L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION Tome 10, N° 2, 1981, pp. 225-232

 $\left\{ \left(u \stackrel{\partial L}{\partial t} \right) = \sum_{i=1}^{n-1} \frac{\partial u(u)}{\partial u_i} \frac{\partial L}{\partial u_i} \left(\frac{\partial u}{\partial u} \right) \left((0,0) du + \frac{1}{2} \int_{\mathbb{R}^n} g(u,t) f(u,t) du \right) du \right\} = 0$

Unfortunitely in the peach of Lemma 2.2 in 12 (an consequently in that ot Lourni S. I in (3), too) their is a miglake (Sewithdow, the lournes) the Journal of the main

SOME PROPERTIES OF A NONLINEAR PARABOLIC -uklasi a sen aw and DIFFERENCE SCHEME and someofice 2

gular algebraith the step k in the Ox_i , i = 1,2 directions and z in the

ilquis 104 O ERVIN SCHECHTER 19 () O to united-them of T not bus string-from to stass of (Cluj-Napoca) for many officers that you wife

A Law was donated the continue of the same 1. Introduction. In [1], [2], [3] we considered the nonlinear degene-Consider the following idifference problems: 11 meldorq arrangement

(1.1)
$$\frac{\partial \mathbf{u}}{\partial t} = \Delta \varphi(\mathbf{u}) + a(\mathbf{x}, t) \text{ on } Q = \Omega \times]0, T[$$

$$(1.2) u(x, 0) = u_0(x) x \in \Omega$$

(1.2)
$$u(x, 0) = u_0(x) x \in \Omega$$

(1.3) $u(x, t) = u_1(x, t) \text{ on } S = \partial \Omega \times]0, T[,$

under the assumption or the assumption (A)(b) = (A)(b) + (

(i)
$$u_0 \in C(\overline{\Omega}), \quad u_1 \in C(S), \quad a \in C(\overline{Q})$$

(2.0)
$$M = \{ X : (L, L, u_0, u_1, a \ge 0) \mid L, (x, y) \mid x = L, (H) \}$$

Suppose that MyNaresements subhithat wit sign & Manares vet it

(ii)
$$\varphi \in C^2(\mathbf{R}_+), \ \varphi(u), \ \varphi'(u) > 0, \ \text{for} \ u > 0$$

$$\varphi(0) = \varphi'(0) = 0, \quad \varphi''(u) \ge 0.$$

 $\Omega \subset \mathbb{R}^2$ is a regular, bounded, convex, domain. Under certain conditions on the data we proved, using an explicit difference scheme, that (1.1) -(1-3) has a unique weak solution in the following sense:

(i)
$$u \in L^{\infty}(Q), \ u \geqslant 0, \quad \partial \varphi(u)/\partial x_i \in L^2(Q)$$

(ii) (1.2), (1.3) are fulfilled in the generalized sense

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ET DE TREMETE DE L'APTROXIMATION (iii) For any $f \in H^1(Q)$ such that $f|_{S_1} = 0$, Tome 10, Nº 2, 1981, pp. 225-222

$$\int_{0}^{\infty} \left(u \frac{\partial f}{\partial t} - \sum_{i=1}^{2} \frac{\partial \varphi(u)}{\partial x_{i}} \frac{\partial f}{\partial x_{i}} \right) dx dt + \int_{\Omega} u_{0}(x) f(x, 0) dx + \int_{Q} a(x, t) f(x, t) dx dt = 0$$

Here
$$S_1 = S \cup \{(x, T) | x \in \Omega\}.$$

Unfortunately in the proof of Lemma 2.2 in [2] (an consequently in that of Lemma 3.1 in [3], too) there is a mistake. Nevertheless, the lemmas themselves are correct as it is shown in §2, Theorem 2.1. Our main result is given in §3 where the conditions on the initial function is less restrictive as in the above mentioned papers.

2. Difference inequalities. As in our previous papers we use a rectangular mesh with the step h in the Ox_i , i=1,2 directions and τ in the

The mesh-points of Ω , Q etc., will be denoted by Ω_h , Q_h , etc.. For simplicity we shall use the same notation for the sets of mesh-points and for the rectangular domains determined by them. We put $\Gamma_h = \partial \Omega_h$ and Δ_h for the classical five-point discretization of Δ . rare problem ! Consider the following difference problems:

$$(2.1) U_i(k) = \Delta_k \varphi(U(k-1)) + a(k) \quad \text{on} \quad Q_k$$

$$(2.2) U(0) = u_{oh} on \Omega_h C(3.1)$$

(2.2)
$$U|_{\Gamma_h} = u_2(x, k\tau) \qquad x \in \Gamma_h, k = 0, \dots, K = \left[\frac{T}{\tau}\right];$$

and

and
$$(2.4) V_{\tilde{i}}(k) = \Delta_{h} \varphi(V(k-1)) + b(k) \text{on } Q_{h}$$

$$(2.5) V(0) = V_{oh} \text{ on } \Omega_h \Omega$$

(2.6)
$$V|_{\Gamma_h} = v_2(x, k\tau), \quad x \in \Gamma_h, k = 0, 1, ..., K = \left[\frac{T}{\tau}\right].$$

Suppose that M,N are constants such that $u_0, u_2, a \leq M$;

 $v_0, v_2, b \leq N$; $M_0 = (1 + T)M$, $N_0 = (1 + T)N$, $M_0 < N_0$. Denote:

$$\lambda=4rac{ au}{h^2}\; arphi'(M_0), \qquad \mu=4rac{ au}{h^2}\; arphi'(N_0).$$

1.EMMA 2.1. ([1], [2]) Suppose that U is the solution of problem (2.4)—(2.6). Assume that (A) is valid.

Then. if
$$\lambda \leq 1$$
: $(Q)^{*}J = \chi G(n) \circ G = 0 \leq n$, $(Q)^{*}J \Rightarrow u$

$$0\leqslant U\leqslant M_0\quad\text{for all }(x,t)\leqslant Q_k. \quad \text{(2.1)}\quad \text{(2.1)}$$

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LEMMA 2.2. Suppose that: The lifty Boulden and The an Impose off all

- (i) Assumption (A) is valid
- (ii) μ ≤ 1
- (iii) $u_{oh} \leq v_{oh}$, $u_2 \leq v_2$, $a \leq b$, on their domains of definition.
- (iv) U satisfies (2.1)-(2.3); V satisfies (2.4)-(2.6). with Then, with the the secretary days to the temperature (181 11) 1983 Themen

$$(2.7) U \leqslant V on Q_{h}$$

Proof: Denote $W_{ij}(k) = V_{ij}(k) - U_{ij}(k)$. Assume that (2.7) is not true. Then there exists a triplet of indices (m, n, l) such that $W_{mn}(l) < 0$ for an $l \ge 1$ and

$$W_{ij}(k) \ge 0$$
 for any i, j and $k < l$.

Put

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Put
$$\overline{W}_{ij}(k) = rac{1}{ au} \left(W_{ij}(k) - W_{ij}(k-1)
ight).$$

 $\overline{V}_{ij}(k)$ and $\overline{U}_{ij}(k)$ will stand for similar differences of V respectively U. Then.

$$(2.8) W_{mn}(l-1) = \Delta_h(\varphi(V_{mn}(l-1) - \varphi(U_{mn}(l-1))).$$

Because

$$\varphi(V_{ij}(l-1)) - \varphi(U_{ij}(l-1)) > \varphi(V_{mn}(l)) - \varphi(U_{mn}(l)),$$

for any i, j; it follows that:

$$\overline{W}_{mn}(l-1) \geqslant \frac{4}{h^2} \left[\varphi(V_{mn}(l)) - \varphi(U_{mn}(l)) - (\varphi(V_{mn}(l-1)) - \varphi(V_{mn}(l)) \right]$$

$$-\varphi(U_{mn}(l-1)))]=4\frac{\tau}{h^2}\left[\widetilde{\varphi'}(U_{mn}(l-1)\overline{W}_{mn}(l)-\widetilde{\varphi'}(U_{mn}(l))\overline{W}_{mn}(l-1))\right].$$

(2.9)
$$W_{mn}(l) \left(1 - 4 \frac{\tau}{h^2} \widetilde{\varphi}(U_{mn}(l))\right) \leq W_{mn}(l-1) \left(1 - \frac{4}{h^2} \widetilde{\varphi}(U_{mn}(l-1))\right),$$

which by Lemma 2.1 and (ii) contradicts our assumption.

COROLLARY 2.1. If in addition to the conditions of Lemma 2.1, V is also nondecreasing in k, then instead of (ii) we can assume, $\lambda \leq 1$, only. Indeed, if as in the proof of the lemma, we admit that $W_{mn} < 0$: $V_{mn}(l-1) \leqslant V_{mn}(l) \leqslant U_{mn}(l) \leqslant M_0$

$$V_{mn}(l-1) \leqslant V_{mn}(l) \leqslant U_{mn}(l) \leqslant M_0$$

which entail that in (2.9)

$$4\frac{\tau}{h^2}\varphi'(U_{mn}(l-1))$$
 as well as $4\frac{\tau}{h^2}\widetilde{\varphi}'(U_{mn}(l))$,

are less than λ . $\langle (A) V \rangle \psi = \langle (A) V \rangle \psi$ for $\langle (A) V \rangle \psi = \langle (A) V \rangle \psi$

In the sequel u_2 will be defined with the aid of u_1 as follows:

$$u_2(x,t) = u(x^*,t) \quad x \in \Gamma_h, \quad x^* \in \partial \Omega_h$$

where x^* is the nearest point to x. In the same time u_{oh} will stand for the restriction to Ω_h of u_0 .

LEMMA 2.3. ([1], [3]) Suppose that conditions of the previous lemma hold and that $\Delta \varphi(u_0) \geq 0$. Then Then the second of the state of the sta

(i)
$$U_{\overline{l}}(k) \geqslant 0$$
, $k = 1, 2, ..., K$ on $\overline{\Omega}_h$
(ii) $\tau h^2 \sum_{\Omega_i} U_{\overline{l}}(k) \leqslant M_0 m(\Omega)$,

(ii)
$$\tau h^2 \sum_{Q_h} U_{\tilde{t}}(h) \leq M_0 m(\Omega)$$

for h sufficiently small.

THEOREM 2.1. Suppose that:

- (i) Assumption (A) holds and $u_0 \in C^2(\overline{\Omega})$
- (ii) $\lambda \leq 1$ (iii) $\partial u_1/\partial t$ exists and is bounded on S
- (iv) $\varphi(\infty) = \infty$

Then there exists a constant C > 0 such that

$$| \tau h^2 \sum_{Q_h} | \varphi(U(k))_{ar{t}} | \leqslant C,$$

Proof: Let $\varepsilon > 0$ be a constant and $V_0 \in C^2(\overline{\Omega})$ such that:

$$-(\Box - \Box \Delta V_0 \geqslant |\Delta \varphi(u_0)| + \varepsilon \text{ on } \Omega \leq (\Box - \Box)$$

$$(2.10) V_0|_{\Gamma} = \varphi(u_0)|_{\Gamma} \Gamma = \partial\Omega, V_0 > 0.$$

Define v_0 by $v_0 = \varphi^{-1}(V_0)$ and the function $v_1 \in C(S)$ as follows:

$$v_1 \ge 0$$
, $\partial v_1/\partial t$ exists on S

$$v_1 \geqslant 0$$
, $\partial v_1/\partial t$ exists on S
$$\frac{\partial v_1}{\partial t}(x,t) \geqslant c + \varepsilon, \quad t \in \]0, \ T[, \quad v_1(x,0) = v_0|_{\Gamma}$$

where $c \ge \max |\partial u_1/\partial t|$ is some and the modified in the standard section where

With the aid of v_0 , v_1 as data functions and b = a, we construct using the scheme (2.4)-(2.6) the discrete solution $V:Q_h\to \mathbf{R}$. According to Lemma 2.3 we have $V_{\bar{i}} \ge 0$ and

$$\tau h^2 \sum_{Q_k} V_{\bar{i}}(k) \leqslant C \quad k = 1, 2, \ldots, K.$$

For convenience we set $\bar{U}(k) = U_i(k)$ and $\bar{V}(k) = V_i(k)$,

$$\overline{\varphi}(U(k)) = \varphi(U(k))_{\tilde{i}}$$
 and $\varphi(V(k)) = \overline{\varphi}(V(k))_{\tilde{i}}$. The first end one

First we observe that for h small enough: is A ((1.2) araibungo nough

$$\Delta_{h} \varphi(v_{0}) > |\Delta_{h} \varphi(u_{0})|$$

so that
$$|U_{ij}(1)| \leqslant V_{ij}(1) \quad \text{on } \Omega_h.$$

(2.11) $|U_{ij}(1)| \leq V_{ij}(1)$ on Ω_h . Because of the properties of φ and taking (2.11) into account, we get:

$$(2.12) \qquad |\overline{\varphi}(U_{ij}(1))| \leq \overline{\varphi}(\overline{V}_{ij}(1)) \quad \text{on } \Omega_h.$$

Indeed, by Corollary 2.1 we have either Then, there exists a constant C indefreadent of h, such that for any Ω^*

$$U_{ij}(1) \leqslant U_{ij}(0) \leqslant V_{ij}(0) \leqslant V_{ij}(1)$$

or
$$U_{ij}(0) \leqslant U_{ij}(1) \leqslant V_{ij}(0) \leqslant V_{ij}(1),$$

or
$$U_{ij}(0) \leqslant V_{ij}(0) \leqslant V_{ij}(1) \leqslant V_{ij}(1).$$

For $k \ge 2$ we obtain:

Now, we want to prove that for any $k \ge 2$

$$\overline{\varphi}(U_{ij}(k)) \leqslant \overline{\varphi}(V_{ij}(k)) \quad \text{on} \quad \Omega_h.$$

Suppose that this is not true and let l be the greatest subscript such that for any i, j (for l = 2 this is true by (2.11)):

$$\overline{\varphi}(U_{ij}(l-1)) \leqslant \overline{\varphi}(V_{ij}(l-1))$$
 (so on $\Omega_{f A}$

and there is a couple of subscripts m, n such that:

$$(2.14) \qquad \overline{\varphi}(U_{mn}(l)) > \overline{\varphi}(V_{mn}(l))$$

(2.14) $\overline{\varphi}(U_{mn}(l)) > \overline{\varphi}(V_{mn}(l)).$ Since $u_0 \leqslant v_0$, $u_1 \leqslant v_1$, by Corollary 2.1, $U_{mn}(l-1)V_{mn}(l-1)$, $U_{mn}(l)V_{mn}(l)$. Then, in view of (2.14) we must have

(2.15)
$$U_{mn}(l) > V_{mn}(l)$$
.

and similar inequalities when (m + 1, n) is replaced successively by (m - 1, n)(m, n + 1), (m, n - 1). From (2.14), (2.15), (2.16):

$$U_{mn}(l) - \bar{V}_{mn}(l) \leq U_{mn}(l-1) - V_{mn}(l-1) +$$

$$+4\frac{\tau}{h^2}\left(\varphi(U_{mn}(l-1))-\varphi(V_{mn}(l-1))+\varphi(U_{mn}(l-2))-\varphi(U_{mn}(l-2))\right)<0,$$

which contradicts (2.14). A similar argument holds for $-\overline{\varphi}(U_{ii}(k))$, so that:

$$|\overline{\varphi}(U_{ij}(k))| \leq \overline{\varphi}(V_{ij}(k)).$$

The conclusion of the theorem now follows from Lemma 2.3. The previous theorem shows the boundedness of the difference in t of $\varphi(U)$, only. Nevertheless Lemma 2.4 of [2] as well as Theorem 3.2 of [3] remain true as it is shown by the following lemma.

LEMMA 2.4. Suppose that U is the solution of (2.1) - (2.3) and that conditions of Theorem 2.1 hold.

Then, there exists a constant C independent of h, such that for any Ω^* $U_{n}(1) \leq U_{n}(0) \leq V_{n}(0) \leq V_{n}(0)$ with $\overline{\Omega}^* \subset \Omega$,

$$abla h^2 \sum_{k=1}^K \sum_{\Omega_k^*} (\varphi^2(U(k))_{x_1} + \varphi^2(U(k))_{x_2}) < C$$

For L 2 2 we obtain:

provided that $h < h_0(\Omega_k^*)$.

Proof: The argument is mainly the same as in [1]. The only difference is that instead of an estimate for U_t in the discrete L^1 norm we have one for $\varphi(U)_{l}$. So we shall have to transform the scalar product:

$$au h^2 \sum_{k=1}^K \sum_{\Omega_k^*} U_i(k) \varphi(U(k))$$

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into
$$-\tau h^2 \sum_{k=0}^{K-1} \sum_{\Omega_h^*} U(k) \varphi(U(k))_t + \sum_{\Omega_h^*} (U(0) \varphi(U(0)) - U(K) \varphi(U(k))).$$

This is bounded since U is bounded on Ω_h^* .

On the basis of the results of this section we can prove in the same way as in [1], [2], [3], the following theorem. Then, in view of (2.14) we must be

THEOREM 2.2. Under the hypotheses of Theorem 2.1 the problem (1.1)— (1.3) has a unique solution u. On the value hand :

If U_h is the solution of (2.1)-(2.3) then:

- (i) $(\varphi(U_h))' \to \varphi(u)$ in $L^p(Q)$ $p \in [1, +\infty[$
- (ii) $(\varphi(U_k)x_i)' \rightarrow \partial \varphi(u)/\partial x_i$, i = 1, 2 weakly in $L^2(Q)$
- (iv) $U_h' \to u$ in $L^p(Q)$, $p \in [1, +\infty[. u])$ and we sold impose tall this

Here $U_h' \in C(Q_h)$ is the multilinear (finite-element) interpolate of the discrete function U_h .

3. Time independent boundary conditions. The main result of this section is given in the following lemma.

LEMMA 3.1. Suppose that:

- (i) Assumption (A) is valid (ii) $u_0 \in W_1^2(\Omega)$ (iii) $\lambda \leq 1$ (iv) u_1 is constant with regard to t.

Then, there exists a constant C independent of h such that

$$(3.1) h^2 \sum_{\mathcal{O}_{\boldsymbol{h}}} |U_{\boldsymbol{i}}| < C.$$

Proof: (3.1) is valid if there exists a constant C_0 such that

$$h^2 \sum_{\Omega_k} |U_t(k)| < C_0, \qquad k = 1, 2, \ldots, K.$$

For k = 1 this means that

$$h^2 \sum_{\Omega_h} |\Delta_h \varphi(u_0)| < C_0,$$

which is true according to (3.1) and assumption (A). Further we have:

$$U_{\overline{i}}(k-1) = \Delta_{\overline{k}}(\varphi(U(k-1)) - \varphi(U(k-2))),$$

so that

$$\begin{split} \bar{U}_{ij}(k) &= \left(1 - 4 \frac{\tau}{h^2} \, \tilde{\varphi}'_{ij}(k-1) \right) \bar{U}_{ij}(k-1) \, + \\ &+ \frac{\tau}{h^2} \, \left[\tilde{\varphi}'_{i+1,j}(k-1) \bar{U}_{i+1,j}(k-1) + \tilde{\varphi}'_{i-1,j}(k-1) \bar{U}_{i-1,j}(k-1) + \right. \\ &+ \tilde{\varphi}_{i,j+1}(k-1) \bar{U}_{i,j+1}(k-1) + \tilde{\varphi}'_{i,j-1} \bar{U}_{i,j-1}(k-1) \right]. \end{split}$$

As before the sign ~ indicates an apropriate intermediary value. Hence taking into account condition (iv) we get for any integer k > 1:

$$\sum_{\Omega_h} |U_{ij}(k)| \leqslant \sum_{\Omega_h} |U_{ij}(k-1)|,$$

which completes our proof.

Finally we notice that The assertion of Theorem 2.2. remains valid under the conditions of Lemma 3.1.

REMARK. It is readily seen from the proof that condition (ii) can be replaced by $\varphi(u_0) \in W_1^2(\Omega)$.

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Facultatea de Matematică a Universității Babeș-Bolyai str. Kogălniceanu Nr. **1** Cluj-Napoca

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For k = 1 this means that

Prior - The argument is analyty the same of in [1]. The only difference is that undearly is an existing the result of the liberate L' north we have the top q(I). So we foul the control of private and the number private :

which is true according to (3-1), and assumption (4). Further we have:

$$U_i(k-1) = \Delta_k(\varphi(U(k-1)) - \varphi(U(k-2))),$$

into v

teril or

$$-\frac{1}{2}\sum_{i=1}^{n}\left\{ \log \left(\mathcal{Q}_{i}^{i}\right) \left(1+\sum_{i=1}^{n}\left(\mathcal{Q}_{i}^{i}\right) \left(1+\sum_{i=1}^{n}\left(\mathcal{Q}_{i}^{i}\right) \left(1+\sum_{i=1}^{n}\left(1+\sum_{i=1}^{$$

The property of the results of the rection $\int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}}$

 $= i\pi (1 - 1)U_{i,1}(1 - 1)U_{i,2}(1 - 1)U_{i,1}(1 - 1)U_{i,2}(1 - 1)U_{i,2}(1 - 1)U_{i,1}(1 - 1)U_{i,2}(1 - 1)U_$

As before the argu \sim indicates an apropriate infarmedisty value of $V_{\rm eff}$. Hence taking into account candition (iv) we get for any integer $V_{\rm eff}$.

$$\lim_{N \to \infty} \frac{(\psi(U_n))' - \psi(\pi) d(I(m, y)) M(\overline{U_n}) \otimes \psi(y) dy}{(\psi(U_n))' - \psi(\pi) d(I(m, y)) M(\overline{U_n})} \ge \psi(y) dy$$

which completes our proof. $| \cos \phi_{-} | = 0$ and $| \cos \phi_{-} | = 0$ by the proof of the conditions of Lemma 3.1. The conditions of Lemma 3.1. The condition of the matrix as the condition of the restly as a subject of the proof that condition (ii) can be replaced by $\varphi(w_0) \in W^2_{\varphi}(\Omega)$ and a given large of the replaced by $\varphi(w_0) \in W^2_{\varphi}(\Omega)$.