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APPROXIMATION OF CONTINUOUS FUNCTIONS BY THEIR FOURIER SERIES

by

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1. Let Σu_n be a given infinite series with n-th partial sum s_n. Let $A = (a_{nk})$ $(k = 0, 1, \ldots, n;\ n = 0, 1, \ldots)$ be an infinite matrix of real numbers. We denote by τ_n, the A-transform of the series Σu_n where

(1.1) $$\tau_n = \sum_{k=0}^{n} a_{nk}\, s_k \qquad (n = 0, 1, \ldots).$$

Let $f(t)$ be a real periodic function with period 2π, and itegrable (L) over $(0, 2\pi)$. Let the Fourier series of $f(t)$ be given by

(1.2) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

We let $t_n(x)$ be the A-transform of the series (1.2) at $t = x$.

If $\{p_n\}$ is a sequence of real numbers with $P_n = \sum_{r=0}^{n} p_n\ (p_0 > 0)$ and if

(1.3) $$a_{nk} = \begin{cases} \dfrac{p_{n-k}}{P_n} & (k \leq n), \\ 0 & (k > n), \end{cases}$$

or

(1.4) $$a_{nk} = \begin{cases} \dfrac{p_k}{P_n} & (k \leq n), \\ 0 & (k > n), \end{cases}$$

the A-transform of $\sum_{n=0}^{\infty} A_n(t)$ is called the Norlund transform or Riesz transform respectively.

Throughout the paper we shall write

(1.5) $$\bar{a}_{nk} = \sum_{r=0}^{k} a_{nr}\ (k = 0,\ 1, \ldots n;\ n = 0,\ 1, \ldots);\quad a'_{n,k} = \sum_{r=k}^{n} a_{nr}$$

(1.6) $$\bar{a}_n(k) = \bar{a}_{nk};$$

(1.7) $$\Phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

and

(1.8) $C^*[0,\ 2\pi]$ for class of a continuous periodic functions on $(0,\ 2\pi)$ with period 2π.

(1.9) Let $N_n(x)$ and $R_n(x)$ are the Nölund and Riesz transforms of the series (1.2) respectively.

2. In attempting to generalise a result of ALEXITS [1] concerning the degree of approximation of a function $f \in \mathrm{Lip}\ \alpha\ (0 < \alpha \leqslant 1)$ by Cesáro means of the Fourier series of $f(x)$, HOLLAND, SAHNEY and TZIMBALARIO [4], and PREMCHANDRA [2] proved the following:

THEOREM A. *Let $\{p_n\}$ be a non-negative monotonically non-increasing sequence of real numbers. If $\omega(t)$ is the modulus of continuity of $f \in C^*[0,\ 2\pi]$*, then

$$\max_{0 \leqslant x \leqslant 2\pi} |f(x) - N_n(x)| = O\left\{\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k\,\omega(1/k)}{k}\right\}.$$

THEOREM B. *Let $\{p_n\}$ be a real, non-negative, monotonically non-decreasing sequence of real numbers with $P_n \to \infty$. Then for a 2π periodic function $f \in \mathrm{Lip}\ \alpha$ $(0 < \alpha \leqslant 1)$* we have

$$\max_{0 \leqslant x \leqslant 2\pi} |f(x) - R_n(x)| = \begin{cases} O\left\{\left(\dfrac{p_n}{P_n}\right)^{\alpha}\right\} & (0 < \alpha < 1),\\[2mm] O\left\{\dfrac{p_n}{P_n}\log\dfrac{P_n}{p_n}\right\} & (\alpha = 1).\end{cases}$$

Our object in this paper is to obtain estimates for $\max\limits_{0 \leqslant x \leqslant 2\pi} |f(x) - t_n(x)|$ when the entries of the matrix $A = (a_{nk})$ satisfy einther $a_{nk} \leqslant a_{nk+1}$ $(k = 0,\ 1, \ldots;\ n = 0,\ 1, \ldots)$ or $a_{nk} \geqslant a_{nk+1}$ $(k = 0,\ 1, \ldots,\ n = 0,\ 1, \ldots)$.

Our main results are given in the following theorems:

THEOREM 1. *Let $A = (a_{nk})$ satisfy the following:*

(2.1) $$a_{nk} \geqslant 0\ (k = 0,\ 1, \ldots;\ n = 0,\ 1, \ldots)\ \text{and}\ a_{\infty} > 0,\ \sum_{k=0}^{n} a_{nk} = 1,$$

(2.2) $$a_{nk} \geqslant a_{nk+1}\ (k = 0,\ 1, \ldots;\ n = 0,\ 1, \ldots),$$

(2.3) $$\exists\ \text{positive constant } a \text{ such that } \sum_{k=1}^{n} a_{nk} \geqslant a.$$

Then for $f \in C^*[0,\ 2\pi]$,

(2.4) $$E_n(f) = \max_{0 \leqslant x \leqslant 2\pi} |f(x) - t_n(x)| = O\left(\sum_{k=1}^{n}\frac{\omega(1/k)\bar{a}_n(k+1)}{k}\right)$$

where $\omega(t)$ is the modulus of continuity of $f(x)$.

THEOREM 2. *Let $A = (a_{nk})$ satisfy (2.1) and*

(2.5) $$a_{nk-1} \leqslant a_{n,k}\qquad (k = 1, \ldots, n).$$

Then for $f \in C^*[0,\ 2\pi]$,

(2.6) $$\max_{0 \leqslant x \leqslant 2\pi} |t_n(x) - f(x)| = O\left(\omega\left(\frac{1}{n}\right)\right) + O\left(\sum_{k=1}^{n}\frac{\omega\left(\frac{1}{k}\right)}{k}\,a'_{n,n-k}\right).$$

A generalization of Theorem B is the following:

THEOREM 3. *Let $A = (a_{nk})$ satisfy (2.1) and (2.5). Then for $f \in \mathrm{Lip}\ \alpha$ $(0 < \alpha \leqslant 1)$,*

(2.7) $$\max_{0 \leqslant x \leqslant 2\pi} |t_n(x) - f(x)| = \begin{cases} O((a_{nn})^{\alpha}) & (0 < \alpha < 1);\\[2mm] O\left(a_{nn}\log\dfrac{1}{a_{nn}}\right) & (\alpha = 1).\end{cases}$$

It is easy to observe that on taking a_{nk} as in (1.4) we have Theorem B from Theorem 3.

R e m a r k. If $f(x)$ is every-where continuous then from Theorem 1 and Theorem 2 we conclude that the A-transform of the Fourier Series of $f(x)$ is uniformly convergent to $f(x)$. If (a_{nk}) is the matrix of arithmetic means then this conclusion is obtained from Fejér's theorem (See [7] p.p. 89 (3.4)).

3. We shall need the following lemmas for the proof of our theorems.

LEMMA 1. *If $\{a_{nk}\}$ satisfies (2.1) and (2.2) then for $\tau = \left[\frac{\pi}{t}\right]$,*

(3.1) $$\sum_{k=0}^{n} a_{nk}\sin\left(k + \frac{1}{2}\right)t = \bar{a}_n(\tau)$$

with $\bar{a}_n(\tau)$ defined by (1.5) *and* (1.6).

The lemma follows by using arguments similar to that of MC FADDEN [6, p. 182].

LEMMA 2. *If* $\{a_{nk}\}$ *satisfies* (2.1) *and* (2.5) *then*

$$\sum_{k=0}^{n} a_{nk} \sin \left(k+\frac{1}{2}\right) t = O\left(\frac{a_{nn}}{t}\right). \tag{3.2}$$

This can be proved by using Abel's lemma.

LEMMA 3. *Let* (a_{nk}) *satisfy* (2.5) *Then for* $\tau = \left[\frac{\pi}{t}\right]$,

$$\sum_{k=0}^{n} a_{nk} \sin \left(k+\frac{1}{2}\right) t = O(a'_{n,n-\tau}),$$

where

$$a'_{nk} = \sum_{r=k}^{n} a_{nr}.$$

Proof. On subdividing the sum from k equals zero to n into two parts we get

$$\sum_{k=0}^{n} a_{nk} \sin \left(k+\frac{1}{2}\right) t = \left(\sum_{k=0}^{n-\tau} + \sum_{n-\tau+1}^{n}\right) a_{nk} \sin \left(k+\frac{1}{2}\right) t = \Sigma_1 + \Sigma_2, \text{ say}$$

Clearly

$$\Sigma_2 = O(a'_{n,m-\tau}).$$

Since a_{nk} is non-decreasing in k,

$$\Sigma_1 = O(\pi t^{-1} a_{n,n-\tau}).$$

But

$$\pi t^{-1} a_{n,n-\tau} = (\tau+1) a_{n,n-\tau} \leqslant \sum_{k=n-\tau}^{n} a_{nk} = a'_{n,n-\tau}.$$

implies

$$\Sigma_1 = O(a'_{n,n-\tau}).$$

On collecting the estimates for Σ_1 and Σ_2, the lemma follows.

Proof of theorem 1. Since $\Phi(t) \leqslant w(t)$, and $\sin t/2 \geqslant t/\pi$ $(0 \leqslant t \leqslant \pi)$, we have

$$\max_{0 \leqslant x \leqslant \pi} |f(x) - t_n(x)| \leqslant \max_{0 \leqslant x \leqslant 2\pi} I_1 + \max_{0 \leqslant x \leqslant 2\pi} I_2,$$

where

$$I_1 = \int_0^{\pi/n} \frac{\omega(t)}{t} \left|\sum_{k=0}^{n} a_{nk} \sin \left(k+\frac{1}{2}\right) t\right| dt \tag{4.2}$$

and

$$I_2 = \frac{1}{\pi} \int_{\pi/n}^{\pi} \frac{\omega(t)}{t} \left|\sum_{k=0}^{n} a_{nk} \sin \left(k+\frac{1}{2}\right) t\right| dt. \tag{4.3}$$

By (2.1), (2.2), (2.3) and some calculation we get

$$\max_{0 \leqslant x \leqslant 2\pi} I_1 = O\left(\omega\left(\frac{1}{n}\right) \sum_{k=0}^{n} a_{nk}\right) = O\left(\omega\left(\frac{1}{n}\right)\right) = O\left(\sum_{k=1}^{n} \frac{\omega\left(\frac{1}{k}\right) \bar{a}_n(k+1)}{k}\right)$$

since by (2.2) and (2.3)

$$\sum_{k=1}^{n} \frac{\omega\left(\frac{1}{k}\right) \bar{a}_n(k+1)}{k} \geqslant \omega\left(\frac{1}{n}\right) \sum_{k=1}^{n} \left(\frac{k+2}{k}\right) a_{nk+1} \geqslant a\omega(1/n). \tag{4.4}$$

By Lemma 1 and the fact that $\bar{a}_n(x)$ increases with x, we have

$$\max_{0 \leqslant x \leqslant 2\pi} I_2 = O\left\{\int_{\pi/n}^{\pi} \frac{\omega(t)}{t} \bar{a}_n\left(\left[\frac{\pi}{t}\right]\right) dt\right\} = O\left\{\sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} \frac{\omega(t)}{t} \bar{a}_n\left(\left[\frac{\pi}{t}\right]\right) dt\right\} = \tag{4.5}$$

$$= O\left\{\sum_{k=1}^{n} \frac{\omega\left(\frac{1}{k}\right) \bar{a}_n(k+1)}{k}\right\}.$$

On collecting the estimates for I_1 and I_2, Theorem 1 follows.

Proof of theorem 2. Observe that by Lemma 3

$$\int_{\pi/n}^{\pi} \frac{\omega(t)}{t} \left|\sum_{k=0}^{n} a_{nk} \sin \left(k+\frac{1}{2}\right) t\right| dt = O\left(\int_{\pi/n}^{\pi} \frac{\omega(t)}{t} a'_{n,n-\tau}\, dt\right) = \tag{5.1}$$

$$= O\left(\sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} \frac{\omega(t)}{t} a'_{n,n-\tau}\, dt\right) = O\left(\sum_{k=1}^{n} \frac{\omega\left(\frac{1}{k}\right)}{k} a'_{n,n-k}\right),$$

since $\omega(t)$ and $a'_{n,n-\tau}$ are non-decreasing functions of t.

Following the method of proof of Theorem 1 and replacing the estimate (4.5) by (5.1), we get (from (4.1), and the fact that

$$\max_{0 \leqslant x \leqslant 2\pi} I_1 = O\left(\omega\left(\frac{1}{n}\right)\right),$$

$$\max_{0 \leqslant x \leqslant \pi} |f(x) - t_n(x)| = O\left(\omega\left(\frac{1}{n}\right)\right) + O\left(\sum_{k=1}^{n} \frac{\omega\left(\frac{1}{k}\right)}{k} a'_{n,n-k}\right).$$

Proof of theorem 3. Since $\Phi(t) \leqslant \omega(t)$,

$$\max_{0\leqslant x\leqslant 2\pi} |f(x) - t_n(x)| \leqslant \max_{0\leqslant x\leqslant 2\pi} I_1 + \max_{0\leqslant x\leqslant 2\pi} I_2, \tag{5.3}$$

where

$$I_1 = \int_0^{a_{nn}} \frac{\omega(t)}{t} \left| \sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t \right| dt \tag{5.3}$$

and

$$I_2 = \int_{a_{nn}}^{\pi} \frac{\omega(t)}{t} \left| \sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t \right| dt \tag{5.4}$$

By (2.1) and the fact that $\omega(t) = O(t^{\alpha})$, we have

$$I_1 = O(a_{nn}^{\alpha}) \text{ for } 0 < \alpha \leqslant 1. \tag{5.5}$$

By Lemma 2 and $\omega(t) = O(t^{\alpha})$, we get

$$I_2 = O\left(a_{nn} \int_{a_{nn}}^{\pi} t^{\alpha-2}\, dt\right) = \begin{cases} O(a_{nn}^{\alpha}) & \text{for } 0 < \alpha < 1, \\ O\left(a_{nn} \log \dfrac{1}{a_{nn}}\right) & \text{for } \alpha = 1. \end{cases} \tag{5.6}$$

From (5.2), (5.5) and (5.6) the required result follows.

6. Our results can be used to obtain many results by specialising the matrix A.

By setting A to be the Nörlund matrix in Theorem 2, we get Theorem A.

Remark. 1. It must be mentioned here that we can not obtain Theorem A from Theorem 1 since although $\{p_n\}$ is monotonic non-increasing, $\{p_{n-k}/P_n\}$ considered as a sequence in k is non-decreasing in k.

2. In view of the above remark it is clear that Corollary 2 of KATHAL, HOLLAND and SAHNEY [5] can not be derived from their theorem as claimed in page 231 of their paper [5].

Let (a_{nk}) be the Riesz matrix or the $(\bar{N}, p_n)$ matrix (See [3]), then we obtain the following result:

THEOREM 4. *Let* $\{p_n\}$ *satisfy*

$$p_n \geqslant 0 \ (n = 1, 2, \ldots), \ p_0 > 0, \ p_n \geqslant p_{n+1} \ (n = 0, 1, \ldots). \tag{6.1}$$

Then for $f \in C^*[0, 2\pi]$,

$$\max_{0\leqslant x\leqslant 2\pi} |f(x) - R_n(x)| = O\left\{ \frac{1}{P_n} \sum_{k=1}^{n} \frac{\omega\left(\frac{1}{k}\right) P_{k+1}}{k} \right\}. \tag{6.2}$$

Remark. It should be noted that Theorem B does not cover the case $p_n \geqslant p_{k+1}$ $(n = 0, 1, \ldots)$. Our Theorem 4 yields an estimate for this case and thereby complements Theorem B. For the sake of demonstration, if in Theorem 4 we take $p_n = \frac{1}{n}$, $P_n \sim \log n$ and write $L_n(x)$ for $R_n(x)$ then we shall obtain

COROLLARY. *For* $f(x) \in C^*[0, 2\pi]$, *then*

$$\max_{0\leqslant x\leqslant 2\pi} |f(x) - L_n(x)| = O\left[\frac{1}{\log n} \sum_{k=1}^{n}{}' \frac{\omega\left(\frac{1}{k}\right) \log (k+1)}{k} \right]$$

If $f(x) \in \text{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$ then we state the following Corollaries from Theorem 2:

COROLLARY. (See [1]). *Let* $f(x) \in \text{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$ *and* $\sigma_n(x)$ *be the first Cesáro mean of the Fourier series of* $f(x)$. *Then*

$$\max_{0\leqslant x\leqslant 2\pi} |\sigma_n(x) - f(x)| = \begin{cases} O(n^{-\alpha}) & (0 < \alpha < 1), \\ O\left(\dfrac{\log n}{n^{\alpha}}\right) & (\alpha = 1). \end{cases} \tag{6.5}$$

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