MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
TOME 10, N° 1, 1981, pp. 75—79

SUFFICIENT CONDITIONS OF UNIVALENCY FOR COMPLEX FUNCTIONS IN THE CLASS C^1

by

PETRU T. MOCANU
(Cluj-Napoca)

1. Introduction. The following sufficient condition for univalency of an analytic function in a convex domain is well known [2], [4], [5]:

THEOREM A. *If D is a convex domain in the complex plane* $\mathbf{C}$, *and if f is an analytic function in D such that*

$$\operatorname{Re} f'(z) > 0, \quad \text{for all } z \in D, \tag{1}$$

then f is univalent in D.

This result was generalized in [1] and [3] as follows:

THEOREM B. *If f is analytic in a domain D and if there exists an analytic function g which is univalent and convex in D* (i.e. $g(D)$ is a convex domain) *such that*

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > 0, \quad \text{for all} \quad z \in D, \tag{2}$$

then f is univalent in D.

In this note we obtain sufficient conditions of univalency similar to (1) and (2) for complex functions in the class C^1. These conditions yield some simple criteria of homeomorphism in the complex plane.

2. Preliminaries. Let D be a domain in $\mathbf{C}$ and let $f: D \to \mathbf{C}$, with $f(z) = u(x, y) + iv(x, y)$.

We say that the function f belongs to the class $C^1(D)$ if the real functions $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ of the real variables x and y have continuous first order partial derivatives in D.

The directional derivative of the function $f \in C^1(D)$ at the point $z \in D$, in the direction defined by the angle $\alpha \in [0, 2\pi)$ is given by the well known formula

$$f'_\alpha(z) = \frac{\partial f}{\partial z} + e^{-2i\alpha} \frac{\partial f}{\partial \bar{z}},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The Jacobian of the function $f \in C^1(D)$ is given by

$$J(f) = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2.$$

If $J(f) > 0$ in D, then f is a locally homeomorphism preserving the orientation in D.

For f and g belonging to $C^1(D)$, if we define

$$I(f, g) = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar{z}} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial \bar{z}} \end{vmatrix},$$

then $I(\bar{f}, f) = J(f)$, and

$$|I(f, \bar{g})|^2 - |I(f, g)|^2 = J(f) \cdot J(g). \tag{3}$$

3. Main results. The following theorems provide sufficient conditions of homeomorphism similar to (1) and (2).

THEOREM 1. *If D is a convex domain in $\mathbb{C}$ and the function $f \in C^1(D)$ satisfies one of the following equivalent conditions:*

$$\operatorname{Re} f'_\alpha(z) > 0, \textit{ for all } z \in D, \textit{ and all } \alpha \in [0, 2\pi) \tag{4}$$

$$\operatorname{Re} \frac{\partial f}{\partial z} > \left|\frac{\partial f}{\partial \bar{z}}\right|, \quad \textit{for all} \quad z \in D, \tag{5}$$

then f is univalent in D.

Proof. Let $z_1, z_2 \in D$, $z_1 \neq z_2$ and let $z(t) = z_1 + t(z_2 - z_1)$, $t \in [0,1]$. Since D is convex, $z(t) \in D$ for all $t \in [0, 1]$. By integrating along the segment $[z_1, z_2]$, we obtain

$$f(z_2) - f(z_1) = \int_0^1 \frac{d}{dt} f([z(t)]dt = \int_0^1 \left[\frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial \bar{z}}\frac{d\bar{z}}{dt}\right] dt =$$

$$= (z_2 - z_1)\int_0^1 \left[\frac{\partial f}{\partial z} + \frac{\bar{z}_2 - \bar{z}_1}{z_2 - z_1}\frac{\partial f}{\partial \bar{z}}\right] dt = (z_2 - z_1)\int_0^1 f'_\alpha[z(t)]dt,$$

where $\alpha = \arg(z_2 - z_1)$. By using (4) we obtain

$$\operatorname{Re} \frac{f(z_1) - f(z_2)}{z_1 - z_2} = \int_0^1 \operatorname{Re} f'_\alpha[z(t)]dt > 0,$$

which shows that f is univalent in D.

It is easy to check that the conditions (4) and (5) are equivalent.

We next generalize Theorem 1, by obtaining a sufficient condition of univalency similar to (2).

THEOREM 2. *Let $g \in C^1(D)$, where D is a domain in $\mathbb{C}$. Suppose g is univalent, convex and satisfies $J(g) < 0$ in D. If $f \in C^1(D)$ and*

$$\operatorname{Re} I(f, \bar{g}) > |I(f, g)|, \textit{ for all } z \in D, \tag{6}$$

then f is univalent and $J(f) > 0$ in D.

Proof. Let $\Delta = g(D)$ and $\varphi = g^{-1}$. Consider the function $h: \Delta \to \mathbb{C}$ defined by

$$h(w) = f[\varphi(w)], \ w \in \Delta.$$

We have

$$\frac{\partial h}{\partial w} = \frac{\partial f}{\partial z}\frac{\partial \varphi}{\partial w} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{\varphi}}{\partial w}$$

$$\frac{\partial h}{\partial \bar{w}} = \frac{\partial f}{\partial z}\frac{\partial \varphi}{\partial \bar{w}} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{\varphi}}{\partial \bar{w}}$$

and

$$\frac{\partial \varphi}{\partial w} = \frac{1}{J(g)}\frac{\partial \bar{g}}{\partial \bar{z}}, \quad \frac{\partial \varphi}{\partial \bar{w}} = -\frac{1}{J(g)}\frac{\partial g}{\partial \bar{z}}.$$

Hence

$$\frac{\partial h}{\partial w} = \frac{I(f, \bar{g})}{J(g)}, \quad \frac{\partial h}{\partial \bar{w}} = -\frac{I(f, g)}{J(g)}.$$

Since $J(g) > 0$, from (6) we obtain

$$\operatorname{Re} \frac{\partial h}{\partial w} > \left|\frac{\partial h}{\partial \bar{w}}\right|, \quad w \in \Delta.$$

Since Δ is convex, by applying Theorem 1 we deduce that h is univalent in Δ and thus $f = h \circ g$ is univalent in D.

From (6) and (3) we easily deduce $J(f) > 0$, which completes the proof of Theorem 2.

4. Particular Cases.

a) If the function f in Theorem 2 is analytic in D, then the condition (6) becomes

$$\operatorname{Re}\left[f'(z)\frac{\partial \bar{g}}{\partial \bar{z}}\right] > \left|f'(z)\frac{\partial g}{\partial \bar{z}}\right|, \qquad z \in D.$$

If in addition g is analytic in D, we obtain condition (2). If D is convex and $g(z) \equiv z$, we obtain condition (1).

b) If the function g in Theorem 2 is analytic in D, the condition (6) becomes

$$\operatorname{Re}\left[\frac{\partial f}{\partial z}\overline{g'(z)}\right] > \left|\frac{\partial f}{\partial \bar{z}}g'(z)\right|, \qquad z \in D,$$

or

$$\operatorname{Re}\frac{\frac{\partial f}{\partial z}}{g'(z)} > \left|\frac{\frac{\partial f}{\partial \bar{z}}}{g'(z)}\right|, \qquad z \in D. \tag{7}$$

As an example, suppose D is the unit disc U, and $g(z) = z/(1-z)$ or $g(z) = \log[(1+z)/(1-z)]$, $z \in U$. Then we obtain respectively the following particular conditions of univalency for a function $f \in C^1(U)$:

$$\operatorname{Re}\left[(1-z)^2\frac{\partial f}{\partial z}\right] > \left|(1-z)^2\frac{\partial f}{\partial \bar{z}}\right|, \qquad z \in U,$$

$$\operatorname{Re}\left[(1-z^2)\frac{\partial f}{\partial z}\right] > \left|(1-z^2)\frac{\partial f}{\partial \bar{z}}\right|, \qquad z \in U.$$

c) If the function f in Theorem 2 is of the form $f = F + \bar{G}$, where F and G are analytic in D, then condition (7) becomes

$$\operatorname{Re}\frac{F'(z)}{g'(z)} > \left|\frac{G'(z)}{g'(z)}\right|, \qquad z \in D. \tag{8}$$

If in addition D is convex and $g(z) \equiv z$, we obtain the following condition of univalency for the function $f = F + \bar{G}$

$$\operatorname{Re}F'(z) > |G'(z)|, \qquad z \in D.$$

(this also follows directely from Theorem 1).

If $F = g$ and $|G'(z)| < |g'(z)|$ for $z \in D$, then condition (8) holds. Hence we obtain the following condition of univalency

THEOREM 3. *If the function g is analytic univalent and convex in a domain D and G is analytic in D such that $|G'(z)| < |g'(z)|$ for all $z \in D$, then the function $f = g + \bar{G}$ is univalent in D.*

As an example, it is easy to check that the function $f(z) = z + e^{\bar{z}}$ is univalent in the halfplane $D = \{z ; \operatorname{Re} z < 0\}$.

d) Consider $g(z) = z + \lambda z\bar{z}$ with $|\lambda| \leqslant 1/2$. It is easy to show that $J(g) > 0$ and g is convex in U. By Theorem 1 we can show that g is univalent in U. From Theorem 2 we obtain the following sufficient condition of univalency for a function $f \in C^1(U)$:

$$\operatorname{Re}\left(\frac{\partial f}{\partial z} + \bar{\lambda}Df(z)\right) > \left|\frac{\partial f}{\partial \bar{z}} - \lambda Df(z)\right|, \qquad z \in U,$$

where $|\lambda| \leqslant 1/2$ and

$$Df = z\frac{\partial f}{\partial z} - \bar{z}\frac{\partial f}{\partial \bar{z}}.$$

BIBLIOGRAPHY

[1] W. Klaplan, *Close-to-convex schlicht functions.* Michigan Math. J. **1**, 169–185 (1952).

[2] K. Noshiro, On the theory of schlicht functions, J. Fac. Sci. Hokaido Univ., **2**, 124–155 (1934).

[3] S. Ozaki, *On the theory of multivalent functions.* Sci. Rep. Tokyo Bunrika Daigaku, A, 2, **40**, 167–188 (1935).

[4] S. E. Warschawski, *On the higher derivatives at the boundary in conformal mapping.* Trans. Amer. Math. Soc., 38, 310–340 (1935).

[5] J. Wolff, *L'intégrale d'une fonction holomorphe et à partie réelle positive dans un demi-plan est univalente.* C.R. Acad. Sci., Paris, 198, **13**, 1209–1210 (1934).

Received 10. VI. 1979

Department of Mathematics,
„Babeş-Bolyai" University
3400 Cluj-Napoca, Romania