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SOME PROPERTIES OF A NONLINEAR PARABOLIC DIFFERENCE SCHEME

by

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1. *Introduction.* In [1], [2], [3] we considered the nonlinear degenerate problem:

(1.1) $$\frac{\partial u}{\partial t} = \Delta\varphi(u) + a(x, t) \text{ on } Q = \Omega \times]0, T[$$

(1.2) $$u(x, 0) = u_0(x) \qquad x \in \Omega$$

(1.3) $$u(x, t) = u_1(x, t) \text{ on } S = \partial\Omega \times]0, T[,$$

under the assumption

(A)

(i) $$u_0 \in C(\overline{\Omega}), \quad u_1 \in C(S), \quad a \in C(\overline{Q})$$
$$u_0,\ u_1,\ a \geqslant 0$$

(ii) $$\varphi \in C^2(\mathbf{R}_+),\ \varphi(u),\ \varphi'(u) > 0, \text{ for } u > 0$$
$$\varphi(0) = \varphi'(0) = 0, \quad \varphi''(u) \geqslant 0.$$

$\Omega \subset \mathbf{R}^2$ is a regular, bounded, convex, domain. Under certain conditions on the data we proved, using an explicit difference scheme, that (1.1) — (1—3) has a unique weak solution in the following sense:

(i) $$u \in L^\infty(Q),\ u \geqslant 0, \quad \partial\varphi(u)/\partial x_i \in L^2(Q)$$

(ii) (1.2), (1.3) are fulfilled in the generalized sense

(iii) For any $f \in H^1(Q)$ such that $f|_{S_1} = 0$,

(1.4)

$$\int_Q \left(u \frac{\partial f}{\partial t} - \sum_{i=1}^{2} \frac{\partial \varphi(u)}{\partial x_i} \frac{\partial f}{\partial x_i}\right) dx\,dt + \int_\Omega u_0(x) f(x, 0) dx + \int_Q a(x, t) f(x, t) dx\,dt = 0$$

Here $S_1 = S \cup \{(x, T) | x \in \Omega\}$.

Unfortunately in the proof of Lemma 2.2 in [2] (an consequently in that of Lemma 3.1 in [3], too) there is a mistake. Nevertheless, the lemmas themselves are correct as it is shown in §2, Theorem 2.1. Our main result is given in §3 where the conditions on the initial function is less restrictive as in the above mentioned papers.

2. Difference inequalities. As in our previous papers we use a rectangular mesh with the step h in the Ox_i, $i = 1,2$ directions and τ in the Ot direction.

The mesh-points of Ω, Q etc., will be denoted by Ω_h, Q_h, etc.. For simplicity we shall use the same notation for the sets of mesh-points and for the rectangular domains determined by them. We put $\Gamma_h = \partial\Omega_h$ and Δ_h for the classical five-point discretization of Δ.

Consider the following difference problems:

(2.1) $$U_t(k) = \Delta_h \varphi(U(k-1)) + a(k) \quad \text{on} \quad Q_h$$

(2.2) $$U(0) = u_{oh} \quad \text{on} \quad \Omega_h$$

(2.3) $$U|_{\Gamma_h} = u_2(x, k\tau) \qquad x \in \Gamma_h,\ k = 0, \ldots, K = \left[\frac{T}{\tau}\right];$$

and

(2.4) $$V_t(k) = \Delta_h \varphi(V(k-1)) + b(k) \quad \text{on} \quad Q_h$$

(2.5) $$V(0) = v_{oh} \quad \text{on} \quad \Omega_h$$

(2.6) $$V|_{\Gamma_h} = v_2(x, k\tau), \qquad x \in \Gamma_h,\ k = 0, 1, \ldots, K = \left[\frac{T}{\tau}\right].$$

Suppose that M, N are constants such that $u_0, u_2, a \leqslant M$; $v_0, v_2, b \leqslant N$; $M_0 = (1 + T)M$, $N_0 = (1 + T)N$, $M_0 < N_0$. Denote:

$$\lambda = 4 \frac{\tau}{h^2} \varphi'(M_0), \qquad \mu = 4 \frac{\tau}{h^2} \varphi'(N_0).$$

LEMMA 2.1. ([1], [2]) *Suppose that U is the solution of problem* (2.4)–(2.6). *Assume that* (A) *is valid.*

Then. *if* $\lambda \leqslant 1$:

$$0 \leqslant U \leqslant M_0 \quad \text{for} \quad (x, t) \in Q_h.$$

LEMMA 2.2. *Suppose that:*

(i) *Assumption* (A) *is valid*

(ii) $\mu \leqslant 1$

(iii) $u_{oh} \leqslant v_{oh}$, $u_2 \leqslant v_2$, $a \leqslant b$, *on their domains of definition.*

(iv) U *satisfies* (2.1)–(2.3); V *satisfies* (2.4)–(2.6).

Then,

(2.7) $$U \leqslant V \quad on \quad Q_h.$$

Proof: Denote $W_{ij}(k) = V_{ij}(k) - U_{ij}(k)$. Assume that (2.7) is not true. Then there exists a triplet of indices (m, n, l) such that $W_{mn}(l) < 0$ for an $l \geqslant 1$ and

$$W_{ij}(k) \geqslant 0 \text{ for any } i, j \text{ and } k < l.$$

Put

$$\overline{W}_{ij}(k) = \frac{1}{\tau}(W_{ij}(k) - W_{ij}(k-1)).$$

$\overline{V}_{ij}(k)$ and $\overline{U}_{ij}(k)$ will stand for similar differences of V respectively U. Then,

(2.8) $$\overline{W}_{mn}(l-1) = \Delta_h(\varphi(V_{mn}(l-1)) - \varphi(U_{mn}(l-1))).$$

Because

$$\varphi(V_{ij}(l-1)) - \varphi(U_{ij}(l-1)) > \varphi(V_{mn}(l)) - \varphi(U_{mn}(l)),$$

for any i, j; it follows that:

$$\overline{W}_{mn}(l-1) \geqslant \frac{4}{h^2}[\varphi(V_{mn}(l)) - \varphi(U_{mn}(l)) - (\varphi(V_{mn}(l-1)) -$$

$$- \varphi(U_{mn}(l-1)))] = 4\frac{\tau}{h^2}[\tilde{\varphi}'(U_{mn}(l-1)\overline{W}_{mn}(l) - \tilde{\varphi}'(U_{mn}(l))\overline{W}_{mn}(l-1)].$$

Hence

(2.9) $$W_{mn}(l)\left(1 - 4\frac{\tau}{h^2}\tilde{\varphi}'(U_{mn}(l))\right) \leqslant W_{mn}(l-1)\left(1 - \frac{4}{h^2}\tilde{\varphi}'(U_{mn}(l-1))\right),$$

which by Lemma 2.1 and (*ii*) contradicts our assumption.

COROLLARY 2.1. If in addition to the conditions of Lemma 2.1, V is also nondecreasing in k, then instead of (*ii*) we can assume, $\lambda \leqslant 1$, only. Indeed, if as in the proof of the lemma, we admit that $W_{mn} < 0$:

$$V_{mn}(l-1) \leqslant V_{mn}(l) \leqslant U_{mn}(l) \leqslant M_0,$$

which entail that in (2.9)

$$4\frac{\tau}{h^2}\varphi'(U_{mn}(l-1)) \text{ as well as } 4\frac{\tau}{h^2}\tilde{\varphi}'(U_{mn}(l)),$$

are less than λ.

In the sequel u_2 will be defined with the aid of u_1 as follows:

$$u_2(x, t) = u(x^*, t) \qquad x \in \Gamma_h, \qquad x^* \in \partial\Omega_h$$

where x^* is the nearest point to x. In the same time u_{oh} will stand for the restriction to Ω_h of u_0.

LEMMA 2.3. ([1], [3]) *Suppose that conditions of the previous lemma hold and that* $\Delta\varphi(u_0) \geqslant 0$.

Then

(i) $U_{\bar{i}}(k) \geqslant 0, \quad k = 1, 2, \ldots, K$ *on* $\overline{\Omega}_h$

(ii) $\tau h^2 \sum_{Q_h} U_{\bar{i}}(k) \leqslant M_0 m(\Omega)$,

for h sufficiently small.

THEOREM 2.1. *Suppose that:*

(i) *Assumption (A) holds and* $u_0 \in C^2(\overline{\Omega})$

(ii) $\lambda \leqslant 1$

(iii) $\partial u_1/\partial t$ *exists and is bounded on S*

(iv) $\varphi(\infty) = \infty$

Then there exists a constant $C > 0$ *such that*

$$\tau h^2 \sum_{Q_h} |\varphi(U(k))_{\bar{i}}| \leqslant C.$$

Proof: Let $\varepsilon > 0$ be a constant and $V_0 \in C^2(\overline{\Omega})$ such that:

$$\Delta V_0 \geqslant |\Delta\varphi(u_0)| + \varepsilon \text{ on } \Omega$$

$$V_0|_\Gamma = \varphi(u_0)|_\Gamma \qquad \Gamma = \partial\Omega,\ V_0 > 0. \tag{2.10}$$

Define v_0 by $v_0 = \varphi^{-1}(V_0)$ and the function $v_1 \in C(S)$ as follows:

$$v_1 \geqslant 0,\ \partial v_1/\partial t \text{ exists on S}$$

$$\frac{\partial v_1}{\partial t}(x, t) \geqslant c + \varepsilon, \quad t \in\,]0, T[, \quad v_1(x, 0) = v_0|_\Gamma$$

where $c \geqslant \max |\partial u_1/\partial t]$

With the aid of v_0, v_1 as data functions and $b = a$, we construct using the scheme (2.4)—(2.6) the discrete solution $V: Q_h \to \mathbf{R}$. According to Lemma 2.3 we have $V_{\bar{i}} \geqslant 0$ and

$$\tau h^2 \sum_{Q_h} V_{\bar{i}}(k) \leqslant C \quad k = 1, 2, \ldots, K.$$

For convenience we set $\bar{U}(k) = U_{\bar{i}}(k)$ and $\bar{V}(k) = V_{\bar{i}}(k)$,

$$\bar{\varphi}(U(k)) = \varphi(U(k))_{\bar{i}} \text{ and } \varphi(V(k)) = \bar{\varphi}(V(k))_{\bar{i}}.$$

First we observe that for h small enough:

$$\Delta_h\varphi(v_0) > |\Delta_h\varphi(u_0)|$$

so that

$$|U_{ij}(1)| \leqslant V_{ij}(1) \quad \text{on } \Omega_h. \tag{2.11}$$

Because of the properties of φ and taking (2.11) into account, we get:

$$|\bar{\varphi}(U_{ij}(1))| \leqslant \bar{\varphi}(\bar{V}_{ij}(1)) \quad \text{on } \Omega_h. \tag{2.12}$$

Indeed, by Corollary 2.1 we have either

$$U_{ij}(1) \leqslant U_{ij}(0) \leqslant V_{ij}(0) \leqslant V_{ij}(1)$$

or

$$U_{ij}(0) \leqslant U_{ij}(1) \leqslant V_{ij}(0) \leqslant V_{ij}(1),$$

or

$$U_{ij}(0) \leqslant V_{ij}(0) \leqslant U_{ij}(1) \leqslant V_{ij}(1).$$

For $k \geqslant 2$ we obtain:

$$\begin{aligned} \bar{U}_{ij}(k) - \bar{V}_{ij}(k) &= \bar{U}_{ij}(k-1) - \bar{V}_{ij}(k-1) + \\ &+ \tau\Delta_h(\bar{\varphi}(U_{ij}(k-1)) - \bar{\varphi}(V_{ij}(k-1))) \text{ on } \Omega_h. \end{aligned} \tag{2.13}$$

Now, we want to prove that for any $k \geqslant 2$

$$\bar{\varphi}(U_{ij}(k)) \leqslant \bar{\varphi}(V_{ij}(k)) \quad \text{on} \quad \Omega_h.$$

Suppose that this is not true and let l be the greatest subscript such that for any i, j (for $l = 2$ this is true by (2.11)):

$$\bar{\varphi}(U_{ij}(l-1)) \leqslant \bar{\varphi}(V_{ij}(l-1)) \quad \text{on} \quad \Omega_h$$

and there is a couple of subscripts m, n such that:

$$\bar{\varphi}(U_{mn}(l)) > \bar{\varphi}(V_{mn}(l)). \tag{2.14}$$

Since $u_0 \leqslant v_0$, $u_1 \leqslant v_1$, by Corollary 2.1, $U_{mn}(l-1) V_{mn}(l-1)$, $U_{mn}(l) V_{mn}(l)$. Then, in view of (2.14) we must have

$$U_{mn}(l) > V_{mn}(l). \tag{2.15}$$

On the other hand:

$$\begin{aligned} \bar{\varphi}(U_{m+1,n}(l-1)) - \bar{\varphi}(V_{m+1,n}(l-1)) - \bar{\varphi}(U_{mn}(l-1)) + \bar{\varphi}(U_{mn}(l-1)) \leqslant \\ \leqslant \bar{\varphi}(U_{mn}(l)) - \bar{\varphi}(V_{mn}(l)) - \bar{\varphi}(U_{mn}(l-1)) + \varphi(V_{mn}(l-1)); \end{aligned}$$

and similar inequalities when $(m + 1, n)$ is replaced successively by $(m - 1, n)$ $(m, n + 1)$, $(m, n - 1)$. From (2.14), (2.15), (2.16):

$$\begin{aligned} \bar{U}_{mn}(l) - \bar{V}_{mn}(l) &\leqslant U_{mn}(l-1) - V_{mn}(l-1) + \\ + 4\frac{\tau}{h^2}(\varphi(U_{mn}(l-1)) &- \varphi(V_{mn}(l-1)) + \varphi(U_{mn}(l-2)) - \varphi(U_{mn}(l-2))) < 0, \end{aligned}$$

which contradicts (2.14). A similar argument holds for $-\overline{\varphi}(U_{ij}(k))$, so that:

$$|\overline{\varphi}(U_{ij}(k))| \leqslant \overline{\varphi}(V_{ij}(k)).$$

The conclusion of the theorem now follows from Lemma 2.3. The previous theorem shows the boundedness of the difference in t of $\varphi(U)$, only. Nevertheless Lemma 2.4 of [2] as well as Theorem 3.2 of [3] remain true as it is shown by the following lemma.

LEMMA 2.4. *Suppose that U is the solution of* (2.1)–(2.3) *and that conditions of Theorem* 2.1 *hold.*

Then, there exists a constant C independent of h, such that for any Ω^* with $\overline{\Omega}^* \subset \Omega$,

$$\tau h^2 \sum_{k=1}^{K} \sum_{\Omega_h^*} (\varphi^2(U(k))_{x_1} + \varphi^2(U(k))_{x_2}) < C$$

provided that $h < h_0(\Omega_h^*)$.

Proof: The argument is mainly the same as in [1]. The only difference is that instead of an estimate for U_t in the discrete L^1 norm we have one for $\varphi(U)_t$. So we shall have to transform the scalar product:

$$\tau h^2 \sum_{k=1}^{K} \sum_{\Omega_h^*} U_{\bar{t}}(k)\varphi(U(k))$$

into

$$-\tau h^2 \sum_{k=0}^{K-1} \sum_{\Omega_h^*} U(k)\varphi(U(k))_t + \sum_{\Omega_h^*} (U(0)\varphi(U(0)) - U(K)\,\varphi\,(U(k))).$$

This is bounded since U is bounded on Ω_h^*.

On the basis of the results of this section we can prove in the same way as in [1], [2], [3], the following theorem.

THEOREM 2.2. *Under the hypotheses of Theorem 2.1 the problem* (1.1)–(1.3) *has a unique solution u.*

If U_h is the solution of (2.1)–(2.3) then:

(i) $(\varphi(U_h))' \to \varphi(u)$ in $L^p(Q)$ $p \in [1, +\infty[$

(ii) $(\varphi(U_h)x_i)' \to \partial\varphi(u)/\partial x_i$, $i = 1, 2$ weakly in $L^2(Q)$

(iv) $U_h' \to u$ in $L^p(Q)$, $p \in [1, +\infty[$.

Here $U_h' \in C(Q_h)$ is the multilinear (finite-element) interpolate of the discrete function U_h.

3. Time independent boundary conditions. The main result of this section is given in the following lemma.

LEMMA 3.1. *Suppose that:*

(i) *Assumption (A) is valid*

(ii) $u_0 \in W_1^2(\Omega)$

(iii) $\lambda \leqslant 1$

(iv) u_1 *is constant with regard to t.*

Then, there exists a constant C independent of h such that

$$h^2 \sum_{Q_h} |U_{\bar{t}}| < C. \tag{3.1}$$

Proof: (3.1) is valid if there exists a constant C_0 such that

$$h^2 \sum_{\Omega_h} |U_{\bar{t}}(k)| < C_0, \quad k = 1, 2, \ldots, K.$$

For $k = 1$ this means that

$$h^2 \sum_{\Omega_h} |\Delta_h\varphi(u_0)| < C_0,$$

which is true according to (3.1) and assumption (A). Further we have:

$$U_{\bar{t}}(k-1) = \Delta_h(\varphi(U(k-1)) - \varphi(U(k-2))),$$

so that

$$\overline{U}_{ij}(k) = \left(1 - 4\frac{\tau}{h^2}\tilde{\varphi}'_{ij}(k-1)\right)\overline{U}_{ij}(k-1) +$$

$$+ \frac{\tau}{h^2}[\tilde{\varphi}'_{i+1,j}(k-1)\overline{U}_{i+1,j}(k-1) + \tilde{\varphi}'_{i-1,j}(k-1)\overline{U}_{i-1,j}(k-1) +$$

$$+ \tilde{\varphi}_{i,j+1}(k-1)\overline{U}_{i,j+1}(k-1) + \tilde{\varphi}'_{i,j-1}\overline{U}_{i,j-1}(k-1)].$$

As before the sign $\sim$ indicates an apropriate intermediary value. Hence taking into account condition (iv) we get for any integer $k > 1$:

$$\sum_{\Omega_h} |U_{ij}(k)| \leqslant \sum_{\Omega_h} |U_{ij}(k-1)|,$$

which completes our proof.

Finally we notice that The assertion of Theorem 2.2. remains valid under the conditions of Lemma 3.1.

REMARK. It is readily seen from the proof that condition (ii) can be replaced by $\varphi(u_0) \in W_1^2(\Omega)$.

REFERENCES

[1] Schechter, E., *The Convergence of the Explicit Difference approximation to Solve the Cauchy-Dirichlet Problem for Non-linear Parabolic Equations.* Mathematica, vol. **21** 67—81 (1979)

[2] Schechter, E., *On the Numerical Solution in a Nonlinear Parabolic Equation.* Rev. d'Analyse Num. et de la Théorie de l'Approximation vol. **8** 89—97 (1979).

[3] Schechter, E., *Remarks on the Numerical Solution of a Nonlinear Parabolic Equation.* Studia Univ. Babeş-Bolyai, vol **XXV** no. 3, 32—39 (1980).

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