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ON THE CONDITIONS FOR CONVERGENCE OF MULTISTEP METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS

by

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In this paper we shall study the application of multistep methods to the initial problem

(1) $$y'(t) = f(t, y(t)),\ t \in I = [a, b],$$

(2) $$y(a) = \eta.$$

Let k be a positive integer. Let $T_i = \{t_{hj} : j = 0\,(1)\,i\}$, where $t_{hj} = a + jh$, $j = 0\,(1)\,N$ with the uniform step $h = (b - a)/N$. To compute an approximate solution of the problem (1–2) we can apply multistep methods y_h of order k satisfying the conditions

(3) $$\begin{cases} \|y_h(t) - y(t)\| \leqslant \varepsilon_1(t, h),\ t \in [a, a + h(k-1)], \\ \|y_h(t+h) - F(t, h, y_h(t), y_h(t-h), \ldots, y_h(t - h(k-1)))\| \leqslant \\ \qquad \leqslant \varepsilon_2(t, h),\ \text{for } t \in [a + h(k-1), b - h], \end{cases}$$

where y is the exact solution of $(1-2)$ in $[a, a + h(k-1)]$.

The purpose of this paper is to give conditions by which this multistep method y_h is convergent in the Dahliquist's sence to the solution y of (1–2) i.e. for arbitrary $\eta \in \mathbf{R}^m$, $\|y_h(t) - y(t)\| \to 0$ as $h \to 0$ with t fixed, $t \in I$.

ASSUMPTIONS *H. Suppose that*

1° $F : I \times H \times \underbrace{\mathbf{R}^m \times \ldots \times \mathbf{R}^m}_{k} \to \mathbf{R}^m$ *where* $H = [0, h_0]$, $h_0 \in (0, \infty)$

2° *there exist functions* $\lambda_i : I \times H \to \mathbf{R}^1_+ = [0, \infty)$, $\beta : I \times H \to \mathbf{R}^1_+$ *such that for* $x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbf{R}^m$ *we have*

$$\|F(t, h, x_1, x_2, \ldots, x_k) - F(t, h, y_1, y_2, \ldots, y_k)\| \leqslant$$
$$\leqslant \sum_{i=1}^{k} \lambda_i(t, h)\|x_i - y_i\| + \beta(t, h),$$

3° *there exists a function* $\varepsilon_3 : [a + h(k-1), b - h] \times H \to \mathbf{R}^1_+$ *such that*

$$\|F(t, h, z_1, z_2, \ldots, z_k) - y(t+h)\| \leqslant \varepsilon_3(t, h) \tag{4}$$

where $y : [t - h(k-1), t+h] \to \mathbf{R}^m$ *is a solution of Eq.* (1) *satisfying the conditions*

$$y(t - h(i-1)) = z_i, \quad i = 1(1)k.$$

We denote the symbols: $I_r = \{-k, -k+1, \ldots, 0, \ldots, r\}$, $I'_r = \{0, 1, \ldots, r\}$ and

$$\sum_{j=1}^{q}{}' C_j = \begin{cases} 0 & \text{if } 1 > q, \\ \sum_{j=1}^{q} C_j & \text{if } 1 \leqslant q, \end{cases}$$

and

$$\prod_{j=1}^{q}{}' C_j = \begin{cases} 1 & \text{if } 1 > q, \\ \prod_{j=1}^{q} C_j & \text{if } 1 \leqslant q. \end{cases}$$

We then have

LEMMA A. *If*

1° $\lambda_i : I \times H \to \mathbf{R}^1_+$, $i = 1\,(1)\,k$,

2° $\sum_{i=1}^{k} \lambda_i(t, h) \geqslant 1$, $t \in I$, $h \in H$,

3° $p_h : I \times H \to \mathbf{R}^1_+$,

4° $u_h(t) = \varepsilon_1(h)$, $t \in T_{k-1}$,

$$u_h(t_{h,k+i}) = p_h(t_{h,k+i-1}, h) + \sum_{j=1}^{k} \lambda_j(t_{h,k+i-1}, h)\, u_h(t_{h,k+i-j}), \quad i \in I'_{N-k}$$

then

$$u_h(t_{h,k+i}) \leqslant d_h(t_{h,k+i}), \quad i \in I_{N-k} \tag{5}$$

and

$$d_h(t_{h,k+i-1}) \leqslant d_h(t_{h,k+i}), \quad i \in I_{N-k} \setminus \{0\}, \tag{6}$$

where

$$d_h(t_{h,k+1}) = \begin{cases} \varepsilon_1(h), \ i \in I_{-1}, \\ \varepsilon_1(h) \prod_{j=0}^{i} w(t_{h,k-1+j}) + \sum_{j=0}^{i} p_h(t_{h,k-1+j}, h) \prod_{r=j+1}^{i}{}' w(t_{h,k-1+r}) & i \in I'_{N-k} \end{cases}$$

where

$$w(t, h) = \sum_{i=1}^{k} \lambda_i(t, h).$$

Proof. The inequalities (5) and (6) we can prove by induction. It is easy to see that the inequalities (5) and (6) are true for $i \in I_0$. Now, we suppose that (5) and (6) are true for $i \in I_m$, $m \geqslant 0$. Then we have

$$u_h(t_{h,k+m+1}) = p_h(t_{h,k+m}, h) + \sum_{j=1}^{k} \lambda_j(t_{h,k+m}, h)\, u_h(t_{h,k+m+1-j}) \leqslant$$
$$\leqslant p_h(t_{h,k+m}, h) + d_h(t_{h,k+m})\, w(t_{h,k+m}) =$$
$$= p_h(t_{h,k+m}, h) + w(t_{h,k+m}) \Big[\varepsilon_1(h) \prod_{j=0}^{m} w(t_{h,k-1+j}) +$$
$$+ \sum_{j=0}^{m} p_h(t_{h,k-1+j}, h) \prod_{r=j+1}^{m} w(t_{h,k-1+r}) \Big] =$$
$$= p_h(t_{h,k+m}, h) + \varepsilon_1(h) \prod_{j=0}^{m+1} w(t_{h,k-1+j}) +$$
$$+ \sum_{j=0}^{m} p_h(t_{h,k-1+j}, h) \prod_{r=j+1}^{m+1}{}' w(t_{h,k-1+r}) = d_h(t_{h,k+m+1}).$$

Moreover

$$d_h(t_{h,k+m+1}) = p_h(t_{h,k+m}, h) + w(t_{h,k+m}) d_h(t_{h,k+m}) \geqslant d_h(t_{h,k+m}).$$

Now, we obtain (5) and (6) by induction. Thus the proof of the lemma is completed.

Let for $t \in I$ us define the sequence $\{\lambda_h^{(n)}\}$ by the relations

$$\begin{cases} \lambda_h^{(0)}(t) = 1, \\ \lambda_h^{(p)}(t) = \sum_{i=1}^{p} \lambda_i(t, h) \lambda_h^{(p-i)}(t - ih), \quad p = 1, 2, \ldots, \end{cases} \tag{7}$$

where $\lambda_i(t, h) \equiv 0$ if $i > k$.

Put

$$\varphi_h(t, r) = p_h(t, h) + \sum_{j=r}^{k}{}' \lambda_j(t, h) \varepsilon_1(t + (1-j)h, h),$$

where $p_h(t, h) = \varepsilon_2(t, h) + \varepsilon_3(t, h) + \beta(t, h)$.

We then have

THEOREM A. *Let the assumptions* H *are fulfilled and let there exists the unique solution* y *of the problem* (1—2). *Suppose that the inequalities* (3) *are fulfilled for* $\varepsilon_1 : [a, a + h(k-1)] \times H \to \mathbf{R}^1_+$ and $\varepsilon_2 : [a + h(k-1), b - h] \times H \to \mathbf{R}^1_+$. Then we have the estimation

$$(8) \qquad \|y_h(t_{h,k+i}) - y(t_{h,k+i})\| \leqslant u_h(t_{h,k+i}), \quad i \in I_{N-k},$$

where

$$(9) \qquad \begin{cases} u_h(t) = \varepsilon_1(t, h), \quad t \in T_{k-1}, \\ u_h(t_{h,k+i}) = \sum_{j=0}^{i} \varphi_h(t_{h,k+i-j-1}, i + 1 - j)\lambda_h^{(j)}(t_{h,k+i-1}), \quad i \in I'_{N-k}. \end{cases}$$

Moreover if

1° $\varepsilon_1(t, h) = \varepsilon_1(h)$,

2° *there exists a Lebesgue integrable function* $M : I \to \mathbf{R}^1_+$ *and a constant* $c > 0$ *such that*

$$1 \leqslant \sum_{i=0}^{k} \lambda_i(t, h) \leqslant c^{\int_t^{t+h} M(s)\, ds}$$

$$3° \lim_{\substack{N \in \infty \\ \text{or } h \to 0}} \left[\varepsilon_1(h) + \sum_{j=0}^{N-k} p_h(t_{h,k-1+j}, h)\right] = 0$$

then

$$(10) \quad u_h(t_{h,k+i}) \leqslant \varepsilon_1(h) \prod_{j=0}^{i} w(t_{h,k-1+j}) + \sum_{j=0}^{i} p_h(t_{h,k-1+j}, h) \prod_{r=j+1}^{i}{}' w(t_{h,k-1+r}), \qquad i \in I'_{N-k},$$

and the method y_h *of* (3) *is convergent to the unique solution* y *of* (1—2).

Proof. We see that the estimation (8) is true for $i \in I_0$. Let us suppose that the estimation (8) is true for some integer $i \geqslant 0$. Put

$$v_h(t) = \|y_h(t) - y(t)\|, \quad t \in I,$$
$$Q(t, z) = F(t, h, z(t), \ldots, z(t - h(k-1))).$$

By (3) and the assumptions H we have

$$v_h(t_{h,k+i+1}) \leqslant \|y_h(t_{h,k+i+1}) - Q(t_{h,k+i}, y_h)\| +$$
$$+ \|Q(t_{h,k+i}, y_h) - Q(t_{h,k+i}, y)\| + \|Q(t_{h,k+i}, y) - y(t_{h,k+i+1})\| \leqslant$$
$$\leqslant p_h(t_{h,k+i}, h) + \sum_{j=1}^{k} \lambda_j(t_{h,k+i}, h) v_h(t_{h,k+i+1-j}) \leqslant$$

$$\leqslant p_h(t_{h,k+i}, h) + \sum_{j=1}^{\min(i+1,k)} \lambda_j(t_{h,k+i}, h) u_h(t_{h,k+i+1-j}) +$$
$$+ \sum_{j=\min(i+1,k)+1}^{k}{}' \lambda_j(t_{h,k+i}, h)\varepsilon_1(t_{h,k+i+1-j}, h) = \varphi_h(t_{h,k+i}, 1 + \min(i+1, k)) +$$
$$+ \sum_{j=1}^{\min(i+1,k)} \lambda_j(t_{h,k+i}, h) \sum_{s=0}^{i+1-j} \varphi_h(t_{h,k+i-j-s}, i + 2 - j - s)\, \lambda_h^{(s)}(t_{h,k+i-j}).$$

Let $\min(i+1, k) = i + 1$. Then, assembling in the last relation the elements φ_h we have the estimation (8).

If $\min(i+1, k) = k$ then $k \leqslant i + 1$ and $\lambda_j(t, h) \equiv 0$ for $j = k + 1, \ldots, i + 1$. Then we have

$$v_h(t_{h,k+i+1}) \leqslant \varphi_h(t_{h,k+i}, k + 1) + \sum_{j=1}^{k} \lambda_j(t_{h,k+i}, h) \sum_{s=0}^{i+1-j} \varphi_h(t_{h,k+i-j-s}, i + 2 -$$
$$- j - s)\lambda_h^{(s)}(t_{h,k+i-j}) = \varphi_h(t_{h,k+i}, i + 2) +$$
$$+ \sum_{j=1}^{i+1} \lambda_j(t_{h,k+i}, h) \sum_{s=0}^{i+1-j} \varphi_h(t_{h,k+i-j-s}, i + 2 - j - s)\lambda_h^{(s)}(t_{h,k+i-j}).$$

Hence and above we have the estimation (8) with $i + 1$ instead of i. Now, we obtain (8) by induction.

Note the inequality (10) follows from Lemma A. Now using 2° we have

$$\prod_{j=0}^{i} w(t_{h,k-1+j}) \leqslant c^{\int_{t_{h,k-1}}^{t_{h,k+i}} M(s)\, ds} \leqslant c^{\int_a^b M(s)\, ds}, \quad i \in I'_{N-k}.$$

Hence and (10) and 3° we see that the method y_h of (3) is convergent to the unique solution y of (1—2).

Thus the proof of Theorem A is completed.

Remark 1. It is easy to see that the proof of Theorem A remains true if the condition (4) is replaced by

$$\|F(t, h, y(t), \ldots, y(t - h(k-1))) - y(t + h)\| \leqslant \varepsilon_3(t, h).$$

Remark 2. If $k = 1$ then we have one-step method and

$$\lambda_h^{(p)}(t) = \prod_{i=0}^{p-1}{}' \lambda_1(t - ih), \quad p = 0, 1, 2, \ldots$$

Now the sequence $\{u_h\}$ defined by (9) is of the form

$$u_h(t_{h,i+1}) = \varepsilon_1(a, h) \prod_{j=0}^{i}{}' \lambda_1(t_{h,j}, h) + \sum_{j=0}^{i}{}' p_h(t_{h,j}, h) \prod_{r=j+1}^{i}{}' \lambda_1(t_{hr}, h),$$
$$i = -1, 0, \ldots, N - 1.$$

Now if

$$\lambda_1(t, h) \leqslant c^{\int_t^{t+h} M(s)\,ds}, \quad c > 0,$$

and if

$$\lim_{\substack{N\to\infty\\ \text{or } h\to 0}} \left[\varepsilon_1(a, h) + \sum_{j=0}^{N-1} p_h(t_{h,j}, h) \right] = 0$$

then the one-step method y_h of (3) is convergent to the unique solution y of (1–2) (see [3]).

Remark 3. Let $\lambda_i(t, h) = a_i + b_i h$, $a_i, b_i \geqslant 0$, $i = 1(1)k$. If $\sum_{i=1}^{k} a_i = 1$ then the condition 2° is true for $M = \sum_{i=1}^{k} b_i$ and $c = \exp(1)$.

Of course we have

$$\sum_{i=1}^{k} \lambda_i(t, h) = 1 + Mh \leqslant \exp(Mh).$$

Remark 4. It is easy to see that if $p_h(t, h) = h\tilde{p}_h(t, h)$ and $\lim_{h\to 0} \tilde{p}_h(t, h) = \lim_{h\to 0} \varepsilon_1(h) = 0$ then the condition 3° is true.

Remark 5. If

1° the function y' is continuous in the closed interval I $(y'(t) = f(t, y(t))$, $t \in I)$,

2° there exist constants a_i, b_j, $i = 0(1)k$, $j = 1(1)k$, $a_0 \neq 0$, $|a_k| + |b_k| > 0$ and a function $\delta : H \to \mathbf{R}^1_+$, $\lim_{h\to 0} \delta(h) = 0$, such that

(i) $$\sum_{i=0}^{k} a_i = 0, \qquad \sum_{i=0}^{k-1} a_i(k-i) = \sum_{i=1}^{k} b_i,$$

(ii) $$\left\| F(t, h, u_1, \ldots, u_k) + \frac{1}{a_0}\sum_{r=1}^{k} a_r u_r - \frac{h}{a_0}\sum_{r=1}^{k} b_r f(t-(r-1)h, u_r) \right\| \leqslant h\delta(h)$$

then the function ε_3 of (4) has of the form

(11) $$\varepsilon_3(t_{h,m}, h) = h\tau(h),$$

and

(12) $$\lim_{h\to 0} \tau(h) = 0,$$

where

$$\tau(h) = \delta(h) + K\chi(kh),$$

and

$$K = \frac{1}{|a_0|}\left[\sum_{i=0}^{k-1} |a_i|(k-i) + \sum_{i=1}^{k-1} |b_i| \right],$$

$$\chi(\varepsilon) = \max_{\substack{|x_1 - x_2| \leqslant \varepsilon\\ x_1, x_2 \in I}} \|y'(x_1) - y'(x_2)\|.$$

Indeed, we have

$$\|F(t_{h,m+k-1}, h, y(t_{h,m+k-1}), \ldots, y(t_{h,m})) - y(t_{h,m+k})\| \leqslant$$

$$\leqslant \left\| F(t_{h,m+k-1}, h, y(t_{h,m+k-1}), \ldots, y(t_{h,m})) + \frac{1}{a_0}\sum_{i=1}^{k} a_i y(t_{h,m+k-i}) - \right.$$

$$\left. - \frac{h}{a_0}\sum_{i=1}^{k} b_i f(t_{h,m+k-i}, y(t_{h,m+k-i})) \right\| + \left\| \frac{1}{a_0}\sum_{i=0}^{k} a_i y(t_{h,m+k-i}) - \frac{h}{a_0}\sum_{i=1}^{k} b_i y'(t_{h,m+k-i}) \right\|.$$

Hence and [2] (see (5–172), p. 245) we have (11) and (12).

REFERENCES

[1] I. Babuska, M. Prager and E. Vitasek, *Numerical processes in differential equations*. Praha 1966.
[2] P. Henrici, *Discrete variable methods in ordinary differential equations*. New York: John Wiley 1968.
[3] T. Jankowski, *Numerical solution of initial problems for systems of differential equations with discontinuous derivatives* (to appear).
[4] T. Shimizu, *Contribution to the theory of numerical integration of nonlinear differential equations*, VIII. TRU Math. 13, pp. 65–75, (1977).

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