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NUMÉRIQUE ET DE THÉORIE
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CONES OF HOLOMORPHIC FUNCTIONS WITH POSITIVE REAL PART

by

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1. Introduction

In this paper we shall study two classes of holomorphic functions cones with positive real part. We recall some notions and results of cones theory in topological linear spaces.

Let X be a topological linear space and $K \subseteq X$.

DEFINITION 1.1. *The set K is a cone in X if:*

(i) $K + K \subset K$ *(ii)* $\lambda K \subset K$, $\lambda \in R$, $\lambda \geqslant 0$.

If the condition.

(iii) $K \cap -K = \{\theta\}$,

is also satisfied, where θ is the zero element of the space, we call K a proper cone in X.

A cone K endows X with a preorder relation (reflexive and transitive): $x \leqslant_K y$ iff $y - x \in K$. If K is a proper cone, then this relation is an order on X.

Let it be $U_0 \in K$, $U_0 \neq \theta$, a fixed element. We denote $X_{U_0} = \{x \in X : \exists t_1, t_2 \geqslant 0 : -t_1 U_0 \leqslant_K x \leqslant_K t_2 U_0\}$. If $x \in X_{U_0}$ we say that x is U_0 — measurable ([1], p. 16). It is easy to see that X_{U_0} is a linear space. Denote:

(1.1) $\quad \alpha(x) = \inf\{t_1 : -t_1 U_0 \leqslant_K x\}$, $\beta(x) = \inf\{t_2 : x \leqslant_K t_2 U_0\}$.

The map $p_{U_0} : X_{U_0} \to \mathbf{R}_+$, $p_{U_0}(x) = \max\{\alpha(x), \beta(x)\}$, is a seminorm on X_{U_0}. If the condition (iii), Definition 1.1. is satisfied then p_{U_0} is even a norm [1].

Further, we shall denote:

$$U = \{z \in \mathbf{C} : |z| < 1\} \qquad \bar{U} = \{z \in \mathbf{C} : |z| \leqslant 1\}$$
$$H(U) = \{f : U \to \mathbf{C} | f \text{ holomorphic on } U\}$$
$$H(\bar{U}) = \{f : U \to \mathbf{C} | f \text{ holomorphic on } \bar{U}\}.$$

In ([3], p. 88) it is proved that the directed and sufficient family of seminorms $p_n : H(U) \mapsto \mathbf{R}$, given by

$$p_n(f) = \max_{U_n} |f(z)|, \qquad n = 0, 1, 2, \ldots, \tag{1.2}$$

whene $U_n = \left\{z \in \mathbf{C} : |z| \leqslant \frac{n}{n+1}\right\}$, endows $H(U)$ with a structure of a complete metrizable locally convex space. $H(U)$ becomes a Fréchet space which has the Heine-Borel property (any closed and bounded set is compact) but is not a normable space. A metric on $H(U)$ is given by

$$\rho(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{p_n(f-g)}{1 + p_n(f-g)} .$$

$H(\bar{U})$ is a linear normed space with the norm

$$||f|| = \max_{U} |f(z)|, \tag{1.3}$$

For $g \in H(U)$ we consider the set

$$P_g = \{f \in H(U) : \operatorname{Re} g(z) f(z) \geqslant 0, \ z \in U\} \tag{1.4}$$

and for $g \in H(\bar{U})$, the set

$$\tilde{P}_g = \{f \in H(\bar{U}) : \operatorname{Re} g(z) f(z) \geqslant 0, \ z \in \bar{U}\}. \tag{1.5}$$

It is easy to see that P_g and $\tilde{P}_g$ are cones in $H(U)$, respectively $H(\bar{U})$, but they are not proper cones.

The problem of studying these cones was proposed by prof. dr. ELENA POPOVICIU at the course of Approximation Schemes.

We shall give some topological properties of these cones in the spaces $H(U)$ and $H(\bar{U})$.

2. The locally convex space of holomorphic functions on the open disk. The P_g cones.

We shall use the following result of the general theory of topological linear spaces.

THEOREM 2.1. *If X is a topological linear space and $K \subset X$ is a cone with $\mathring{K} \neq \emptyset$ ($\mathring{K}$ being the topological interior of K) then K is a generator cone for X, namely $X = K - K$.*

Proof. $\mathring{K} \neq \emptyset \Rightarrow \exists z_0 \in \mathring{K} \Rightarrow 0 \in \overset{\circ}{\widehat{K - z_0}}$. Then $\forall x \in X$, $\exists \lambda > 0$ such that $\lambda x \in \overset{\circ}{\widehat{K - z_0}} \subset K - z_0$. From here:

$$\lambda x = u - z_0, \ u \in K \Rightarrow x = \frac{1}{\lambda} u - \frac{1}{\lambda} z_0 \in K - K.$$

REMARK 2.2. The condition $\mathring{K} \neq \emptyset$ is very strong. The following example shows that in $H(U)$ there are cones which though rich in elements have an empty interior in any locally convex topology $H(U)$ is endowed with.

Let it be

$$K_n = \left\{f \in H(U) : f(z) = \sum_{i=0}^{\infty} a_i z^i, \ a_n > 0\right\}, \tag{2.1}$$

K_n is evidently a cone. Let τ be a topology for which $(H(U), \tau)$ is a locally convex space. Then $\mathring{K}_n = \emptyset$, otherwise Theorem 2.1. implying $H(U) = K_n - K_n$, which is imposible due to the form of the holomorphic functions from (2.1).

PROPOSITION 2.3. *If $g(z) \neq 0$, $z \in U$, then*

$$P_g = \frac{1}{g} P,$$

where $P = \{f \in H(U) : Re f(z) \geqslant 0\}$, and P_g is the set given in (1.4).

The proof is immediate and we omit it. This proposition shows the importance of studyind the cone P. That is why we shall now study this cone.

THEOREM 2.4. *$\mathring{P} = \emptyset$, in the topology given by the (1.1) family of seminorms and P is a closed set in this topology.*

Proof. For showing that $\mathring{P} = \emptyset$, in the topology induced by the (1.1) seminorms we suppose that there is $f \in P$. Then there are $n \in \mathbf{N}$ and $r > 0$, such as $f + B_{n,r} \subset P$, where

$$B_{n,r} = \{h \in H(U) : p_n(h) \leqslant r\}. \tag{2.2}$$

Two cases are possible.

I. $f \equiv 0$. We put in this case $h(z) = -\frac{rn}{n+1}$, $z \in U$. We see that $h(z) \in B_{n,r}$, $\forall n \in \mathbf{N}$, $\forall r > 0$, but $h \notin P$. This is a contradiction. So $0 \notin P$.

II. $f \not\equiv 0$, on U. Applying the principle of maximum ([2], p. 142), there is $z_0 \in \partial U$ with $f(z_0) \neq 0$. Two cases are to be considered:

a) $z_0 \neq -1$. Let it be then

$$h(z) = \frac{r}{n+1} \frac{f(z)}{p_n(f)} \frac{1}{z - z_0}, \tag{2.3}$$

We show that $h \in B_{n,r}$:

$$p_n(h) = \max_{U_n} \left| \frac{r}{n+1} \frac{f(z)}{p_n(f)} \frac{1}{z-z_0} \right| = \frac{r}{n-1} \max_{U_n} \frac{|f(z)|}{p_n(f)} \frac{1}{|z-z_0|} \leqslant \frac{r}{n+1} \frac{1}{|z-z_0|}.$$

But $|z_0 - z| \geqslant ||z_0| - |z|| \geqslant 1 - \frac{n}{n+1} = \frac{1}{n+1}$, and so

$p_n(h) \leqslant \frac{r}{n+1}(n+1) = r$. We show that $f + h \notin P$. Let it be $z_0 = x_0 + iy_0$, $z = x + iy$. Then for $y = y_0$, and $x \leqslant x_0$, we hawe.

$$\mathrm{Re}\,(f(z) + h(z)) = \mathrm{Re}\, f(z) \left(1 + \frac{r}{n+1} \cdot \frac{1}{x - x_0}\right) < 0,$$

because $\exists \delta > 0$, $|x - x_0| < \delta$ such that $\left(1 + \frac{r}{n+1} \frac{1}{x-x_0}\right) < 0$. So $f + h \notin P$, which is a contradiction.

b) $z_0 = -1$. We put $h(z) = \frac{r}{n+1} \frac{f(z)}{p_n(f)} \frac{1}{z - z_0}$.
Reasoning similarly as in the a) case on shows that for $y = 0$ and $x \geqslant -1$, $\exists \delta > 0$, such that for $x - 1 < \delta$, we have

$$\mathrm{Re}\,(f(z) + h(z)) = \mathrm{Re}\, f(z) \left(1 + \frac{r}{n+1} \cdot \frac{1}{x-1}\right) < 0,$$

that is $f + h \notin P$. From the a) and b) cases it follows that the case II is impossible too. So $\mathring{P} = \emptyset$. It is easy to see that P is closed in the topology given by the (1.2) family of seminorms. The proof is now complete.

Let it be $f_0 \in P$, $f_0 \neq 0$, $z \in U$. Denote:

$$H_{f_0}(U) = \{f \in H(U) : f \text{ is } f_0 - \text{measurable}\}.$$

PROPOSITION 2.5. *$H_{f_0}(U)$ is a locally convex space with respect to the $(q_{n,f_0})_{n \geqslant 0}$ family of seminorms, where*

$$q_{n,f_0}(f) = \max_{U_n} \frac{|\mathrm{Re}\, f(z)|}{\mathrm{Re}\, f_0(z)}, \tag{2.4}$$

each of these seminorms being induced by the cone P.
Proof. Let it be $f \in H_{f_0}(U)$. Then there are $t_1, t_2 \geqslant 0$ such that

$$-t_1 f_0 \leqslant_P f \leqslant_P t_2 f_0 \Leftrightarrow \begin{cases} \mathrm{Re}\, f + t_1 \mathrm{Re}\, f_0 \geqslant 0, \\ t_2 \mathrm{Re}\, f_0 - \mathrm{Re}\, f \geqslant 0 \end{cases}$$

or,

$$-t_1 \leqslant \frac{|\mathrm{Re}\, f|}{\mathrm{Re}\, f_0} \leqslant t_2.$$

Using the principle of maximum of harmonic functions, ([2], p. 142) we see that for $\alpha(f)$ and $\beta(f)$ defined in (1.1), we hawe

$$\alpha(f) = \beta(f) = \max_{U_n} \frac{|\mathrm{Re}\, f(z)|}{\mathrm{Re}\, f_0(z)}.$$

It is easy to see that $q_{n,f_0}(f) = \max_{U_n} \frac{|\mathrm{Re}\, f(z)|}{\mathrm{Re}\, f_0(z)}$, is a seminorm.

PROPOSITION 2.6. *P is not a normal cone, in the sense of definition given in [6], p. 215.*
Proof. If P would be a normal cone, it would follows that P is proper in $H(U)$, ([6], Corollary 1, p. 216).
But, as we hawe already seen, P is not a proper cone.

3. The normed space of holomorphic functions on the closed disk. The $\widetilde{P}_g$ cones.

THEOREM 3.1. *$\mathring{\widetilde{P}}_g \supset \{f \in H(\bar{U}) : \mathrm{Re}\, f(z)\, g(z) \geqslant 0, z \in \bar{U}\}$, and the second set is open in the topology given by (1.3), where $g \in H(\bar{U})$. P_g is closed.*

Proof. Let it be $h_0 \in \mathring{\widetilde{P}}_g$. The continuity of $\mathrm{Re}\, gh_0$ on $\bar{U}$ implies that there exists $\delta = \min_{z \in U} \mathrm{Re}\, g(z)\, h_0(z) > 0$.

We consider $B = \left\{h \in H(\bar{U}) : ||h - h_0|| < \frac{\delta}{2G}\right\}$, where $G = \max_{\bar{U}} |g(z)|$.

We show that $B \subset \widetilde{P}_g$. We hawe:

$$|h(z) - h_0(z)| < \frac{\delta}{2G}, \; z \in \bar{U}$$

and

$$|g(z)|\, |h(z) - h_0(z)| \leqslant G |h(z) - h_0(z)| < \frac{\delta}{2}, \; z \in \bar{U}.$$

$$|\mathrm{Re}\, gh - \mathrm{Re}\, gh_0| < \frac{\delta}{2}.$$

So

$$\mathrm{Re}\, gh > \mathrm{Re}\, gh_0 - \frac{\delta}{2} > \frac{\delta}{2} > 0, \; z \in \bar{U}.$$

Similarly to Proposition 2.3 on proves that for $g \in H(\bar{U})$, $g(z) \neq 0$, $z \in \bar{U}$, we hawe $\widetilde{P}_g = \frac{1}{g} \widetilde{P}$, where

$$\widetilde{P} = \{f \in H(\bar{U}) : \mathrm{Re}\, f(z) \geqslant 0, \; z \in \bar{U}\}.$$

COROLLARY 3.2. $H(\bar{U}) = \widetilde{P}_g - \widetilde{P}_g = \frac{1}{g} \widetilde{P} - \frac{1}{g} \widetilde{P}$.

The proof is an immediate consequence of the theorems 2.1 and 3.1. Denote $H_{f_0}(\overline{U}) = \{f \in H(\overline{U}) : f \text{ is } f_0 - \text{measurable}\}$.

PROPOSITION 3.3. *$H_{f_0}(\overline{U})$ is a seminormed space where the seminorm is induced by the preorder of $\widetilde{P}$ and is given by*

$$p(f) = \max_{\overline{U}} \frac{|\mathrm{Re}\, f(z)|}{\mathrm{Re}\, f_0(z)}.$$

REMARK 3.4. *$\widetilde{P}$ is not a normal cone in the normed space $H(\overline{U})$, in the sense of the definition from* [1], *p.* 17.

The following example shows this.

Let it be $f_n, g_n \in \widetilde{P}$, $f_n(z) = \frac{1}{2n} + i\sqrt{1 - \frac{1}{4n^2}}$, $g_n(z) = \frac{1}{2n} - i\sqrt{1 - \frac{1}{4n^2}}$. Evidently $||f_n|| = ||g_n|| = 1$, $n = 1, 2, \ldots$, but $||f_n + g_n|| = \frac{1}{n} \to 0$ $(n \to \infty)$.

4. Extremal directions in the cone P.

DEFINITION 4.1. *We call $x_0 \neq 0$ an extremal point of the cone K iff $x_0 = \alpha x + (1 - \alpha)y$, $x, y \in K$, $\alpha \in (0, 1)$, implies $x = y$.*

DEFINITION 4.2. *The set $R_{x_0} = \{\lambda x_0 : \lambda \geqslant 0\}$ is called an extremal ray of the cone K, generated by the extremal point x_0.*

For finding the extremal directions in the cone P, the following result will be helpful.

LEMA 4.3. *If $f : \overline{U} - \{z_0\} \to \mathbf{C}$, with $z_0 \in \partial U$, is holomorphic in U and continuous on $\partial U - \{z_0\}$, then :*

$$f(z) = 0,\ z \in \partial U - \{z_0\} \Rightarrow f(z) = 0,\ z \in \overline{U} - \{z_0\}.$$

Proof. Let it be $f = u + iv$. Then u, v are continuous on $\partial U - \{z_0\}$. Denote $\Delta_n = \left\{z \in \mathbf{C} : |z_0 - z| < \frac{1}{n}\right\}$, $\Omega_n = U \cap \Delta_n$, $C = \partial\Omega_n$. We consider $f_n : U - \Omega_n \to \mathbf{C}$, $f_n(z) = f(z)$, $z \in \overline{U - \Omega_n}$, $n = 1, 2, \ldots$. We have $\max_{U - \Omega_n} |f_n(z)| = M_n$, which is reached on $\partial(U - \Omega_n)$, and $M_n \to 0$, when $n \to \infty$. But by the principle of maximum ([2], p. 142) we hawe $|f(z)| = |f_n(z)| \leqslant M_n$, $z \in \overline{U - \Omega_n}$. So $|f(z)| = 0$, $z \in \overline{U} - \{z_0\}$. Then $f(z) = 0$, $z \in \overline{U} - \{z_0\}$.

THEOREM 4.4. *The functions*

$$e_t = \frac{1 + ze^{-it}}{1 - ze^{-it}}, \quad t \in [0, 2\pi] \tag{4.3}$$

are extremal points in the cone P.

Proof. Let's suppose that for a fixed t from $[0, 2\pi]$ we hawe the representation

$$\frac{1 + ze^{-it}}{1 - ze^{-it}} = \alpha f(z) + (1 - \alpha)\, g(z), \tag{4.4}$$

where $z \in U$, $\alpha \in (0,1)$, $f, g \in P$ and $f(z) \neq g(z)$, $z \in U$. It follows from here that

$$\frac{1 + ze^{-it}}{1 - ze^{-it}} = \alpha f(z) + (1 - \alpha)\, g(z), \quad z \in \overline{U} - \{e^{it}\},$$

which implies

$$0 = \alpha\, \mathrm{Re} f(z) + (1 - \alpha)\, \mathrm{Re}\, g(z),\ z \in \partial U - \{e^{it}\} \tag{4.5}$$

For $f, g \in P$ it follows that $\mathrm{Re} f(z) \geqslant 0$, $\mathrm{Re}\, g(z) \geqslant 0$. Now, using (4.4) and $\alpha \in (0,1)$ we hawe $\mathrm{Re} f(z) = \mathrm{Re}\, g(z) = 0$, $z \in \partial U - \{e^{it}\}$. Using Lemma 4.3. it follows $\mathrm{Re} f(z) = \mathrm{Re}\, g(z) = 0$, $z \in \overline{U} - e^{it}$. But, $\mathrm{Re} \frac{1 + ze^{-it}}{1 - ze^{-it}} \neq 0$, $z \in U$, which is a contradiction. So (4.4) holds iff $f(z) = g(z)$, $z \in U$.

COROLLARY 4.5. *The functions λe_t, $\lambda \geqslant 0$, are extremal directions in P, where e_t are the functions defined in* (4.3).

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