde{P}(x) + y \qquad (x \in X,\ y \in Y) \tag{9}$$

$$\|\tilde{P}(x)\|_Y \leqslant \lambda \|x\|_X \qquad (x \in X) \tag{10}$$

THEOREM 1. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed linear spaces with $Y \subset X$ and let $\lambda \geqslant 1$. Suppose that for each $y \in Y$ we have $\|y\|_Y \leqslant \lambda \|x\|_X$. Then $\tilde{P}(X, Y, \lambda) \neq \emptyset$ if and only if $Y^{\lambda}_{Y,X} = Y$.*

Proof. Suppose there exists $\tilde{P} \in \tilde{P}(X, Y, \lambda)$. Then for each $x \in X \setminus Y$ and each $y \in Y$ we have by (8)—(10)

$$\|\tilde{P}(x) - y\|_Y = \|\tilde{P}(x - y)\|_Y \leqslant \lambda \|x - y\|_X$$

which shows that $x \notin Y^{\lambda}_{Y,X}$ and so $Y^{\lambda}_{Y,X} \subset Y$. By Remark 1c) it follows $Y^{\lambda}_{Y,X} = Y$.

Conversely, suppose $Y^{\lambda}_{Y,X} = Y$. If $X = Y$, then for each $x \in X$, we define $\tilde{P}(x) = x$. Then (8), (9) are obviously fulfilled, while (10) follows since $\|\tilde{P}(x)\|_Y = \|x\|_Y \leqslant \lambda \|x\|_X$. Suppose now $Y \neq X$. Let X/Y be the quotient space and we denote by $\hat{x}$ the equivalence class of $x \in X$. For each one-dimensional subspace G of X/Y we shall fix an element $\hat{x}_G$ such that $G = \mathrm{sp}\,[\hat{x}_G]$. Choose a representative $x \in \hat{x}_G$. Since $x_G \in X \setminus Y$, by hypothesis $x_G \notin Y^{\lambda}_{Y,X}$, and so

$$\bigcap_{y \in Y} B_Y(y, \lambda \|x_G - y\|_X) \neq \emptyset. \tag{11}$$

Choose y_G in this intersection. Let now $\hat{x} \in G$. Then $\hat{x} = \alpha \hat{x}_G$ for some $\alpha \in \mathbf{R}$. If $x \in \hat{x}$, then $x = \alpha x_G + y$ for some $y \in Y$. We define $\tilde{P}(x) = \alpha y_G + y$. So we have defined $\tilde{P}$ for the elements of the equivalence classes of each one-dimensional subspace of X/Y, whence $\tilde{P}: X \to Y$ is well defined (but not always unique). Then clearly $\tilde{P}$ defined as above satisfies (8)—(10). Let us note that (10) holds using either the fact that y_G belongs to the left hand side of (11) (for $x \in X \setminus Y$) or the assumption $\|y\|_Y \leqslant \lambda \|y\|_X$ for each $y \in Y$ (for $x \in Y$).

When $\|y\|_Y = \|y\|_X$ for each $y \in Y$ and $\lambda = 1$, we gave in [3] the following result. Let us denote $S_X = \{x \in X : \|x\|_X = 1\}$, $\mathrm{sm}\, S_X$ the set of all $x \in S_X$ such that there exists a unique $x^*_x \in S_{X^*}$ with $x^*_x(x) = 1$, and $P(X, Y, \lambda)$ the set of all linear projections P of X onto Y with $\|P\| \leqslant \lambda$.

THEOREM 2. ([3], *Theorem* 2) *Let $(X, \|\cdot\|)$ be a normed linear space and Y a closed linear subspace of X such that $S_Y \subset \mathrm{sm}\, S_X$. Then $P(X,Y,1)$ contains at most one element and $P(X,Y,1) \neq \emptyset$, if and only if $Y^1_{Y,X} = Y$.*

By the first part of this result follows the implication 1) $\Rightarrow$ 3) of Theorem 3.2 [8] which states that if E is a very smooth Banach space (i.e., $S_E \subset S_{E^{**}}$) then each closed subspace of E is the range of at most one contractive projection in E^{**}.

COROLLARY 1. *Let $(X, \|\cdot\|)$ be a normed linear space and Y a closed linear subspace of X such that $S_Y \subset \mathrm{sm}\, S_X$. Then $\tilde{P}(X,Y,1) = P(X,Y,1)$ and $\tilde{P}(X,Y,1)$ contains at most one element.*

Proof. Let $\tilde{P} \in \tilde{P}(X,Y,1)$. By Theorem 1 it follows $Y^1_{Y,X} = Y$ and by Theorem 2, $P(X, Y, 1)$ contains exactly one element, say P. Since for each $x \in X$, both $P(x)$ and $\tilde{P}(x)$ belong to the left hand side of (11) (for $\lambda = 1$, $x_G = x$ and $\|y\|_Y = \|y\|_X = \|y\|$ for each $y \in Y$), which, as we have shown in the proof of Theorem 2 [3], is a singleton. Therefore $\tilde{P}(x) = P(x)$, which completes the proof.

Let $(X, \|\cdot\|)$ be a normed linear space and Y a linear subspace of X. For each $x \in X$ let $P_Y(x)$ be the set of all best approximations of x out of Y, i.e., $P_Y(x) = \{y_0 \in Y : \|x - y_0\| = \mathrm{dist}\,(x, Y)\}$. Y is called (see e.g., [7]) a *proximinal* subspace of X if $P_Y(x) \neq \emptyset$ for each $x \in X$, and a *Čebyšev* subspace of X if $P_Y(x)$ contains exactly one element for each $x \in X$. We shall denote the elements of $P_Y(x)$ by $p_Y(x)$.

COROLLARY 2. *Let Y be a proximinal subspace of the normed linear space $(X, \|\cdot\|)$ such that $S_Y \subset \mathrm{sm}\, S_X$ and $\|p_X(x)\| \leqslant \|x\|$ for each $p_Y(x) \in P_Y(x)$ and each $x \in X$. Then Y is a Čebyšev subspace of X and p_Y is linear. Moreover $P(X, Y, 1) = \{p_Y\}$.*

Proof. If Y is not a Čebyšev subspace of X, then there exists $x_0 \in X \setminus Y$ and $p^{(1)}_Y(x_0), p^{(2)}_Y(x_0) \in P_Y(x_0)$, $p^{(1)}_Y(x_0) \neq p^{(2)}_Y(x_0)$. In the quotient space X/Y, let $G_0 = \mathrm{sp}\,[\hat{x}_0]$ and for each one-dimensional subspace $G \subset X/Y$, $G \neq G_0$, let $\hat{x}_G \in G$ such that $G = \mathrm{sp}\,[\hat{x}_G]$. Choose $x_G \in \hat{x}_G$ and $p_Y(x_G) \in P_Y(x_G)$. We denote $\hat{x}_{G_0} = \hat{x}_0$ and $x_{G_0} = x_0$.

Let now G be a one-dimensional subspace of X/Y and $\hat{x} \in G$. Then $\hat{x} = \alpha \hat{x}_G$ for some $\alpha \in \mathbf{R}$. If $x \in \hat{x}$, then $x = \alpha x_G + y$ for some $y \in Y$. We define for $G = G_0$, $\tilde{P}_i(x) = \alpha p^{(i)}_Y(x_0) + y$, $i = 1, 2$ and for $G \neq G_0$, $\tilde{P}_i(x) = \alpha p_Y(x_G) + y$, $i = 1, 2$. Then $\tilde{P}_i : X \to Y$, $i = 1, 2$ are well defined and it is easy to show that $\tilde{P}_i (i = 1, 2)$ satisfy (8) and (9). Since $\tilde{P}_i(x) \in P_Y(x)$, using the hypothesis $\|p_Y(x)\| \leqslant \|x\|$ for each $p_Y(x) \in P_Y(x)$ and each $x \in X$, we have $\tilde{P}_i \in \tilde{P}(X,Y,1)$, $i = 1, 2$. This contradicts Corollary 1 since $\tilde{P}_1(x_0) \neq \tilde{P}_2(x_0)$. Therefore Y is a Čebyšev subspace of X and since $p_Y \in \tilde{P}(X, Y, 1)$, using again Corollary 1 it follows that p_Y is linear and $P(X, Y, 1) = \{p_Y\}$.

When Y is a proximinal subspace of X, using an argument similar with that of Corollary 2, we can choose a selection $p_Y(x) \in P_Y(x)$, $(x \in X)$ satisfying (8), (9), and since $\|p_Y(x)\| \leqslant 2\|x\|$ for each $x \in X$, $\tilde{P}(X,Y,\lambda) \neq \emptyset$

for each $\lambda \geqslant 2$, and so by Theorem 1, $Y^{\lambda}_{Y,X} = Y$ for each $\lambda \geqslant 2$. Using this remark and a now classical result of J. Lindenstrauss and L. Tzafriri [5] it follows that for $\lambda \geqslant 2$ we have not a result similar with Theorem 2.

We conclude this paper with the following possible generalization: for each $M \subset Y$ and each $\lambda \geqslant 1$, let M^{λ}_{YX} be the set of all $x \in X$ with the property that there exists no $y \in Y$ such that $\|y - m\|_Y < \lambda \|x - m\|_X$ for each $m \in M$. Clearly, this set is larger than that studied in this paper. When $Y = X$, $\|x\|_Y = \|x\|_X$ for each $x \in X$ and $\lambda = 1$ then M^1_{XX} is the set of all „closest points to M" (see e.g., [6], where a characterization of a Hilbert space of dimension at least three is given using this notion).

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