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INTERPOLATING CONVEX POLYNOMIALS

by
RADU PRECUP
(Cluj-Napoca)

1. In this paper we define the piecewise retarded convex interpolating polynomials and piecewise anticipated convex interpolating polynomials. An existence theorem is proved by a generalized Young's method.

2. Let be $n \in \mathbf{N}$, $n \geqslant 2$ and

$$(1) \qquad 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1,$$

$$y_0, y_1, \ldots, y_{n-1}, y_n \in \mathbf{R}.$$

We say that a polynomial P is a piecewise retarded convex interpolating polynomial if

$$(2) \qquad P(x_i) = y_i, \; i = 0, 1, \ldots, n$$

$$(3) \qquad P''(x)[x_{i-2}, x_{i-1}, x_i ; y] \geqslant 0, \; x \in [x_{i-1}, x_i], \; i = 3, \ldots, n$$

$$P''(x)[x_0, x_1, x_2 ; y] \geqslant 0, \; x \in [0, x_2].$$

A polynomial Q is called a piecewise anticipated convex interpolating polynomial if it satisfies (2) and

$$(3') \qquad Q''(x)[x_{i-2}, x_{i-1}, x_i ; y] \geqslant 0, \; x \in [x_{i-2}, x_{i-1}], \; i = 2, \ldots, n-1$$

$$Q''(x)[x_{n-2}, x_{n-1}, x_n ; y] \geqslant 0, \; x \in [x_{n-2}, 1].$$

In the following we suppose that the knots $x = (x_i | \, i = 0, 1, \ldots, n)$ are equidistant, more precisely $x_i = i/n$, $i = 0, 1, \ldots, n$, $x_{-1} = -1/n$ and we note:

$$\Delta_i^{(3)}(y) = 2[x_i, x_{i+1}, x_{i+2} ; y]$$

$$\Delta_i^{(2)}(y) = (1/n)[x_{i+1}, x_{i+2} ; y], \; i = -1, 0, 1, \ldots, n-2$$

where $[x_i, x_{i+1}, x_{i+2}; y]$ and $[x_{i+1}, x_{i+2}; y]$ are divided differences and $y_{-1} = y_0$.

On the values y from (1) we suppose that there exists $p \in Z$, $p \geqslant 0$ and the indices $0 = k_0 < k_1 < \ldots < k_p < k_{p+1} = n$ such that y will be convex and increasing on the discrets intervals

(4) $$[x_{k_j - 1}, x_{k_{j+1}}], \text{ respectively } [x_{k_j}, x_{k_{j+1}}],$$

for the even values of $j \in \{0, 1, \ldots, p\}$, and concave and decreasing on the discrets intervals (4) for the odd values of j, or conversely: convex and increasing for the odd values of j and concave and decreasing for the even values of j.

We have the following:

THEOREM. *If the above conditions on x and y are satisfied, there exists then P a piecewise retarded convex interpolating polynomial.*

Proof. We may assume $y_0 = 0$. Under the above presumtion on y we have $\Delta_i^{(3)}(y) \neq 0$, $\Delta_i^{(2)} \neq 0$, $i = -1, 0, \ldots, n-2$ and, which is important:

(5) $$\operatorname{sign} \Delta_{k_j-1}^{(3)}(y) = \operatorname{sign} \Delta_{k_j-1}^{(2)}(y),\ j = 0, 1, \ldots, p.$$

Denote

$$\Delta_i(y) = \begin{cases} \Delta_i^{(3)}(y) \text{ if } i+1 \notin \{k_0, k_1, \ldots, k_p\} \\ \Delta_i^{(2)}(y) \text{ if } i+1 \in \{k_0, k_1, \ldots, k_p\}, \quad i = -1, 0, \ldots, n-2. \end{cases}$$

Using (5) we have $\operatorname{sign} \Delta_i(y) = \operatorname{sign} \Delta_i^{(3)}(y)$, $i = -1, 0, \ldots, n-2$. We denote by D the following convex cone of polynomials

(6) $$D = \{Q \in \mathfrak{P} \mid Q(x)[x_{i-2}, x_{i-1}, x_i; y] \geqslant 0,\ x \in [x_{i-1}, x_i],\ i = 3, \ldots, n, \quad Q(x)[x_0, x_1, x_2; y] \geqslant 0,\ x \in [0, x_2]\}$$

and we define a linear function $F: D \to \mathbf{R}^n$ by

(7) $$F(Q) = \left(\int_0^{x_i} dt \int_0^t Q(x)\,dx\right)_{i=1}^n.$$

Clearly $F(D)$ is a convex cone in $\mathbf{R}^n$. We will prove that $y = (y_1, y_2, \ldots, y_n) \in F(D)$.

For each $1 \leqslant i \leqslant n$ we construct a vector $\Phi_i \in \mathbf{R}^n$ in the following manner:

(8) $$\Phi_i = (0, 0, \ldots, 0, 1, 2, \ldots, k_{j+1} - i + 1, k_{j+1} - i + 1, \ldots, k_{j+1} - i + 1),$$

where $k_j + 1 \leqslant i \leqslant k_{j+1}$, the first $i - 1$ components are 0 and the last $n - k_{j+1} + 1$ components are equals to $k_{j+1} - i + 1$. We remark that

$\Phi_1, \Phi_2, \ldots, \Phi_n$ are linearly independent and therefore they form a base in $\mathbf{R}^n$. Moreover, for each $z \in \mathbf{R}^n$:

(9) $$z = \sum_{i=1}^n \Delta_{i-2}(z)\,\Phi_i$$

where $z_0 = 0$. For each $i \in \{1, 2, \ldots, n\}$ denote $\lambda_i = (\operatorname{sign} \Delta_{i-2}(y))\,\Phi_i$. We have $\Phi_i = (\operatorname{sign} \Delta_{i-2}(y))\,\lambda_i$, $i = 1, 2, \ldots, n$ and using (9):

(10) $$y = \sum_{i=1}^n |\Delta_{i-2}(y)|\,\lambda_i,\ |\Delta_{i-2}| > 0,\ i = 1, 2, \ldots, n.$$

We construct another base θ in $\mathbf{R}^n$ which is defined by

$$\theta_i = \begin{cases} \lambda_i - \lambda_{i+1} & \text{if } i \notin \{k_1, k_2, \ldots, k_p\} \\ \lambda_i & \text{if } i \in \{k_1, k_2, \ldots, k_p\}, \quad i = 1, 2, \ldots, n \end{cases}$$

Clearly if $z = \sum_{i=1}^n b_i \lambda_i$, $b_i > 0$, $i = 1, 2, \ldots, n$ then $z = \sum_{i=1}^n a_i\,\theta_i$, $a_i > 0$, $i = 1, 2, \ldots, n$ and in particular using (10).

(11) $$y = \sum_{i=1}^n a_i\,\theta_i, \quad a_i > 0, \quad i = 1, 2, \ldots, n.$$

Let us prove that for any $i \in \{1, 2, \ldots, n\}$ θ_i is an accumulation point of $F(D)$. Indeed, consider $Q \in D$. By (9) we have

(12) $$F(Q) = \sum_{i=1}^n \Delta_{i-2}(F(Q))\,(\operatorname{sign} \Delta_{i-2}(y))\,\lambda_i.$$

Using the Weierstrass' polynomial approximation theorem we find for each interval $[x_{i-1}, x_i]$ a polynomial P, $P > 0$ on $[0, 1]$ such that

$$P(x) \cong 0,\ x \notin [x_{i-1}, x_i]$$

$$P(x) \cong 1 \Big/ \left| \int_{x_{i-1}}^{x_i} dt \int_{x_{i-1}}^{t} Q(x)\,dx \right|,\ x \in [x_{i-1}, x_i]$$

Therefore

$$P(x)\,Q(x) \cong 0,\ x \notin [x_{i-1}, x_i]$$

$$P(x)\,Q(x) \cong Q(x) \Big/ \left| \int_{x_{i-1}}^{x_i} dt \int_{x_{i-1}}^{t} Q(x)\,dx \right|,\ x \in [x_{i-1}, x_i]$$

Then we have

$$\Delta_{i-2}(F(PQ)) \cong \int_{x_{i-1}}^{x_i} \mathrm{d}t \int_{x_{i-1}}^{t} \left(Q(x) \Big/ \left| \int_{x_{i-1}}^{x_i} \mathrm{d}t \int_{x_{i-1}}^{t} Q(s)\mathrm{d}s \right| \right) \mathrm{d}x =$$

$$= \operatorname*{sign}_{x \in (x_{i-1}, x_i)} Q(x) = \operatorname{sign} \Delta_{i-2}(y)$$

and if $i \notin \{k_1, k_2, \ldots, k_p\}$:

$$\Delta_{i-1}(F(PQ)) \cong - \int_{x_{i-1}}^{x_i} \mathrm{d}t \int_{x_{i-1}}^{t} \left(Q(x) \Big/ \left| \int_{x_{i-1}}^{x_i} \mathrm{d}t \int_{x_{i-1}}^{t} Q(s)\mathrm{d}s \right| \right) \mathrm{d}x =$$

$$= - \operatorname*{sign}_{x \in (x_{i-1}, x_i)} Q(x) = - \operatorname{sign} \Delta_{i-2}(y) = - \operatorname{sign} \Delta_{i-1}(y).$$

If $i \in \{k_1, k_2, \ldots, k_p\}$ then

$$\Delta_{i-1}(F(PQ)) = \Delta^{(2)}_{i-1}(F(PQ)) \cong 0.$$

Thus $F(PQ)) \cong \lambda_i - \lambda_{i+1}$ if $i \notin \{k_1, \ldots, k_p\}$ and $F(PQ) \cong \lambda_i$ if $i \in \{k_1, k_2, \ldots, k_p\}$ and consequently for any $i \in \{1, 2, \ldots, n\}$ θ_i is an accumulation point of $F(D)$. So, going back to (11) we can find a base θ in $\mathbf{R}^n$ formed by elements of $F(D)$ such that $y = \sum_{i=1}^{n} b_i \theta_i$, $b_i > 0$, $i = 1, 2, \ldots, n$ and consequently $y = (y_1, y_2, \ldots, y_n) \in F(D)$. Thus, $y = F(Q)$ for some $Q \in D$ and

$$P(x) = \int_0^x \mathrm{d}t \int_0^t Q(s)\,\mathrm{d}s$$

is the desired polynomial.

REMARKS. 1. The piecewise retarded convex interpolating polynomial P found in the proof of our theorem is monotone on each of the intervals $[x_{k_j+1}, x_{k_{j+1}}]$, $j = 0, 1, \ldots, p$.

2. If y is convex on the equidistant points x, there exists then an interpolating convex polynomial. Indeed, let us add to y a linear polynomial P_1 such that $y + P_1$ be increasing on x. Clearly the convexity of y is not affected by P_1. Now, if we apply the above theorem to $y + P_1$ we find a polynomial P. It is easy to see that $P - P_1$ is the desired interpolating convex polynomial.

3. If f is a convex continuous real function on $[0, 1]$, then f is the uniform limit of a sequence of convex polynomials on $[0, 1]P_n$ which interpolates f on the knots $\{i/n | i = 0, 1, \ldots, n\}$ respectively ($n \in \mathbf{N}$).

4. Similar results can be obtained for the piecewise anticipated convex interpolating polynomials.

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Grupul şcolar Traian Vuia
Bul. Lenin
Cluj-Napoca