MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 10, Nº 1, 1981, pp. 57–74

FREE SETS ASSOCIATED TO A FINITE ORIENTED GRAPH

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In this paper, we consider a vector spece $\mathfrak{X}$ of finite dimensions with $\mathfrak{B} = \{b_1, b_2, \ldots, b_n\}$ as a bases and **R** (the field of real numbers) as a range of values; we shall introduce the concept of *free set* $\mathfrak{L}$ ($\mathfrak{L} \subseteq \mathfrak{B}$) related to a nonull subspace $\mathfrak{W}$ of $\mathfrak{X}$. Using this concept we shall characterize the bases of the subspaces Ker Δ and Im ∇ defined in [17]; we make also some remarks concerning the free sets associated to Ker Δ and Im ∇.

1. Introduction. The results in this paper are natural consequences of what we got in [16], [17] and [18]. They make it easy for us to tackle some concepts in the graph theory using a powerful accurate mathematical apparatus, linear algebra.

Since we have defined the concepts of free sets related to a nonull subspace of a linear space of finite dimensions, we shall continue the study of the subspaces KerΔ and Im∇ that were introduced in [17], and we shall concurrently point out the necessary remarks concerning the free sets associated to this spaces.

2. Preliminary considerations. This whole paragraph is meant to present some definitions and results of [17] and [18] that are necessary to follow and understand the further notes.

Iu this article we shall consider the notion of *finite oriented graph* defined in [5] where $\mathfrak{N} = \{n_1, n_2, n_3, \ldots, n_p\}$ is the set of *nodes* and $\mathfrak{A} = \{a_1, a_2, \ldots, a_q\}$ is the set of *arcs*. An arc $a \in \mathfrak{A}$ will be marked $a = \langle n, m \rangle$, $n, m \in \mathfrak{N}$, and to the graph $G = \langle \mathfrak{N}, \mathfrak{A} \rangle$ we shall relate (see [17]) the functions $\Delta^+ : \mathfrak{A} \to$ and $\Delta^- : \mathfrak{A} \to \mathfrak{N}$, where $\Delta^+(a) = n$ and $\Delta^-(a) = m$.

We call *incidence matrix* (see [5]) of the graph G, the matrix $\Lambda = (\Lambda_k^i) = 1, 2, \ldots, p$; $k = 1, 2, \ldots, q$, defined as follows:

$$\Lambda_k^i = \begin{cases} 1, \text{ if } n_i = \triangle^+(a_k) \neq \triangle^-(a_k), \\ -1, \text{ if } n_i = \triangle^-(a_k) \neq \triangle^+(a_k), \\ 0, \text{ otherwise.} \end{cases}$$

We shall denote by $V[\mathcal{A}, \mathbf{R}]$, (see [17]) the set of the linear forms X with $\mathcal{A}$ as bases and $\mathbf{R}$ as value range (we denote by $\mathbf{R}$ the field of the real numbers), $X = \sum_{k=1}^{q} x_k a_k$, $x_k \in \mathbf{R}$, where $\pm a_i$ is the vector with all the $x_k = 0$, except for $x_i = \pm 1$ and $\theta_{\mathcal{A}}$ the vector with all the $x_k = 0$. Similarly, we define the vector speace $V[\mathcal{N}, \mathbf{R}]$ as consisting in all the vectors of the form $Z = \sum_{i=1}^{p} z_i n_i$, $z_i \in \mathbf{R}$, where $\theta_{\mathcal{N}}$ is the vector with all $z_i = 0$ and $\pm n_i$ is the vector with all $z_j = 0$ except for $z_i = \pm 1$ Practically, $V[\mathcal{A}, \mathbf{R}]$ is $\mathbf{R}^q$ and $V[\mathcal{N}, \mathbf{R}]$ is $\mathbf{R}^p$.

To the above defined vector spaces the following applications will be attached (see [17]): $\triangle : V[\mathcal{A}, \mathbf{R}] \to V[\mathcal{N}, \mathbf{R}]$, $\nabla : V[\mathcal{N}, \mathbf{R}] \to V[\mathcal{A}, \mathbf{R}]$ defined by $\triangle(X) = \sum_{i=1}^{p}\left(\sum_{k=1}^{q} \Lambda_k^i x_k\right) \cdot n_i$, $\nabla(Z) = \sum_{k=1}^{q}\left(\sum_{i=1}^{p} \Lambda_k^i z_i\right) \cdot a_k$, where

$$X = \sum_{k=1}^{q} x_k a_k \in V[\mathcal{A}, \mathbf{R}] \text{ and } Z = \sum_{i=1}^{p} z_i n_i \in V[\mathcal{N}, \mathbf{R}].$$

Out of the definitions of the functions $\triangle$ and ∇, results the fact that $\triangle(a_k) = \sum_{i=1}^{p} \Lambda_k^i n_i$, $k = 1, 2, \ldots, q$ and $\nabla(n_i) = \sum_{k=1}^{q} \Lambda_k^i a_k$, $i = 1, 2, \ldots, p$ which means that $\triangle$ and ∇ are two homomorphisms between $V[\mathcal{A}, \mathbf{R}]$ and $V[\mathcal{N}, \mathbf{R}]$. They have as a matrix for linear transformation, the incidence matrix of the graph $G = \langle \mathcal{N}, \mathcal{A} \rangle$, and according to [15] it means that $\dim V[\mathcal{A}, \mathbf{R}] = \dim \operatorname{Ker} \triangle + (\dim \operatorname{Im} \nabla = \dim \operatorname{Im} \triangle)$ or:

$$q = \dim \operatorname{Ker} \triangle + [\operatorname{rank}(\Lambda) = \dim \operatorname{Im} \nabla] \quad (2.1),$$

where

$$\operatorname{Ker} \triangle = \{X \in [\mathcal{A}, \mathbf{R}] \mid \nabla(X) = \theta_{\mathcal{N}}\} \text{ and } \operatorname{Im} \nabla = \nabla(V[\mathcal{N}, \mathbf{R}])$$

are linear subspaces in $V[\mathcal{A}, \mathbf{R}]$, (see [17]).

In [17] we have considered a vector space of the finite dimensions $\mathcal{X}$ with $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ as basis and $\mathbf{R}$ the values range, and, subsequently, we have introduced the preorder relation „$\sqsubseteq$", where $X \sqsubseteq Y$ if and only if $\mathcal{B}^+(X) \subseteq \mathcal{B}^+(Y)$ and $\mathcal{B}^-(X) \subseteq \mathcal{B}^-(Y)$, with $\mathcal{B}^+(X) = \{b_i \in \mathcal{B} \mid x_i > 0\}$, $\mathcal{B}^-(X) = \{b_i \in \mathcal{B} \mid x_i < 0\}$, $\mathcal{B}(X) = \mathcal{B}^+(X) \cup \mathcal{B}^-(X)$, and $X = \sum_{i=1}^{n} x_i b_i \in \mathcal{X}$.

It was in [17] as well that we have defined the notion of *minimal** related to a nonull subspace $\mathcal{Y}$ of $\mathcal{X}$ and we showed that any nonull vector $X \in \mathcal{Y}$ is of the form $X = \sum_{i=1}^{m} \overline{X}_i$ with $\overline{X}_i \in \mathcal{Y}$, minimal related to $\mathcal{Y}$ and $\overline{X}_i \sqsubseteq X$ for any $i = 1, 2, \ldots, m$.

As to a vector $X = \sum_{i=1}^{n} x_i b_i \in \mathcal{X}$ we may say it is *elementary* as compared to $\mathcal{Y}$ if it is minimal related to $\mathcal{Y}$ and $x_i \in \{-1, 0, 1\}$ for any $i = 1, 2, \ldots, n$.

In [17] we particularized $\mathcal{X}$ to $V[\mathcal{A}, \mathbf{R}]$ and $\mathcal{Y}$ to $\operatorname{Ker} \triangle$ or/and $\operatorname{Im} \nabla$, we proved that any nonull vector $X \in \operatorname{Ker} \triangle$ is of the form $X = \sum_{i=1}^{m} \alpha_i \overline{X}_i$ with $\alpha_i \in \mathbf{R} - \{0\}$ and $\overline{X}_i$ elementary related to $\operatorname{Ker} \triangle$, and any nonull vector $Y \in \operatorname{Im} \nabla$ is of the form $Y = \sum_{i=1}^{\tilde{m}} \beta_i \overline{Y}_i$ with $\beta_i \in \mathbf{R} - \{0\}$ and $\overline{Y}_i$ elementary related to $\operatorname{Im} \nabla$. (2.2).

3. Remarks on the subspaces $\operatorname{Ker} \triangle$ and $\operatorname{Im} \nabla$. In this paragraph we shall define the *free set* related to a nonull subspace of a linear space of finite dimensions; subsequently, making use of this notion we shall characterize the bases of the subspaces $\operatorname{Ker} \triangle$ and $\operatorname{Im} \nabla$, simultaneously making the necessary remarks on the free sets as compared to these subspaces.

DEFINITION 3.1. Let us consider $\mathcal{X}$ as a space of finite dimensions and $\mathcal{B}$ its bases. A set $\mathcal{L} \subseteq \mathcal{B}$, $\mathcal{L} \neq \emptyset$ is said to be *free* related to a nonull subspace $\mathcal{Y}$ of $\mathcal{X}$, if and only if for every nonull $X \in \mathcal{X}$ with $\mathcal{B}(X) \subseteq \mathcal{L}$ we have $X \notin \mathcal{Y}$.

Remark 3.1. One can presently notice that if $\mathcal{L}$ is free and $\mathcal{C} \subseteq \mathcal{L}$, $\mathcal{C} \neq \emptyset$ then $\mathcal{C}$ is free.

THEOREM 3.1. *If $\mathcal{F} \subseteq \mathcal{A}$ is free related to* $\operatorname{Ker} \triangle$ *and maximal* (as compared to the inclusion of sets) *with this property and $a_k \in \mathcal{A} \setminus \mathcal{F}$, then, there exists an unique $X^{[k]} \in$* $\operatorname{Ker} \triangle$, *elementary, so that $X^{[k]} = a_k + \sum_{a_\alpha \in \mathcal{F}} x_\alpha^{[k]} a_\alpha$, with $x_\alpha^{[k]} \in \{-1, 0, 1\}$.*

Proof. If $a_k \in \mathcal{A} \setminus \mathcal{F}$, then, having in view the maximality of $\mathcal{F}$, the set $\mathcal{F} \cup \{a_k\}$ is no longer free related to $\operatorname{Ker} \triangle$; that means that there is $X \in \operatorname{Ker} \triangle$, $X \neq \theta_{\mathcal{A}}$ with $\mathcal{A}(X) \subseteq \mathcal{F} \cup \{a_k\}$. (3.1)

Taking into account (2.2), it means that $X = \sum_{i=1}^{m} \alpha_i \overline{X}_i$ with $\alpha_i \in \mathbf{R} - \{0\}$, $\overline{X}_i$ elementary in $\operatorname{Ker} \triangle$ and $\overline{X}_i \sqsubseteq \overline{X}$ for all $i = 1, 2, \ldots, m$.

* a vector $X \in \mathcal{X}$ is said to be *minimal* related to a nonull subspace $\mathcal{Y}$ of $\mathcal{X}$, if and only if X is nonull, $X \in \mathcal{Y}$ and for any $Y \in \mathcal{X}$ with $\mathcal{B}(Y) \subset \mathcal{B}(X)$, we have $Y \notin \mathcal{Y}$.

But if $\overline{X}_i \sqsubseteq X$, then, $\mathcal{A}(\overline{X}_i) \subseteq \mathcal{A}(X)$ and according to the relation 3.1. we have $\mathcal{A}(\overline{X}_i) \subseteq \mathcal{F} \cup \{a_h\}$, which means that $\overline{X}_i$, being elementary, is of the form $\overline{X}_i = x_k^{[i]} a_k + \sum_{a_\alpha \in \mathcal{F}} x_\alpha^{[i]} a_\alpha$, with $x_\alpha^{[i]} \in \{-1, 0, 1\}$ and $x_k^{[i]} \in \{-1, 1\}$.

If $x_k^{[i]} = 0$, then, $\mathcal{A}(\overline{X}_i) \subseteq \mathcal{F}$, i.e. $\mathcal{F}$ is not free related to Ker Δ; contradiction with the above hypothesis.

If there exists $i_0 \in \{1, 2, \ldots, m\}$, so that $x_k^{[i_0]} = 1$, then we take $\overline{X}^{[k]} = \overline{X}_{i_0}$. But, if $x_k^{[i]} = -1$ for all $i = 1, 2, \ldots, m$, we consider the vectors $\overline{X}_i^* = -\overline{X}_i$, $i = 1, 2, \ldots, m$, which are obviously elementary in Ker Δ (see (1.21) in [17]) and for which $\mathcal{A}(\overline{X}_i^*) \subseteq \mathcal{F} \cup \{a_k\}$ for every $i = 1, 2, \ldots, m$. Of course, there exists now $i_0 \in \{1, 2, \ldots, m\}$ for which $x_k^{*[i_0]} = 1$ and we take $X^{[k]} = X_{i_0}^*$.

So, it exists $X^{[k]} = a_k + \sum_{a_\alpha \in \mathcal{F}} x_\alpha^{[k]} a_\alpha$, elementary in Ker Δ. Let us prove now that $X^{[k]}$ is unique. To this purpose let us consider $\tilde{X} \in$ Ker Δ, elementary, with $\mathcal{A}(\tilde{X}) \subseteq \mathcal{F} \cup \{a_k\}$ of the form $\tilde{X} = a_k + \sum_{a_\alpha \in \mathcal{F}} \tilde{x}_\alpha a_\alpha$.

Since Ker Δ is a linear subspace, there results the fact that $\tilde{X} - X^{[k]}$ belongs to Ker Δ, which means that $\sum_{a_\alpha \in \mathcal{F}} (\tilde{x}_\alpha - x_\alpha^{[k]})\, a_\alpha \in$ Ker Δ. As $\mathcal{A}(\tilde{X} - X^{[k]}) \subseteq \mathcal{F}$ and $\mathcal{F}$ is free related to Ker Δ, it means that $\tilde{X} = X^{[k]}$ (Q.E.D.).

Let us consider $\bar{k} = |\mathcal{A} \setminus \mathcal{F}|$ and let us re-mark the set $\mathcal{A}$ as follow: $\mathcal{A} = \{V_1, V_2, \ldots, V_{\bar{k}}, U_1, U_2, \ldots, U_{\bar{m}}\} = \{a_1, a_2, \ldots, a_q\}$, where $V_\alpha \in \mathcal{A} \setminus \mathcal{F}$, $\alpha = 1, 2, \ldots, \bar{k}$, and $U_\beta \in \mathcal{F}$, $\beta = 1, 2, \ldots, \bar{m}$, with $\bar{m} = q - \bar{k}$.

R e m a r k 3.2. Having in view the theorem 3.1, it means that for any $\alpha \in \{1, 2, \ldots, \bar{k}\}$, there exists an unique $X^{[\alpha]} = V_\alpha + \sum_{\beta=1}^{\bar{m}} x_\beta^{[\alpha]} U_\beta$, elementary in Ker Δ with $x_\beta^{[\alpha]} \in \{-1, 0, 1\}$.

DEFINITION 3.2. Let $\mathcal{X}$ be a linear space over $\mathbf{R}$ and $\langle X_1, X_2, \ldots, X_n\rangle$ a vector system of $\mathcal{X}$. We call the system $\langle X_1, X_2, \ldots, X_n\rangle$ *a systeme of generating vectors* of $\mathcal{X}$, if any vector $X \in \mathcal{X}$ can be expressed as a linear combination of these vectors: $X = \sum_{i=1}^{n} r_i X_i$ with $r_i \in \mathbf{R}$, $X_i \in \mathcal{X}$, $i = 1, 2, \ldots, n$.

THEOREM 3.2. *The vectors* $X^{[1]} X^{[2]}, \ldots, X^{[\bar{k}]}$, *make up a system of generating vectors of the space* Ker Δ.

Proof. Let X be a vector of Ker Δ. According to the way re-marked the set $\mathcal{A}$, it means that $X = \sum_{\alpha=1}^{\bar{k}} x_\alpha V_\alpha + \sum_{\beta=1}^{\bar{m}} x_\beta U_\beta$, with $x_\alpha, x_\beta \in \mathbf{R}$, $\alpha = 1, 2, \ldots \bar{k}$; $\beta = 1, 2, \ldots, \bar{m}$.

Using the components of the vector X we define the vector

$$\tilde{X} = \sum_{\alpha=1}^{\bar{k}} x_\alpha X^{[\alpha]} = \sum_{\alpha=1}^{\bar{k}} x_\alpha V_\alpha + \sum_{\alpha=1}^{\bar{k}} \sum_{\beta=1}^{\bar{m}} x_\alpha x_\beta^{[\alpha]} U_\beta,$$

vector for which we have:

$$X - \tilde{X} = \sum_{\beta=1}^{\bar{m}} x_\beta U_\beta - \sum_{\alpha=1}^{\bar{k}} \sum_{\beta=1}^{\bar{m}} x_\alpha x_\beta^{[\alpha]} U_\beta = \sum_{\beta=1}^{\bar{m}} x_\beta U_\beta - \sum_{\beta=1}^{\bar{m}} \sum_{\alpha=1}^{\bar{k}} x_\alpha x_\beta^{[\alpha]} U_\beta =$$

$$= \sum_{\beta=1}^{\bar{m}} \left(x_\beta - \sum_{\alpha=1}^{\bar{k}} x_\alpha x_\beta^{[\alpha]} \right) U_\beta. \quad (3.2)$$

But, $(X - \tilde{X}) \in$ Ker Δ and taking into account the relation (3.2) it means that $\mathcal{A}(X - \tilde{X}) \subseteq \{U_1, U_2, \ldots, U_{\bar{m}}\} = \mathcal{F}$. Since $\mathcal{F}$ is a free set related to Ker Δ, it results that $X - \tilde{X} = \theta_{\mathcal{A}}$.

So, $X = \sum_{\alpha=1}^{\bar{k}} x_\alpha X^{[\alpha]}$. (Q.E.D.).

DEFINITION 3.3. Let $\mathcal{X}$ be a linear space over $\mathbf{R}$, and $\langle X_1, X_2, \ldots, X_n\rangle$ a vector system of $\mathcal{X}$. The system $\langle X_1, X_2, \ldots, X_n\rangle$ is *linear independent* if for any null linear combination $\sum_{i=1}^{n} r_i X_i = \theta_{\mathcal{B}}$, it results that $r_i = 0$, $i = 1, 2, \ldots, n$, where $r_i \in \mathbf{R}$ and $\theta_{\mathcal{B}}$ is null vector of the space $\mathcal{X}$.

DEFINITION 3.4. Let $\mathcal{X}$ be a linear space over $\mathbf{R}$ and $\langle X_1, X_2, \ldots, X_n\rangle$ a vector system of $\mathcal{X}$. The system $\langle X_1, X_2, \ldots, X_n\rangle$ is *a bases* of the space $\mathcal{X}$ if and only if $\langle X_1, X_2, \ldots, X_n\rangle$ constitutes a linear independent system of generating vectors of the space $\mathcal{X}$.

THEOREM 3.3. *The vectors* $X^{[1]}, X^{[2]}, \ldots, X^{[\bar{k}]}$ *represent a linear independent system in the space* Ker Δ.

Proof. Let $r_1X^{[1]} + r_2X^{[2]} + \ldots + r_{\bar{k}}X^{\bar{k}} = \theta_{\mathfrak{A}}$ a null linear combination of the vectors $X^{[1]}, X^{[2]}, \ldots, X^{[\bar{k}]}$. Having in view the remark 3.2. we obtain:

$$\theta_{\mathfrak{V}} = r_1V_1 + r_1x_1^{[1]}U_1 + r_1x_2^{[1]}U_2 + \ldots + r_1x_{\bar{m}}^{[1]}U_{\bar{m}} +$$
$$+ r_2V_2 + r_2x_1^{[2]}U_1 + r_2x_2^{[2]}U_2 + \ldots + r_2x_{\bar{m}}^{[2]}U_{\bar{m}} +$$
$$\vdots$$
$$+ r_{\bar{k}}V_{\bar{k}} + r_{\bar{k}}x_1^{[\bar{k}]}U_1 + r_{\bar{k}}x_2^{[\bar{k}]}U_2 + \ldots + r_{\bar{k}}x_{\bar{m}}^{[\bar{k}]}U_{\bar{m}} =$$
$$= r_1V_1 + r_2V_2 + \ldots + r_{\bar{k}}V_{\bar{k}} + \left(\sum_{j=1}^{\bar{k}} r_jx_1^{[j]}\right)U_1 + \left(\sum_{j=1}^{\bar{k}} r_jx_2^{[j]}\right)U_2 + \ldots +$$
$$+ \left(\sum_{j=1}^{\bar{k}} r_jx_{\bar{m}}^{[j]}\right)U_{\bar{m}}. \quad (3.3)$$

Denoted by $\omega_i = \sum_{j=1}^{\bar{k}} r_jx_i^{[j]}$, $i = 1, 2, \ldots, \bar{m}$, and taking into account the relation (3.3) we obtain: $\theta_{\mathfrak{A}} = r_1V_1 + r_2V_2 + \ldots + r_{\bar{k}}V_{\bar{k}} + \omega_1U_1 + \omega_2U_2 + \ldots + \omega_{\bar{m}}U_{\bar{m}}$. (3.4)

But, $\{V_1, V_2, \ldots, V_{\bar{k}}, U_1, U_2, \ldots, U_{\bar{m}}\} = \{a_1, a_2, \ldots, a_q\} = \mathfrak{A}$, and having in mind that $\mathfrak{A}$ is a bases of $V[\mathfrak{A}, \mathbf{R}]$ it means that the system $\langle V_1, V_2, \ldots, V_{\bar{k}}, U_1, U_2, \ldots, U_{\bar{m}}\rangle$ is linear independent and from (3.4) we have: $r_1 = r_2 = \ldots = r_{\bar{k}} = \omega_1 = \omega_2 = \ldots = \omega_{\bar{m}} = 0$. (3.5)

Thus, if $r_1X^{[1]} + r_2X^{[2]} + \ldots + r_{\bar{k}}X^{[\bar{k}]} = \theta_{\mathfrak{A}}$, then, having in view (3.5), it means that the vectors $X^{[1]}, X^{[2]}, \ldots, X^{[\bar{k}]}$ make up a linear independent system in Ker Δ. (Q.E.D.).

R e m a r k 3.3. Having in mind the theorems 3.3, 3.2, and taking into account the definition 3.4, it means that the vectors $X^{[1]}, X^{[2]}, \ldots, X^{[\bar{k}]}$ form up a bases of the space Ker ∇.

DEFINITION 3.5. A linear space $\mathfrak{X}$ is considered to have the *dimension n* if:

a) there exists in this space a system with n linear independent vectors (n is a nonull integer) and

b) every vector system of $\mathfrak{X}$ which contains more than n vector is not linear independent, better to say it is *linear dependent*.

R e m a r k 3.4. We can also add the fact that the dimension of a space $\mathfrak{X}$ reprezents the maximal number of linear independent vectors of this space.

R e m a r k. 3.5. Having in mind the remark 3.3. and the fact that if a linear space admits a basis with n vectors, then it has the dimension n, we can say that the dimension of the subspace Ker Δ is equal to $\bar{k}$, i,e. dim Ker $\Delta = \bar{k}$ (3.6.).

R e m a r k 3.6. The above demonstrated things lead us to the following conclusions: using the concept of free set related to a nonull subspace $\mathfrak{W}$ of a linear space $\mathfrak{X}$ of finite dimensions, and particularizing the space $\mathfrak{X}$ to $V[\mathfrak{A}, \mathbf{R}]$ and the subspace $\mathfrak{W}$ to Ker Δ, having in view the theorem 3.1., the remarks 3.2. and 3.3., we obtain a bases for the sub-space Ker Δ, which is made up of the vectors $X^{[1]}, X^{[2]}, \ldots, X^{[\bar{k}]}$. These vectors are uniquely determined related to the set $\mathfrak{T} \subseteq \mathfrak{A}$ ($|\mathfrak{T}| = \bar{m} = q - \bar{k}$), which is free in Ker Δ and maximal (as compared to the inclusion of sets) with this property.

Practically, the determination of bases for Ker Δ is to find a set $\mathfrak{T} \subseteq \mathfrak{A}$, free related to Ker Δ and maximal with this property; the construction of $X^{[\alpha]}$, $\alpha = 1, 2, \ldots, \bar{k}$ following from the remark 3.6.

R e m a r k 3.7. Let us consider $\mathfrak{T} \subseteq \mathfrak{A}$, free related to Im ∇ and maximal with this property. By means of a theorem analogous to the theorem 3.1, we can prove that if $a_k \in \mathfrak{A} \setminus \mathfrak{T}$, then there is an unique $Y^{[k]} \in$ Im ∇, elementary, so that

$$Y^{[k]} = a_k + \sum_{a_\alpha \in \mathfrak{T}} y_\alpha^{[k]} a_\alpha, \text{ with } y_\alpha^{[k]} \in \{-1, 0, 1\}.$$

If $|\mathfrak{A} \setminus \mathfrak{T}| = \bar{s}$ and re-marking the set $\mathfrak{A}$ so that $\mathfrak{A} = \{W_1, W_2, \ldots, W_{\bar{s}}, T_1, T_2, \ldots, T_{\bar{r}}\} = \{a_1, a_2, \ldots, a_q\}$, $W_\beta \in \mathfrak{A} \setminus \mathfrak{T}$, $\beta = 1, 2, \ldots, \bar{s}$ and $T_\alpha \in \mathfrak{T}$, $\alpha = 1, 2, \ldots, \bar{r}$, with $\bar{r} = q - \bar{s}$, we obtain the vector unique system $\langle Y^{[1]}, Y^{[2]}, \ldots, Y^{[\bar{s}]}\rangle$ with $Y^{[\beta]} = W_\beta + \sum_{\alpha=1}^{\bar{r}} y_\alpha^{[\beta]} T_\alpha$, with $y_\alpha^{[\beta]} \in \{-1, 0, 1\}$, $\beta = 1, 2, \ldots, \bar{s}$. This vector system can be proved, by a reasouing similar to that made in the theorems 3.2. and 3.3. to constitute a basis for the space Im ∇ and, thus, dim Im $\nabla = \bar{s}$. (3.7.).

Taking into account the relations (3.6), (3.7) and (2.1) we get: $q = \bar{k} + \bar{s}$ and considering the fact that $\bar{m} = q - \bar{k}$ and $\bar{r} = q - \bar{s}$, it results that $\bar{m} = \bar{s}$ and $\bar{r} = \bar{k}$ and thus,

$$\mathfrak{A} = \{W_1, W_2, \ldots, W_{\bar{m}}, T_1, T_2, \ldots, T_{\bar{k}}\} =$$
$$= \{V_1, V_2, \ldots, V_{\bar{k}}, U_1, U_2, \ldots, U_{\bar{m}}\} = \{a_1, a_2, \ldots, a_q\}. \quad (3.8)$$

R e m a r k 3.8. Practically speaking, the determination fo a basis for the subspace Im ∇, goes down upon the discovery of a set $\mathfrak{T} \subseteq \mathfrak{A}$, free related to Im ∇ and maximal with this property. According to it, the construction of the vectors $Y^{[1]}, Y^{[2]}, \ldots, Y^{[\bar{m}]}$ uniquely determined as compared to $\mathfrak{T}$, is imediate if we have in mind the remark 3.7.

R e m a r k 3.9. From the above mentioned thus we notice that if $\mathcal{T} \subseteq \mathcal{A}$ is free related to Ker $\triangle$ and maximal with this property, then, $|\mathcal{T}| = \bar{m} = \dim \operatorname{Im} \nabla = \operatorname{rank}(\Lambda)$, and if $\bar{\mathcal{T}} \subseteq \mathcal{A}$ is free related to Im ∇ and maximal with this property, then $\bar{\mathcal{T}} = \bar{k} = \dim \operatorname{Ker} \triangle$.

LEMMA 3.1. *If $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is free related to* Ker $\triangle$, *then there exists $\mathcal{T}_{\tilde{\mathcal{A}}} \subseteq \mathcal{A}$, free related to* Ker $\triangle$ *and maximal with this property, so that $\tilde{\mathcal{A}} \subseteq \mathcal{T}_{\tilde{\mathcal{A}}}$.*

Proof. Let $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ be free related to Ker $\triangle$. If $\tilde{\mathcal{A}}$ is maximal, then we take $\mathcal{T}_{\tilde{\mathcal{A}}} = \tilde{\mathcal{A}}$ and the theorem is proved.

If $\tilde{\mathcal{A}}$ is not maximal, then there exists at least an arc $a \in \mathcal{A} \setminus \tilde{\mathcal{A}}$ so that $\tilde{\mathcal{A}} \cup \{a\}$ to be further on free related to Ker $\triangle$.

Since $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ and $\mathcal{A}$ is finite, it means that there exists a finite number of arcs, be them $\bar{\mathcal{A}} = \{\bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_r}\}$, $\bar{\mathcal{A}} \subseteq \mathcal{A} \setminus \tilde{\mathcal{A}}$, so that $\tilde{\mathcal{A}} \cup \bar{\mathcal{A}}$ to be free related to Ker $\triangle$ and for any $a \in \mathcal{A} \setminus (\tilde{\mathcal{A}} \cup \bar{\mathcal{A}})$, the set $\tilde{\mathcal{A}} \cup \bar{\mathcal{A}} \cup \{a\}$ has no longer this property. In this case, if we take $\mathcal{T}_{\tilde{\mathcal{A}}} = \tilde{\mathcal{A}} \cup \bar{\mathcal{A}}$, it results that $\mathcal{T}_{\tilde{\mathcal{A}}}$ is free related to Ker $\triangle$, maximal with this property and $\tilde{\mathcal{A}} \subseteq \mathcal{T}_{\tilde{\mathcal{A}}}$. (Q.E.D.)

LEMMA 3.2. *If $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is free related to* Im ∇, *then there exists $\bar{\mathcal{T}}_{\tilde{\mathcal{A}}} \subseteq \mathcal{A}$, free related to* Im ∇ *and maximal with this property, so that $\tilde{\mathcal{A}} \subseteq \bar{\mathcal{T}}_{\tilde{\mathcal{A}}}$.*

Proof. Analogous to the proof of the lemma 3.1.

R e m a r k 3.10. Taking into account to the remark 3.9. and lemma 3.1, it immediatly results that if $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is free related to Ker $\triangle$ and $|\tilde{\mathcal{A}}| = \bar{m}$, then $\tilde{\mathcal{A}}$ is free related to Ker $\triangle$ and maximal with this property. Similarly, if $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is free relatet to Im ∇ and $|\tilde{\mathcal{A}}| = \bar{k}$, then, according to the lemma 3.2 and the remark 3.9, it results that the set $\tilde{\mathcal{A}}$ is free related to Im ∇ and maximal with this property. According to the relation (2.1) we have also got: $q = \bar{k} + \bar{m}$. (3.9)

THEOREM 3.6. *If $\mathcal{T} \subseteq \mathcal{A}$ is free related to* Ker $\triangle$ *and maximal with this property, then $\mathcal{A} \setminus \mathcal{T}$ is free related to* Im ∇ *and maximal with this property.*

Proof. Having in mind the remark 3.9, it means that $|\mathcal{T}| = \bar{m}$ and thus $|\mathcal{A} \setminus \mathcal{T}| = \bar{k}$ (see the relation (3.9)).

Let us prove now that the set $\mathcal{A} \setminus \mathcal{T}$ is free related to Im ∇. With this purpose, let us consider $Y \in \operatorname{Im} \nabla$, arbitrary, so that $\mathcal{A}(Y) \subseteq \mathcal{A} \setminus \mathcal{T}$. If $\mathcal{A}(Y) \subseteq \mathcal{A} \setminus \mathcal{T}$ it means that $Y = \sum_{\alpha=1}^{\bar{k}} y_\alpha V_\alpha$, with $y_\alpha \in \mathbf{R}$, $\alpha = 1, 2, \ldots, \bar{k}$. Taking into account the inner product within the linear space $V[\mathcal{A}, \mathbf{R}]$ (see [17]) and using the results obtined within the paragraph 4 in [17], we get: $[X^{[\alpha]}|Y] = 0$, for all $\alpha = 1, 2, \ldots, \bar{k}$. But, $0 = [X^{[\alpha]}|Y] = \left[V_\alpha + \sum_{\beta=1}^{\bar{m}} x_\beta^{[\alpha]} U_\beta \,\middle|\, \sum_{\alpha=1}^{\bar{k}} y_\alpha V_\alpha\right] = y_\alpha$, for all $\alpha = 1, 2, \ldots, \bar{k}$ and thus $Y = \theta_{\mathcal{A}}$. This fact leads us to the conclusion that the set $\mathcal{A} \setminus \mathcal{T}$ is free related to Im ∇.

Consequently we have $\mathcal{A} \setminus \mathcal{T}$ free related to Im ∇ and $|\mathcal{A} \setminus \mathcal{T}| = \bar{k}$; under such circumstances, according to the remark 3.10, the theorem is proved.

THEOREM 3.7. *If $\bar{\mathcal{T}} \subseteq \mathcal{A}$ is free related to* Im ∇ *and maximal with this property, then $\mathcal{A} \setminus \bar{\mathcal{T}}$ is free related to* Ker $\triangle$ *and maximal with this property.*

Proof. Similar to the proof of the theorem 3.6.

R e m a r k 3.11. The theorems 3.6 and 3.7 allow us to assert that the sets $\mathcal{T}$ and $\bar{\mathcal{T}}$ make up a partition of the set $\mathcal{A}$, and, therefore, according to the remark 3.7 we have got the followings: $\mathcal{T} = \{U_1, U_2, \ldots, U_{\bar{m}}\} = \{W_1, W_2, \ldots, W_{\bar{m}}\}$ and $\bar{\mathcal{T}} = \{V_1, V_2, \ldots, V_{\bar{k}}\} = \{T_1, T_2, \ldots, T_{\bar{k}}\}$.

R e m a r k 3.12. Taking into consideration the remarks 3.6, 3.8 and 3.11 we may assert that the moment a bases has been determined for Ker $\triangle$ we automatically have a bases for Im ∇ as well, and, reciprocally having determined a bases for Im ∇ we implicitely have a bases for Ker $\triangle$.

THEOREM 3.8. *Let $\mathcal{A}_1 \subseteq \mathcal{A}$ be free related to* Ker $\triangle$ *and $\mathcal{A}_2 \subseteq \mathcal{A}$ free related to* Im ∇, *so that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Under such circumstances there exist $\mathcal{T} \subseteq \mathcal{A}$ free related to* Ker $\triangle$ *and maximal with this property and $\bar{\mathcal{T}} \subseteq \mathcal{A}$ free related to* Im ∇ *and maximal with this property, so that $\mathcal{A}_1 \subseteq \mathcal{T}$ and $\mathcal{A}_2 \subseteq \bar{\mathcal{T}}$.*

Proof. Let us suppose that $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$. According to the lemma 3.1 there exists $\mathcal{T}_{\mathcal{A}_1} \subseteq \mathcal{A}$ free related to Ker $\triangle$ and maximal with this property so that $\mathcal{A}_1 \subseteq \mathcal{T}_{\mathcal{A}_1}$, and according to the lemma 3.2 there exists $\bar{\mathcal{T}}_{\mathcal{A}_2} \subseteq \mathcal{A}$ free relatet to Im ∇ and maximal with this property so that $\mathcal{A}_2 \subseteq \bar{\mathcal{T}}_{\mathcal{A}_2}$.

But, according to the remark 3.11, we have $\mathcal{T}_{\mathcal{A}_1} \cup \bar{\mathcal{T}}_{\mathcal{A}_2} = \mathcal{A}$ and $\mathcal{T}_{\mathcal{A}_1} \cap \bar{\mathcal{T}}_{\mathcal{A}_2} = \emptyset$; which obviously leads us to $\mathcal{T}_{\mathcal{A}_1} = \mathcal{A}_1$ and $\bar{\mathcal{T}}_{\mathcal{A}_2} = \mathcal{A}_2$, where, taking $\mathcal{T} = \mathcal{A}_1$ and $\bar{\mathcal{T}} = \mathcal{A}_2$, the theorem is proved. Let us suppose now that $\mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathcal{A}$ and let $\bar{\mathcal{A}} = \{\bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_r}\} = \mathcal{A} - (\mathcal{A}_1 \cup \mathcal{A}_2)$ be. We shall prove now either that $\mathcal{A}_1 \cup \{\bar{a}_{i_1}\}$ is free related to Ker $\triangle$, or that $\mathcal{A}_2 \cup \{\bar{a}_{i_1}\}$ is free related to Im ∇.

For this purpose, let us suppose now, against all reson, that $\mathcal{A}_1 \cup \{\bar{a}_{i_1}\}$ is not free related to Ker $\triangle$ and $\mathcal{A}_2 \cup \{\bar{a}_{i_1}\}$ is not free related to Im ∇.

Under such circumstances (see the definition 3.1) there exists $X \in \operatorname{Ker} \triangle$, $X \neq \emptyset_{\mathcal{A}}$ and $Y \in \operatorname{Im} \nabla$, $Y \neq \theta_{\mathcal{A}}$, so that $\mathcal{A}(X) \subseteq \mathcal{A}_1 \cup \{\bar{a}_{i_1}\}$,

$\mathcal{A}(Y) \subseteq \mathcal{A}_2 \cup \{\bar{a}_{i_1}\}$, where $X = x_{i_1}\bar{a}_{i_1} + \sum_{a_\alpha \in \mathcal{A}_1} x_\alpha a_\alpha$ and $Y = y_{i_1}\bar{a}_{i_1} + \sum_{a_\beta \in \mathcal{A}_2} y_\beta a_\beta$, with $x_{i_1} \neq 0$ and $y_{i_1} \neq 0$.

Having in view the inner product of $V[\mathcal{A}, \mathbf{R}]$ (see [17]) and using the results obtained in the paragraph 4 from [17], we obtain: $[X|Y] = 0$, which leads us to $0 = [X|Y] = x_{i_1}\bar{a}_{i_1} + \sum_{a_\alpha \in \mathcal{A}_1} x_\alpha a_\alpha | y_{i_1}\bar{a}_{i_1} + \sum_{a_\beta \in \mathcal{A}_2} y_\beta a_\beta] = x_{i_1} y_{i_1} \neq 0$. We get thus a contradiction, and therefore, either $\mathcal{A}_1 \cup \{\bar{a}_{i_1}\}$ is free related to Ker $\triangle$, or $\mathcal{A}_2 \cup \{\bar{a}_{i_1}\}$ is free related to Im ∇. We shall put $\mathcal{A}_1^{[1]} = \mathcal{A}_1 \cup \{\bar{a}_{i_1}\}$ if $\mathcal{A}_1 \cup \{\bar{a}_{i_1}\}$ is free related to Ker $\triangle$ and $\mathcal{A}_2^{[1]} = \mathcal{A}_2 \cup \{\bar{a}_{i_1}\}$ if $\mathcal{A}_2 \cup \{a_{i_1}\}$ is free related to Im ∇.

By a similar reasoning we can show that $\mathcal{A}_1^{[1]} \cup \{\bar{a}_{i_2}\}$ is free related to Ker $\triangle$ or $\mathcal{A}_2^{[1]} \cup \{\bar{a}_{i_2}\}$ is free related to Im ∇, similarly building up one of the sets $\mathcal{A}_1^{[2]} = \mathcal{A}_1^{[1]} \cup \{\bar{a}_{i_2}\}$ or $\mathcal{A}_2^{[2]} = \mathcal{A}_2^{[1]} \cup \{\bar{a}_{i_2}\}$.

Going on just this way, we obtain the sets $\mathcal{A}_1^{[f]}$ and $\mathcal{A}_2^{[g]}$ with $f, g \in \{0, 1, 2, \ldots, r\}$, $\mathcal{A}_1^{[0]} = \mathcal{A}_1$, $\mathcal{A}_2^{[0]} = \mathcal{A}_2$, for which $\mathcal{A}_1^{[f]}$ is free related to Ker $\triangle$, $\mathcal{A}_2^{[g]}$ is free related to Im ∇, $\mathcal{A}_1^{[f]} \cap \mathcal{A}_2^{[g]} = \emptyset$, $\mathcal{A}_1^{[f]} \cup \mathcal{A}_2^{[g]} = \mathcal{A}$, $\mathcal{A}_1 \subseteq \mathcal{A}_1^{[f]}$ and $\mathcal{A}_2 \subseteq \mathcal{A}_2^{[g]}$; so we find the hypothesis of the first part of the demonstration, where teaking $\mathcal{T} = \mathcal{A}_1^{[f]}$ and $\bar{\mathcal{T}} = \mathcal{A}_2^{[g]}$, the theorem is proved.

R e m a r k 3.12. The theorem 3.8 represents an extension of the theorems 3.6 and 3.7. Practically it's a stronger representation of these two theorems.

4. On the free sets related to Ker $\triangle$. Within this paragraph we shall give a theorem who characterizes the free sets related to the subspace Ker $\triangle$.

DEFINITION 4.1. We call *cycle* of length r (see [17]) a sequence of $r + 1$ nodes $(n_{i_0}, n_{i_1}, n_{i_2}, \ldots, n_{i_r})$ and a sequence of r arcs with sign $(\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \ldots, \varepsilon_r a_{k_r})$ where $\varepsilon_j = 1$ or -1, so that for each $j = 1, 2, \ldots, r$ we have:

$$n_{i_{j-1}} = \begin{cases} \triangle^+(a_{k_j}), & \text{if } \varepsilon_j = 1, \\ \triangle^-(a_{k_j}), & \text{if } \varepsilon_j = -1, \end{cases} \quad \text{and} \quad n_{i_j} = \begin{cases} \triangle^-(a_{k_j}), & \text{if } \varepsilon_j = 1, \\ \triangle^+(a_{k_j}), & \text{if } \varepsilon_j = -1. \end{cases}$$

We denote a cycle by $\omega[n_{i_0}]$, and its set of arcs by $\mathcal{A}(\omega[n_{i_0}])$.

THEOREM 4.1. *Let* $\mathcal{L} \subseteq \mathcal{A}$, $\mathcal{L} \neq \emptyset$ *be. The set* $\mathcal{L}$ *is free related to* Ker $\triangle$, *if and only if there is no cycle* $\omega[n]$ *so that* $\mathcal{A}(\omega[n]) \subseteq \mathcal{L}$.

Proof. Let $\mathcal{L} \subseteq \mathcal{A}$, $\mathcal{L} \neq \emptyset$ be, free related to Ker $\triangle$ and let us prove that there is no cycle $\omega[n]$ with $\mathcal{A}(\omega[n]) \subseteq \mathcal{L}$.

We suppose against all reason that there exists a cycle $\omega[n]$ with $\mathcal{A}(\omega[n]) = \{a_{k_1}, a_{k_2}, \ldots, a_{k_r}\} \subseteq \mathcal{L}$. According to the lemma 3.1. from [17], the vector $X = \sum_{j=1}^{r} \varepsilon_j a_{k_j}$ is nonull in Ker $\triangle$, and as $\mathcal{A}(X) = \mathcal{A}(\omega[n]) \subseteq \mathcal{L}$, it means that $\mathcal{L}$ is not free related to Ker $\triangle$; which is contradictory to

the above — stated hypothesis. Reciprocally, let us suppose that there is no cycle $\omega[n]$ with $\mathcal{A}(\omega[n]) \subseteq \mathcal{L}$, and let us prove that $\mathcal{L}$ is free related to Ker $\triangle$. Against all reason we suppose that $\mathcal{L}$ is not free related to Ker $\triangle$ which leads us to the existence of a nonull vector $X \in$ Ker $\triangle$, with $\mathcal{A}(X) \subseteq \mathcal{L}$.

According (2.2) the vector X is of the form $X = \sum_{i=1}^{m} \alpha_i \bar{X}_i$, with $\alpha \in \mathbf{R} - \{0\}$ and $\bar{X}_i$ elementary in Ker $\triangle$, and $\bar{X}_i \sqsubseteq X$ for all $i = 1, 2, \ldots, m$. Because $\bar{X}_i$ is elementary in Ker $\triangle$, then, according to the theorem 3.3 from [17], it exists a cycle $\omega[n]$ with $\mathcal{A}(\omega[n]) = \{a_{k_1}, a_{k_2}, \ldots, a_{k_{r_i}}\}$ so that $\bar{X}_i = \pm \sum_{j=1}^{r_i} \varepsilon_j a_{k_j}$. But since $\bar{X}_i \sqsubseteq X$ and $\mathcal{A}(X) \subseteq \mathcal{L}$, then $\mathcal{A}(\bar{X}_i) \subseteq \mathcal{L}$ and, thus, there exists a cycle $\omega[n]$ with $\mathcal{A}(\omega[n]) \subseteq \mathcal{L}$; which is contradictory to the above — stated hypothesis.

5. On the free sets related to Im ∇. Within this paragraph we shall give two theorems to characterize (from the point of view of the graph theory) the free sets related to the subspace Im ∇.

DEFINITION 5.1. Let $\mathcal{N}^* \subset \mathcal{N}$, $\mathcal{N}^* \neq \emptyset$. We call *section* (see [17]) induced by $\mathcal{N}^*$ a set of arcs with sign $\mathcal{S} = \{\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \ldots, \varepsilon_r a_{k_r}\}$ $(\varepsilon_j = \pm 1$, $j = 1, 2, \ldots, r)$, so that for any a_{k_j}, $j = 1, 2, \ldots, r$ one of the following relations is verified:

a) $\triangle^+(a_{k_j}) \in \mathcal{N}^*$ and $\triangle^-(a_{k_j}) \in \mathcal{N} \setminus \mathcal{N}^*$, if $\varepsilon_j = 1$,

b) $\triangle^-(a_{k_j}) \in \mathcal{N}^*$ and $\triangle^+(a_{k_j}) \in \mathcal{N} \setminus \mathcal{N}^*$, if $\varepsilon_j = -1$,

the set $\mathcal{A}^* = \{a_{k_1}, a_{k_2}, \ldots, a_{k_r}\}$ beeing maximal (related to the inclusion of sets) with this property.

THEOREM 5.1. *Let* $\bar{\mathcal{L}} \subseteq \mathcal{A}$, $\bar{\mathcal{L}} \neq \emptyset$ *be. The set* $\bar{\mathcal{L}}$ *is free related to* Im ∇, *if and only if there is no section* $\mathcal{S}$ *with* $\mathcal{A}^* \subseteq \mathcal{L}$.

Proof. Let $\bar{\mathcal{L}} \subseteq \mathcal{A}$, $\bar{\mathcal{L}} \neq \emptyset$ be, free related to Im ∇ and let us prove that there is no section $\mathcal{S}$ with $\mathcal{A}^* \subseteq \mathcal{L}$.

Against all reson we suppose that there exists a section $\mathcal{S} = \{\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \ldots, \varepsilon_r a_{k_r}\}$ with $\mathcal{A}^* = \{a_{k_1}, a_{k_2}, \ldots, a_{k_r}\} \subseteq \bar{\mathcal{L}}$. According to the theorem 2.1 from [17], the vector $Y = \sum_{j=1}^{r} \varepsilon_j a_{k_j}$ is nonull in Im ∇, and since $\mathcal{A}(Y) = \mathcal{A}^* \subseteq \bar{\mathcal{L}}$ it means that $\bar{\mathcal{L}}$ is not free related to Im ∇; which is contradictory to the above — stated hypothesis.

Reciprocally, let us suppose that there is no section $\mathcal{S}$ with $\mathcal{A}^* \subseteq \bar{\mathcal{L}}$ and let us prove that $\bar{\mathcal{L}}$ is free related to Im ∇.

Against all reason, we suppose that $\bar{\mathcal{L}}$ is not free related to Im ∇, which leads us to the existence of a nonull vector $Y \in$ Im ∇ with $\mathcal{A}(Y) \subseteq \bar{\mathcal{L}}$.

According to (2.2), the vector Y is of the form $Y = \sum_{i=1}^{\bar{m}} \beta_i \bar{Y}_i$ with $\beta_i \in \mathbf{R} - \{0\}$ and $\bar{Y}_i$ elementary in Im ∇ and $\bar{Y}_i \sqsubseteq Y$ for all $i = 1, 2, \ldots, \bar{m}$.

If $\bar{Y}_i$ is elementary in Im ∇, then, according to the theorem 2.4 from [17], there exists a section $\mathcal{S} = \{\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \ldots, \varepsilon_{r_i} a_{k_{r_i}}\}$ so that $\bar{Y}_i = \pm \sum_{j=1}^{r_i} \varepsilon_j a_{k_j}$. But since $\bar{Y}_i \sqsubseteq Y$ and $\mathcal{A}(Y) \subseteq \bar{\mathfrak{L}}$, then $\mathcal{A}(\bar{Y}_i) \subseteq \bar{\mathfrak{L}}$, and, thus, there exists a section $\mathcal{S}$ with $\mathcal{A}(\bar{Y}_i) = \mathcal{A}^* \subseteq \bar{\mathfrak{L}}$; which is contradictory to the above — stated hypothesis.

DEFINITION 5.2. We call *chain* with the length r (see [17]) from the node n to the node m a sequence of $r + 1$ nodes $(n_{i_0}, n_{i_1}, \ldots, n_{i_r})$ and a sequence of r arcs with sign $(\varepsilon_1 a_{k_1}, \varepsilon_2 a_{k_2}, \ldots, \varepsilon_r a_{k_r})$, where $n_{i_0} = n$, $n_{i_r} = m$, $\varepsilon_j = 1$ or -1, so that for every $j = 1, 2, \ldots, r$ we have:

$$n_{i_{j-1}} = \begin{cases} \Delta^{+}(a_{k_j}), & \text{if } \varepsilon_j = 1, \\ \Delta^{-}(a_{k_j}), & \text{if } \varepsilon_j = -1, \end{cases} \quad \text{and} \quad n_{i_j} = \begin{cases} \Delta^{-}(a_{k_j}), & \text{if } \varepsilon_j = 1, \\ \Delta^{+}(a_{k_j}), & \text{if } \varepsilon_j = -1. \end{cases}$$

Particularly, a single node (and an empty sequence of arcs) is regarded as a chain with the length zero from the nod to itself.

We denote a chain from n to m by $\gamma[n,m]$ and its set of arcs of arcs by $\mathcal{A}(\gamma[n, m])$.

DEFINITION 5.3. Two nodes n and m are *mutually connected* (see [16]) and we denote this by $n \sim m$ if and only if there exists $\gamma[n, m]$. Evidently, „$\sim$" is an equivalence on $\mathcal{N}$ and induces a partition in the clases $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_s$, called *connected components* (see [16]).

THEOREM 5.2. *Let $\bar{\mathfrak{L}} \subseteq \mathcal{A}$, $\bar{\mathfrak{L}} \neq \emptyset$ be, and $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_s$, the connected components of the graph $G = \langle \mathcal{N}, \mathcal{A} \rangle$. If the set $\bar{\mathfrak{L}}$ is free related to the subspace* Im ∇, *then, for all $i \in \{1, 2, \ldots, s\}$ and for any $n, m \in \mathcal{N}_i$, there exists a chain $\gamma[n, m]$ with $\mathcal{A}(\gamma[n, m]) \subseteq \mathcal{A} \setminus \bar{\mathfrak{L}}$.*

Proof. Let $\bar{\mathfrak{L}} \subseteq \mathcal{A}$, $\bar{\mathfrak{L}} \neq \emptyset$ be free related to Im ∇, $i_0 \in \{1, 2, \ldots, s\}$ arbitrary fixed and $n, m \in \mathcal{N}_{i_0}$. We denote by $\mathcal{D}_{(n,m)}^{[0]}$ the set of all the chains from the node n to the node m. Evidently, $\mathcal{D}_{(n,m)}^{[0]}$ is not empty because $n, m \in \mathcal{N}_{i_0}$.

Against all reson we suppose that for any $\gamma[n, m] \in \mathcal{D}_{(n,m)}^{[0]}$ we have not $\mathcal{A}(\gamma[n, m]) \subseteq \mathcal{A} \setminus \bar{\mathfrak{L}}$, which leads to the existence of an arc (at least one) $a \in \mathcal{A}$ for which $a \in \mathcal{A}(\gamma[n, m])$ and $a \in \bar{\mathfrak{L}}$. Let us consider $\gamma_0[n, m]$ such a chain, $\mathcal{A}(\gamma_0[n, m]) = \{a_{k_1}^{[0]}, a_{k_2}^{[0]}, \ldots, a_{k_r}^{[0]}\}$ its set of arcs and $\{a_{k_{t_1}}^{[0]}, a_{k_{t_2}}^{[0]}, \ldots, a_{k_{t_u}}^{[0]}\} \subseteq \mathcal{A}(\gamma_0[n, m])$ the set of arcs for which $a_{k_{t_j}}^{[0]} \in \mathcal{A}(\gamma_0[n, m])$ and $a_{k_{t_j}}^{[0]} \in \bar{\mathfrak{L}}$, $j = 1, 2, \ldots, u$.

We define $n_{\alpha_j}^{[0]} = \begin{cases} \Delta^{+}(a_{k_{t_j}}^{[0]}), & \text{if } \varepsilon_{t_j}^{[0]} = 1, \\ \Delta^{-}(a_{k_{t_j}}^{[0]}), & \text{if } \varepsilon_{t_j}^{[0]} = -1 \end{cases}$ and we consider the vector $Z = \sum_{j=1}^{u} n_{\alpha_j}^{[0]}$, vector which obviously belongs to $V[\mathcal{N}, \mathbf{R}]$.

Having in view the construction of $n_{\alpha_j}^{[0]}$ and Λ we have:

$$\nabla(Z) = \nabla\left(\sum_{j=1}^{u} n_{\alpha_j}^{[0]}\right) = \sum_{j=1}^{u} \nabla(n_{\alpha_j}^{[0]}) = \sum_{j=1}^{u}\left(\sum_{k=1}^{q} \Lambda_k^{\alpha_j} a_k\right) =$$

$$= \sum_{j=1}^{u} \Lambda_{k_{t_j}}^{\alpha_j} a_{k_{t_j}}^{[0]} = \sum_{j=1}^{u} \varepsilon_{t_j}^{[0]} a_{k_{t_j}}^{[0]}. \tag{5.1}$$

Putting $Y = \sum_{j=1}^{u} \varepsilon_{t_j}^{[0]} \cdot a_{k_{t_j}}^{[0]}$ we have, according to (5.1), $Y \in$ Im ∇, $Y \neq \theta_{\mathcal{A}}$, $\mathcal{A}(Y) \subseteq \bar{\mathfrak{L}}$ and subsequently, $\bar{\mathfrak{L}}$ is not free related to Im ∇; which is contradictory to the above-stated hypothesis.

Therefore, if $\bar{\mathfrak{L}}$ is free related to the Im ∇, then, there exists $\gamma[n, m] \in \mathcal{D}_{(n,m)}^{[0]}$, so that $\mathcal{A}(\gamma[n, m]) \subseteq \mathcal{A} \setminus \bar{\mathfrak{L}}$. (Q.E.D)

6. Matrix associated to the free sets. Let us consider $\mathcal{F} \subseteq \mathcal{A}$ a free set related to Ker Δ and maximal with this property. According to the theorem 3.6, the set $\bar{\mathcal{F}} = \mathcal{A} \setminus \mathcal{F}$ is free related to Im ∇ and maximal with this property.

According to the theorem 3.7 and the remark 3.8, if $\bar{\mathcal{F}}$ is free related to Im ∇ and maximal with this property, then, we can construct the vectors $Y^{[1]}, Y^{[2]}, \ldots, Y^{[\bar{m}]}$, which form a basis for the subspace Im ∇.

Because the vectors $Y^{[\beta]}$, $\beta = 1, 2, \ldots, \bar{m}$ belong to $Y[\mathcal{A}, \mathbf{R}]$, they may be written as a linear combination of the vectors $a_1, a_2, \ldots, a_q$, which means that there exists a matrix $\Omega = (\Omega_{\beta t})$, $\beta = 1, 2, \ldots, \bar{m}$; $t = 1, 2, \ldots, q$ (matrix with $\bar{m}$ rows and q columns) so that we have:

$$Y^{[\beta]} = \sum_{t=1}^{q} \Omega_{\beta t}\, a_t; \quad \beta = 1, 2, \ldots, \bar{m} \tag{6.1}$$

Having in view the way the vectors $Y^{[\beta]}$, $\beta = 1, 2, \ldots, \bar{m}$, have been construct (see remark 3.7) we can assert the fact that $\Omega_{\beta t} \in \{-1, 0, 1\}$, for all $\beta = 1, 2, \ldots, \bar{m}$; $t = 1, 2, \ldots, q$.

Considering another pair $(\mathcal{F}', \bar{\mathcal{F}}')$ of free maximal sets related to KerΔ and Im∇ respectively, we obtain another basis $\tilde{Y}^{[\beta]}$, $\beta = 1, 2, \ldots, \bar{m}$, for the subspace Im$\nabla$ for which we have, according to (6.1) another matrix $\tilde{\Omega}$.

LEMMA 6.1. *If* Ω *and* $\tilde{\Omega}$ *are two matrices associated to the bases* $Y^{[\beta]}$, $\tilde{Y}^{[\beta]}$, $\beta = 1, 2, \ldots, \bar{m}$, *then, there exists a square matrix* Γ, *so that* $\tilde{\Omega} = \Gamma\Omega$ *and* $\det(\Gamma) = \pm 1$.

Proof. Writing $\tilde{Y}^{[\beta]}$ as a combination of the vectors $Y^{[\beta]}$, we obtain:

$$\tilde{Y}^{[\beta]} = \sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} Y^{[\gamma]}, \quad \beta = 1, 2, \ldots, \bar{m}, \tag{6.2}$$

where $\Gamma = (\Gamma_{\beta\gamma})$ $\beta = 1, 2, \ldots, \bar{m}$, $\gamma = 1, 2, \ldots, \bar{m}$ is a square matrix with elements in $\{-1, 0, 1\}$.

Now, writing $Y^{[\beta]}$ as a combination of the vectors $\tilde{Y}^{[\beta]}$ we obtain:

$$Y^{[\gamma]} = \sum_{\alpha=1}^{\bar{m}} \tilde{\Gamma}_{\gamma\alpha} \tilde{Y}^{[\alpha]}, \quad \gamma = 1, 2, \ldots, \bar{m}, \tag{6.3}$$

where $\tilde{\Gamma} = (\tilde{\Gamma}_{\gamma\alpha})$ $\gamma = 1, 2, \ldots, \bar{m}$; $\alpha = 1, 2, \ldots, \bar{m}$; is a matrix similar to Γ. From (6.2) and (6.3) we obtain:

$$\tilde{Y}^{[\beta]} = \sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} \left(\sum_{\alpha=1}^{\bar{m}} \tilde{\Gamma}_{\gamma\alpha} \tilde{Y}^{[\beta]} \right) = \sum_{\gamma=1}^{\bar{m}} \sum_{\alpha=1}^{\bar{m}} \Gamma_{\beta\gamma} \tilde{\Gamma}_{\gamma\alpha} \tilde{Y}^{[\alpha]} = \sum_{\alpha=1}^{\bar{m}} \left(\sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} \tilde{\Gamma}_{\gamma\alpha} \right) \tilde{Y}^{[\alpha]}. \tag{6.4}$$

Because the vectors $\tilde{Y}^{[\beta]}$, $\beta = 1, 2, \ldots, \bar{m}$ are linear independent, then, according to (6.4) we must have:

$$\sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} \tilde{\Gamma}_{\gamma\alpha} = \delta_{\beta\alpha} = \begin{cases} 1, & \text{if } \beta = \alpha, \\ 0, & \text{if } \beta \neq \alpha. \end{cases} \tag{6.5}$$

From (6.5) we have: $1 = \det[(\delta_{\beta\alpha})\ \beta = 1, 2, \ldots, \bar{m};\ \alpha = 1, 2, \ldots, \bar{m}] = \det(\Gamma\tilde{\Gamma}) = \det(\Gamma)\det(\tilde{\Gamma})$, which does not mean anything else but that $\det(\Gamma)$ and $\det(\tilde{\Gamma})$ are concurently equal to 1 or -1. Consequently, $\det(\Gamma) = \pm 1$. (Q.E.D.)

Rewriting the relation (6.2) we obtain: $\sum_{t=1}^{q} \tilde{\Omega}_{\beta t} a_t = \tilde{Y}^{[\beta]} = \sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} Y^{[\gamma]} = \sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} \left(\sum_{t=1}^{q} \Omega_{\gamma t} a_t \right) = \sum_{t=1}^{q} \left(\sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} \Omega_{\gamma t} \right) a_t$, which leads to $\tilde{\Omega}_{\beta t} = \sum_{\gamma=1}^{\bar{m}} \Gamma_{\beta\gamma} \Omega_{\gamma t}$, i.e. $\tilde{\Omega} = \Gamma\Omega$. (Q.E.D.)

We denote by $\mathfrak{D}_{[t_1, t_2, \ldots, t_{\bar{m}}]}$ a subdeterminant of the matrix Ω, a subdeterminant made up with the $\bar{m}$ rows of Ω and the columns $t_1, t_2, \ldots, t_{\bar{m}}$ of Ω.

THEOREM 6.1. *If the set* $\{a_{t_1}, a_{t_2}, \ldots, a_{t_{\bar{m}}}\}$ *is free related to Ker* Δ *and maximal with this property, then,* $\mathfrak{D}_{[t_1, t_2, \ldots, t_{\bar{m}}]} = \pm 1$ *and if* $\{a_{t_1}, a_{t_2}, \ldots, a_{t_{\bar{m}}}\}$ *is not free related to* Ker Δ *and maximal with this property, then,* $\mathfrak{D}_{[t_1, t_2, \ldots, t_{\bar{m}}]} = 0$.

Proof. Let us suppose that the set $\{a_{t_1}, a_{t_2}, \ldots, a_{t_{\bar{m}}}\}$ is free related to Ker Δ and maximal with this property. According to the remarks 3.7 and 3.8 we can construct a bases from the subspace Im ∇, using the set $\mathcal{A} \setminus \{a_{t_1}, a_{t_2}, \ldots, a_{t_{\bar{m}}}\}$, who is free related to Im ∇ and maximal with this property. Let us consider $\tilde{\Omega}$ the matrix associated to this bases according to the procedure described at the begining of this paragraph.

According to the lemma 6.1. it exists a square matrix Γ so that:

$$\tilde{\Omega} = \Gamma\Omega \quad \text{and} \quad \det(\Gamma) = \pm 1. \tag{6.6}$$

Because $\tilde{\Omega} = \Gamma\Omega$, then, we can write:

$$\tilde{\mathfrak{D}}_{[t_1, t_2, \ldots, t_{\bar{m}}]} = \det(\Gamma)\, \mathfrak{D}_{[t_1, t_2, \ldots, t_{\bar{m}}]}. \tag{6.7}$$

But, having in mind the way the matrix $\tilde{\Omega}$ has been defined we may assert that:

$$\tilde{\Omega}_{\beta t_i} = \begin{cases} 0, & \text{if } \beta \neq i, \\ 1, & \text{if } \beta = i, \end{cases} \quad \beta = 1, 2, \ldots, \bar{m}$$

which means that (developing according the diagonal line)

$$\tilde{\mathfrak{D}}_{[t_1, t_2, \ldots, t_{\bar{m}}]} = \pm 1. \tag{6.8}$$

From (6.6), (6.7) and (6.8) we obtain: $\mathfrak{D}_{[t_1, t_2, \ldots, t_{\bar{m}}]} = \pm 1$. (Q.E.D). Let us suppose now that the set $\{a_{t_1}, a_{t_2}, \ldots, a_{t_{\bar{m}}}\}$ is not free related to Ker Δ and maximal with this property, fact that implies that there exists $X = \sum_{i=1}^{\bar{m}} x_{t_i} a_{t_i} \in$ Ker Δ, nonull, so that (see the remark 4.2 from [17]) $[X | Y^{[\beta]}] = 0$, for all $\beta = 1, 2, \ldots, \bar{m}$. (6.9). Rewriting (6.9) we obtain:

$$0 = [X | Y^{[\beta]}] = \left[\sum_{i=1}^{\bar{m}} x_{t_i} a_{t_i} \middle| \sum_{t=1}^{q} \Omega_{\beta t} a_t \right] = \sum_{i=1}^{\bar{m}} x_{t_i} \Omega_{\beta t_i} = \sum_{i=1}^{\bar{m}} \Omega_{\beta t_i} x_{t_i}, \quad \beta = 1, 2, \ldots, \bar{m} \tag{6.10}$$

The relation (6.10) represents a homogeneous linear system of $\bar{m}$ equations with $\bar{m}$ unknown quantites (the unknown quantites are $x_{t_1}, x_{t_2}, \ldots, x_{t_{\bar{m}}}$).

But, as not all the unknown quantites are null (we have supposed that the vector X is nonull) it means that the system of equations (6.10) admits a nonull solution fact that implies:

$$\det[(\Omega_{\beta t_i})\ \beta = 1, 2, \ldots, \bar{m};\ i = 1, 2, \ldots, \bar{m}] = \mathfrak{D}_{[t_1, t_2, \ldots, t_{\bar{m}}]} = 0. \quad \text{(Q.E.D.)}.$$

R e m a r k 6.1. If $(\mathcal{F}, \bar{\mathcal{F}})$ is a pair of free sets, maximal related to Ker Δ and Im ∇ respectively, then, according to the remark 3.6 we can construct the vectors $X^{[\alpha]}$, $\alpha = 1, 2, \ldots, \bar{k}$ which form a bases for Ker Δ and we may associated to it the matrix $\Omega^* = (\Omega^*_{\alpha t})$ $\alpha = 1, 2, \ldots, \bar{k}$; $t = 1, 2, \ldots q$; similar to the matrix Ω. Denoting by $\mathfrak{D}^*_{[t_1, t_2, \ldots, t_{\bar{k}}]}$ a subdeterminant of Ω^* (subdeterminant made up with the $\bar{k}$ rows of Ω^* and the columns $t_1, t_2, \ldots, t_{\bar{k}}$ of Ω^*) we obtain, by a similar resoning to that previously expounded, the following theorem:

THEOREM 6.2. *If the set* $\{a_{t_1}, a_{t_2}, \ldots, a_{t_{\bar{k}}}\}$ *is free related to* Im ∇ *and maximal with this property, then* $\mathfrak{D}^*_{[t_1, t_2, \ldots, t_{\bar{k}}]} = \pm 1$, *and if* $\{a_{t_1}, a_{t_2}, \ldots, a_{t_{\bar{k}}}\}$ *is not free related to* Im ∇ *and maximal with this property, then,* $\mathfrak{D}^*_{[t_1, t_2, \ldots, t_{\bar{k}}]} = 0$.

7. Exemple. For the graph $G = \langle \mathcal{N}, \mathcal{A} \rangle$ with $\mathcal{N} = \{n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$ and $\mathcal{A} = \{a_1 = \langle n_1, n_2 \rangle, a_2 = \langle n_2, n_1 \rangle, a_3 = \langle n_2, n_3 \rangle, a_4 = \langle n_3, n_3 \rangle, a_5 = \langle n_3, n_1 \rangle, a_6 = \langle n_1, n_3 \rangle, a_7 = \langle n_4, n_1 \rangle, a_8 = \langle n_3, n_4 \rangle, a_9 = \langle n_5, n_4 \rangle$ $a_{10} = \langle n_7, n_6 \rangle, a_{11} = \langle n_6, n_7 \rangle\}$ the set $\mathcal{T} = \{a_1 = U_1, a_5 = U_2, a_7 = U_3, a_9 = U_4, a_{11} = U_5\}$ is free related to Ker Δ and maximal with this property; the set $\bar{\mathcal{T}} = \{a_2 = V_1, a_3 = V_2, a_4 = V_3, a_6 = V_4, a_8 = V_5, a_{10} = V_6\}$ is free related to Im ∇ and maximal with this property.

8. Conclusions. The results of this paper and [16] — [18] represent a generalization and an extension of those obtained in [1] — [13]. More exactly: in [1] — [13] is developed a theory in the *particular case when the graph is connected*; in our paper and [16] — [18] we consider and develope a theory in the *general case when the graph is not necessary connected*. Evidently, the results of [1]—[13] become natural consequences of what we got in [16] — [18] and this paper. It is obviously to see that for a connected graph $G = \langle \mathcal{N}, \mathcal{A} \rangle$ a set $\mathcal{T}$ is free related to Ker Δ and maximal with this property, if and only if $\mathcal{T}$ is a *spanning tree* of G. (see [1] — [13]). Similarly, a set $\bar{\mathcal{T}}$ is free related to Im ∇ and maximal with this property, if and only if $\bar{\mathcal{T}}$ is a *spanning cotree* of G. (see [1] — [13]). Moreover, Ker Δ is the *space of cycles* and Im ∇ the *space of cocycles*. (see [1] — [13]). Indeed, from [16] and (2.1) it results:

$$\dim \text{Ker}\, \Delta = q - p + s,$$

where s is the number of connected components of G. Hence

$$\dim \text{Ker}\, \Delta = \nu[G],$$

where $\nu[G]$ is the *cyclomatic number* of G. (see [5]).

So, according to remarks 3.6 and 3.9 we obtain: *if* $\mathcal{T}$ *is free related to* Ker Δ *and maximal with this property, then*

$$|\mathcal{T}| = \text{rank } (\Lambda);$$

if $\bar{\mathcal{T}}$ *is free related to* Im ∇ *and maximal with this property, then*

$$|\bar{\mathcal{T}}| = \nu[G].$$

But, if G is connected, e.g., $s = 1$, then, according with [16], it results:

$$|\mathcal{T}| = p - 1,$$

e.g., $\mathcal{T}$ is a spanning tree of G. (see [1] — [13]).

Similarly, if G is connected, then

$$|\bar{\mathcal{T}}| = \nu[G] = q - p + 1,$$

e.g., $\bar{\mathcal{T}}$ is a spanning cotree of G. (see [1] — [13]). Evidently, in our general theory from [16] — [18] and this paper, if G is not connected, then every spanning tree is free and maximal related to Ker Δ, but reciprocaly is not true. (see exemple 7). Similarly for a spanning cotree. Look, what for is necessary a general theory when the graph G is not connected.

9. Opened problem. A research concerning the link between the *free sets* (introduced in this paper) and the *independent systems* from matroids theory.

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Received 1. XII. 1980

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