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THE BOUNDARY ELEMENT METHOD – A REVIEW.

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Abstract

This report presents the Boundary Element Method applied to the solution of Laplace's equation and the problems of plane elastostatics. The characteristics of the method are presented together with an indication of how it can be applied in a wide variety of problems for which many are referenced.

1. Introduction

Many problems in mathematical physics and engineering can be formulated as integral equations, although not all have been traditionally posed in this way. It is usual to seek an integral equation in which the integrals are taken over the boundary of the region. The boundary is discretised as in the finite element method, this enabling the integral equation to be approximated by a set of algebraic equations which can be solved numerically. Such, in essence, is the technique of solution known as the Boundary Element Method.

The transformation to an integral equation is usually from a differential equation and is effected by the use of an appropriate reciprocal identity (e.g. Green's identity for Laplace's equation) together with the use of a singularity function. Such a procedure has properties that make it the equivalent of satisfying the governing equation throughout the region of applicability and determining some approximation to the boundary conditions. This is in contrast to many approximate techniques, including some applications of the finite element method, which approximate the governing equations but satisfy exactly the boundary conditions. There

are alternative ways of determining the transformation to an integral equation (e.g. Weighted Residual Methods), but in each case it is the boundary conditions which are approximated.

Because a singularity solution is used in the transformation procedure, the resulting equations are singular integral equations. Provided adequate care is taken not too much difficulty is encountered in the numerical solution of these integral equations. The real advantage, however, of the Boundary Element Method, is that the dimension of the original problem is reduced by one as compared with, say, the Finite Element Method because the integral equation is restricted to the boundary. By way of disadvantage, the determination of unknowns at interior points requires further integrations and where one would expect sparsity in matrices in other methods it is the case that in the Boundary Element Method the matrices (although smaller) are dense. It is also very critical to be aware of singularities on the boundary which could be caused by discontinuity of the given data or irregularity of the boundary.

Exploitation of these techniques was made by Jaswon and Ponter (1), Jaswon (2) and Symm (3) for applications governed by Laplace's equation —applications using reciprocal work theorems or their equivalent in elasticity were developed by Rizzo (4), Cruse and Rizzo (5) and Cruse (6, 7) Subsequent to these papers there have been many publications together with applications in diverse fields (see e.g. 8, 9, 10).

The objective of this paper is to present the Boundary Element Method in as straightforward manner as possible. Indications of how the method has been applied to a variety of situations are given. A detailed account of the method, approached by weighted residual techniques, is given in the book by Brebbia (11).

2. Laplace's Equation

By way of detailed exposition, the technique of solution will be applied to Laplace's equation. Consider

$$\nabla^2 u = 0 \qquad u \in \Omega, \tag{2.1}$$

where Ω is the interior of a domain bounded by Γ. Suppose that

$$\left.\begin{aligned} u &= \bar{u}, \qquad u \in \Gamma_1 \\ \frac{\partial u}{\partial n} &= \bar{g}, \qquad u \in \Gamma_2 \end{aligned}\right\} \quad \text{and} \tag{2.2}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ $(\Gamma_1 \cap \Gamma_2 = \emptyset)$ and n is the direction of the outward normal to Γ.

Green's theorem states that

$$\int_\Omega (u\nabla^2 v - v\nabla^2 u)\mathrm{d}\Omega = \int_\Gamma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\Gamma. \tag{2.3}$$

Suppose that v satisfies the equation

$$\nabla^2 v = -\delta^\alpha \tag{2.4}$$

where $\delta^\alpha = \delta(r^\alpha - r)$ is the Dirac delta function and $r^\alpha \in \Omega$. Call the solution of (2.4) v^α, and substituting v^α for v in (2.3) gives

$$\int_\Omega (-u\delta^\alpha - v^\alpha \nabla^2 u)d\Omega = \int_\Gamma \left(u\frac{\partial v^\alpha}{\partial n} - v^\alpha \frac{\partial u}{\partial n}\right) d\Gamma. \tag{2.5}$$

If u satisfies (2.1), then (2.5) becomes

$$-u^\alpha = \int_\Gamma \left(u\frac{\partial v^\alpha}{\partial n} - v^\alpha \frac{\partial u}{\partial n}\right) d\Gamma. \tag{2.6}$$

This equation enables the value of u at any point of Ω to be determined in terms of the values of u and v^α (and their normal derivatives) at the boundary Γ.

If, in equation (2.5), δ^α is chosen so that

$$\delta^\alpha = \delta(r^i - r) = \delta^i, \quad r^i \in \Gamma \tag{2.7}$$

then, provided the boundary is smooth, it can easily be shown that

$$-\frac{1}{2}u^i = \int_\Gamma \left(u\frac{\partial v^i}{\partial n} - v^i \frac{\partial u}{\partial n}\right) d\Gamma, \tag{2.8}$$

which is an integral equation relating values of u on the boundary Γ only. From (2.2) the value of u or $\frac{\partial u}{\partial n}$ is known at each point on Γ, and thus the equation determines the value of u or $\frac{\partial u}{\partial n}$, whichever is unknown at that point. Thus the solution to (2.1) and (2.2) will be obtained by solving the integral equation (2.8) and then using (2.6) to determine values of u in Ω.

For ease of illustration, Ω is restricted to a two dimensional region and Γ is a closed contour. In this case the solution of (2.4) is

$$v = -\frac{1}{2\pi}\log r \tag{2.9}$$

where $r = |r - r^\alpha|$. Equation (2.8) can be thus rearranged as

$$\frac{1}{2}u^i + \int_\Gamma u\frac{\partial}{\partial n}\left(-\frac{1}{2\pi}\log r\right) d\Gamma = -\int_\Gamma \frac{1}{2\pi}\frac{\partial u}{\partial n}\log r \, d\Gamma \tag{2.10}$$

and $r = |r - r^i|$. Note that (2.10) is a singular integral equation, and that where singularities occur the principal value is taken. In order to solve (2.10), the boundary Γ is divided into discrete elements E_i $(i = 1 \ldots N)$ and the simplest such arrangement is shown in figure 1.

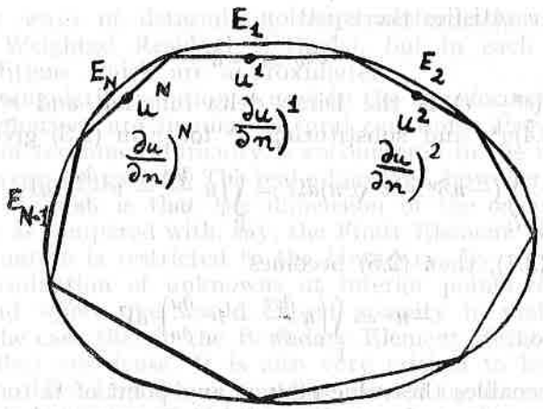

Figure 1

The values of u and $\frac{\partial u}{\partial n}$ are regarded as constant throughout the element E_i, and are those taken at the mid-point of the element.

Equation (2.10) is approximated by taking $\Gamma = \bigcup_{i=1}^{N} E_i$ and using the constant mid-point values in each E_i. Thus equation (2.10) becomes

$$\frac{1}{2}u^i + \sum_{j=1}^{N} u^j \int_{E_j} \frac{\partial}{\partial n}\left(-\frac{1}{2\pi}\log r_i\right)d\Gamma = \sum_{j=1}^{N} \frac{\partial u}{\partial n}\Big)^j \int_{E_j} -\frac{1}{2\pi}\log r_i d\Gamma \quad (i = 1 \ldots N) \tag{2.11}$$

where r_i represents the distance from the mid-point of Element E_i, and (2.11) can be written as

$$\sum_{j=1}^{N} X_{ij}u^j = \sum_{j=1}^{N} Y_{ij}\frac{\partial u}{\partial n}\Big)^j \qquad (i = 1, \ldots, N) \tag{2.12}$$

where

$$\left.\begin{aligned} X_{ij} &= \frac{1}{2}\delta_{ij} + \int_{E_j} \frac{\partial}{\partial n}\left(-\frac{1}{2\pi}\log r_i\right)d\Gamma, \\ Y_{ij} &= -\int_{E_j} \frac{1}{2\pi}\log r_i d\Gamma. \end{aligned}\right\} \tag{2.13}$$

Orthogonality of r and n demands that $\int_{E_i} \frac{\partial}{\partial n}\left(-\frac{1}{2\pi}\log r_i\right)d\Gamma = 0$ and thus

$$X_{ii} = \frac{1}{2}, \quad (i = 1 \ldots N) \tag{2.14}$$

$Y_{ii} = -\int_{E_i} \frac{1}{2\pi}\log r_i d\Gamma$, and assuming the length of element E_i to be $2l$,

$$Y_{ii} = -\frac{1}{\pi}\int_0^{l_1} \log r\, dr \tag{2.15}$$

$$= \frac{l_1}{\pi}(1 - \log l_1). \tag{2.16}$$

In this particular formulation the singularity in the determination of X_{ii} vanishes and is non-existent in Y_{ii}. The remaining integrals can be determined numerically, hence producing the matrices X and Y.

Returning to (2.12) — at any value of j either u^j or $\frac{\partial u}{\partial n}\Big)^j$ are known from (2.2) (but not both) and so (2.12) gives a set of N linear simultaneous equations in N unknowns. In solving them, the values of u^i and $\frac{\partial u}{\partial n}\Big)^i$ are known for all i, which means that (2.6) is approximated to give the following result for values of $u \in \Omega$

$$u^\alpha = \sum_{i=1}^{N} \int_{E_i} \left\{u^i \frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log r\right) - \frac{1}{2\pi}\log r \frac{\partial u}{\partial n}\Big)^i\right\} d\Gamma \tag{2.17}$$

where r is the distance from r^α, the internal point in Ω. Noting that u^i and $\frac{\partial u}{\partial n}\Big)^i$ are constant within $\tilde{E}_i$, integrals of the same form as (2.13) are required to evaluate u. All these integrals are probably best calculated using Gaussian Quadrature, and a useful resource for such formulae is the text by Stroud and Secrest (12).

Although the integral equation (2.8) is written in notation convenient for the particular approximate technique above it is worth noting that it could have been written as

$$\frac{1}{2}u(t) = \int_\Gamma \left(u(s)\frac{\partial}{\partial n}v(s, t) - v(s, t)\frac{\partial u(s)}{\partial n}\right) dS \tag{2.18}$$

where s measures arc length along Γ as also does t. The solution above therefore represents a collocation technique using the midpoints of the elements E_i as collocation points. An alternative method could have chosen the N collocation points to be the extreme points of the elements,

although care would have to be taken in dealing with integrals on the two adjacent elements which define any particular node. Other solutions could be constructed using the various approaches for the numerical solution of integral equations (see (13)).

Interpolation errors occur on the boundary as a result of the fact that u and $\frac{\partial u}{\partial n}$ are constant over each element, and such errors will occur whatever discretisation is used for the boundary values of u and its normal derivative. The above solution has approximated the boundary by straight line segments, and should the boundary be a smooth curve then errors will inevitably result. In the above case, these errors could have been eliminated by integrating over the true boundary and this is always an option. In addition to these errors in the formulation of the discretised problem, the quadrature and solution of the linear equations produce further potential error.

Singularities due to the discontinuity of given boundary values in terms of value or type are a particular hazard, as also is non-uniqueness particularly when $\frac{\partial u}{\partial n}$ is given everywhere on the boundary. Singularities can be handled by introducing particular solutions to take account of their local behaviour, see Wendland (14). Lack of care about non-uniqueness, which can often be dealt with in a simple way, will usually produce nonsensical and highly inaccurate solutions.

3. Further Refinements

The discretisation of the boundary values can easily be seen to be the type of discretisation used in the finite element method in which the next natural step is to consider more complex elements. So consider N straight line elements as before but let there be a linear variation of u and $\frac{\partial u}{\partial n}$ within the element. The unknowns now become the values of u and $\frac{\partial u}{\partial n}$ at each end of the element, and in element E_i the extreme points are $\underset{\sim}{r}^i$ and $\underset{\sim}{r}^{i+1}$ at which the u and $\frac{\partial u}{\partial n}$ take values u^i, $\left(\frac{\partial u}{\partial n}\right)^i$ and u^{i+1}, $\left(\frac{\partial u}{\partial n}\right)^{i+1}$ respectively. Because the boundary is closed $\underset{\sim}{r}^1$ and $\underset{\sim}{r}^{N+1}$ represent the same node and so $N+1$ is replaced by 1.

It is convenient to use local coordinates for the element E_i, as in figure 2.

Figure 2 Element E_i

Within the element

$$\left.\begin{aligned} u(\xi) &= (\varphi_1^i,\ \varphi_2^i)\begin{pmatrix} u^i \\ u^{i+1}\end{pmatrix} \\ \frac{\partial u(\xi)}{\partial n} &= (\varphi_1^i,\ \varphi_2^i)\begin{pmatrix} \left(\frac{\partial u}{\partial n}\right)^i \\ \left(\frac{\partial u}{\partial n}\right)^{i+1}\end{pmatrix}\end{aligned}\right\} \tag{3.1}$$

where

$$\left.\begin{aligned} \varphi_1^i &= \frac{1}{2l_i}(l_i - \xi) \\ \varphi_2^i &= \frac{1}{2l_i}(l_i + \xi)\end{aligned}\right\} \tag{3.2}$$

Equation (2.10) is now approximated by

$$\frac{1}{2}u^i + \sum_{j=1}^{N}\int_{E_j}(\varphi_1^j,\ \varphi_2^j)\frac{\partial}{\partial n}\left(-\frac{1}{2\pi}\log r_i\right)\begin{pmatrix} u^i \\ u^{i+1}\end{pmatrix}d\Gamma$$

$$= \sum_{j=1}^{N}\int_{E_j}(\varphi_1^j,\ \varphi_2^j)\left(-\frac{1}{2\pi}\log r_i\right)\begin{pmatrix} \left(\frac{\partial u}{\partial n}\right)^i \\ \left(\frac{\partial u}{\partial n}\right)^{i+1}\end{pmatrix}d\Gamma, \qquad (i = 1, \ldots N) \tag{3.3}$$

where r_i is the distance from $\underset{\sim}{r}^i$. The integrals which have to be evaluated are

$$\int_{E_j}(\varphi_1^j,\ \varphi_2^j)\frac{\partial}{\partial n}\left(-\frac{1}{2\pi}\log r_i\right)d\Gamma$$

and

$$\int_{E_j}(\varphi_1^j,\ \varphi_2^j)\left(-\frac{1}{2\pi}\log r_i\right)d\Gamma.$$

It is worth looking to see what happens if $i = j$. The first integral will vanish because the normal derivative is zero, and the second integral can be evaluated analytically and has no singularity at all. It is clear that whilst straight elements are used, however complex, the normal derivative term will always vanish and the second integral will be easily determined analytically. However, when the elements are curved the normal derivative will not necessarily vanish.

The simultaneous equations are assembled from (3.3) as before with appropriate care if there are, in particular, mixed boundary conditions.

These linear elements would expect to reduce the interpolation errors and, incidentally, demonstrate why collocation at the extreme ends of elements using constant valued elements would be pointless.

Further refinements lead to the use of quadratic and cubic elements. Hermitian cubic elements have been proposed by Watson (15) and appear to have computational advantages. The opportunities for complexity are legion, but caution should be exercised as further complexity does not necessarily lead to computational advantage.

4. Poisson's Equation

Consider the equation

$$\nabla^2 u = \rho(x, y), \qquad u \in \Omega \tag{4.1}$$

together with boundary conditions as in (2.2). This can be solved by finding a particular integral and thus reducing the numerical problem to that of Laplace's equation. Determining the particular integral can cause problems and it is therefore necessary to solve the Poisson equation numerically.

Choosing v as in (2.4), substitution into Green's theorem (2.3) gives

$$-u^\alpha - \int_\Omega \rho v^\alpha d\Omega = \int_\Gamma u \frac{\partial v^\alpha}{\partial n} - v^\alpha \frac{\partial u}{\partial n} d\Gamma. \tag{4.2}$$

Proceeding as in § 2, the boundary integral equation becomes

$$\frac{1}{2} u^i + \int_\Gamma u \frac{\partial}{\partial n}\left(-\frac{1}{2\pi} \log r\right) d\Gamma + \int_\Omega \rho v^i d\Omega = \int_\Gamma -\frac{1}{2\pi} \frac{\partial u}{\partial n} \log r \, d\Gamma. \tag{4.3}$$

The only difference from Laplace's equation is that there is an additional term $\int_\Omega \rho v^i d\Omega$ which is independent of the discretisation on the boundary but dependent only on the node and requires an integral throughout the region Ω. It is most unlikely that this integral can be determined analytically and thus some form of twodimensional numerical quadrature will be used for its evaluation. Having determined the value of the integral for each u^i, the solution proceeds as for Laplace's equation.

5. Generalisations

Consider first the solution of (2.1) and (2.2) in three dimensions. The appropriate solution to (2.4) is

$$v = \frac{1}{4\pi r}, \tag{5.1}$$

resulting in the integral equation

$$\frac{1}{2} u^i + \int_\Gamma u \frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) d\Gamma = \int_\Gamma \frac{1}{4\pi r} \frac{\partial u}{\partial n} d\Gamma. \tag{5.2}$$

The surface Γ is divided up into plane triangular (finite) elements E_i in which a linear variation fo u and $\frac{\partial u}{\partial n}$ is assumed. Thus, with the usual shape functions φ_i the values of u and $\frac{\partial u}{\partial n}$ on Γ are given in terms of the nodal values by

$$\left.\begin{aligned} u &= \sum_{i=1}^{N} \varphi_i u^i \\ \frac{\partial u}{\partial n} &= \sum_{i=1}^{N} \varphi_i \frac{\partial u}{\partial n}\Big)^i \end{aligned}\right\} \tag{5.3}$$

The shape functions φ_i are zero except in the elements which have the node i as a vertex. Equation (5.2), when the nodes are made to be the collocation points, becomes

$$\frac{1}{2} u^i + \int_\Gamma \sum_{j=1}^{N} \varphi_j u^j \frac{\partial}{\partial n}\left(\frac{1}{4\pi r_i}\right) d\Gamma = \int_\Gamma \sum_{j=1}^{N} \frac{1}{4\pi r_i} \varphi_j \frac{\partial u}{\partial n}\Big)^j d\Gamma \tag{5.4}$$

where r_i is the distance from node i. Singularities can only occur when $i = j$, in which case the integral on the left hand side vanishes as for two dimensions. The integrals on the right hand side can be shown to ele finite, and again no singularities occur. More complex elements may be used to discretise the boundary and these are familiar in the finite bement method. If it had been Poisson's equation under discussion then integrals throughout Ω of the form $\int_\Omega \rho \frac{1}{4\pi r_i} d\Omega$ would have been necessary.

A number of problems in mathematical physics can, by suitable arrangements and manipulations, be formulated so that the unknown function(s) have to satisfy Laplace's equation. Thus the Boundary Element Method can be used. This approach represents the so-called group of indirect methods, named particularly because the unknowns have no physical significance and further calculations have to be made in order to find the physically significant variables. An example of this in two-dimensional elasticity is Symm and Bhattacharyya (16).

Further extensions can be made by using rather more general Green's formulae (see 17)) for the solution of general second order elliptic partial differential equations.

6. Direct Methods in Elastostatics

As an illustration of direct methods, consider the method as applied to two-dimensional elastostatic problems. The important physical variables are the displacements and the stresses, and in terms of these quantities on the boundaries it is necessary to consider displacements and tractions. The approach here is that based on Rizzo (4) although this is by no means the only approach. Using the usual suffix notation, the equations for displacements under no body forces can be written

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} = 0, \tag{6.1}$$

where i, j etc. run from 1 to 2. Hooke's Law gives an expression for the stress tensor,

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{6.2}$$

Thus the traction, t_i, on the boundary with outward normal n_i is given by

$$t_i = \sigma_{ij}n_j = \lambda u_{k,k}n_i + \mu n_j(u_{i,j} + u_{j,i}). \tag{6.3}$$

The integral equations for the components of displacement and traction are set up using Betti's reciprocal work theorem, which states that

$$\int_C (t_i^{(1)}u_i^{(2)} - t_i^{(2)}u_i^{(1)})dS = 0 \tag{6.4}$$

where $t_i^{(1)}$, $u_i^{(1)}$ and $t_i^{(2)}$, $u_i^{(2)}$ are twoself-equilibrating systems and the integral is taken over the boundary C.

Consider the stress/displacement system corresponding to a unit point force in the direction of the x_i-axis. These are given by (18),

$$u_i^{(j)} = -\frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}\{\delta_{ij}\log r + Mr_{,i}r_{,j}\}, \tag{6.5}$$

$$t_i^{(j)} = -\frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}\left\{[k\delta_{ij} - \lambda 4\mu M r_{,i}r_{,j}]\frac{\partial}{\partial n}(\log r) + k[(\log r)_{,i}n_j - (\log r)_{,j}n_i]\right\} \tag{6.6}$$

where $k = \frac{2\mu^2}{\lambda + 3\mu}$, $M = -\frac{\lambda + \mu}{\lambda + 3\mu}$ and $u_i^{(j)}$ and $t_i^{(j)}$ represent the i^{th} component of the displacement and boundary traction vectors resulting from the unit force in the, x_j direction.

$$r = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]^{1/2} \tag{6.7}$$

and (ξ_1, ξ_2) is the location of these forces, with all derivatives being taken with respect to the coordinates x_i.

Betti's theorem (6.4) is applied with the above singularity system, to give

$$\int_{C+m} (u_i t_i^{(j)} - t_i u_i^{(j)})dC = 0 \tag{6.8}$$

where C is the boundary and m is a small circle surrounding the point ξ_i. The radius of the circle m is reduced to zero and the point ξ_i is taken to the boundary as in section 2, to give the following singular integral equation

$$u_j = \frac{1}{\pi(2\mu M - k)}\int_C [t_i u_i^{(j)} - u_i t_i^{(j)}]dC. \tag{6.9}$$

A suitable discretisation on the boundary C is made and an approximate numerical solution is obtained as before. The displacements at interior points can be evaluated from (6.8) applied to a general point

$$u_j = \frac{1}{2\pi(2\mu M - k)}\int_C (t_i u_i^{(j)} - u_i t_i^{(j)})\, dC, \tag{6.10}$$

and the stresses can thus be evaluated from (6.2).

The corresponding method for three dimensions is given in Cruse (7) — the technique is the same as above, but the singularity functions are, of course, different.

7. Other Applications

It can be seen from sections 2 and 6 that provided there is some appropriate reciprocal relation, a system of partial differential equations may be solved numerically using the Boundary Element Method. All that is required is to be able to determine a singularity solution which enables the integral equation on the boundary to be formed. Without this, the method, of course, cannot be applied.

The solution of the problem of plane elastostatics assumed that there were no body forces. Should body forces exist then the basic differential equations for the displacement would become

$$L\underset{\sim}{u} = \underset{\sim}{f} \tag{7.1}$$

where L is a differential operator, $\underset{\sim}{u}$ the displacement vector and $\underset{\sim}{f}$ a term determined by the body force. The solution procedure would then follow that of section 4, in which the resulting integral equations would have a term involving an integral throughout the region. This situation also arises in solid mechanics when complex forms of deformation occur. Examples in elastoplasticity by Cathie and Banerjee (19) and in inelastic deformation by Mukherjee (20) illustrate the technique in which the more complex parts of the deformation can be made the mathematical equivalent of body forces.

The Finite Element Method poses problems when infinite regions are under consideration, but because the Boundary Element Method is reduced to boundary integrals which, if suitable „equilibrium" conditions occur, are

only in the finite parts of the region. Although integrations will be required throughout the region if there are ,,body forces'' or their equivalent, these calculations can be done using the appropriate quadrature formulae. However, this gives the potential for hybrid techniques whereby the finite element method can be used for part of the finite region and the boundary element method for the remaining infinite region. Such approaches have been made by Zienkiewicz et al (21) and Johnson and Nedelec (22).

The general approach to time dependent problems has been to take the Laplace Transform with respect to the time variable, thereby producing elliptic equations. These can be solved as explained above, and these results are transformed back using some numerical Laplace inversion technique. Rizzo and Cruse (5) used this technique and a useful paper with references to the problems of numerical inversion is that of Beskos (23).

8. Further Comments

The singularity method, which forms the basis of the Boundary Element Method, is well known and has a long history, but it is the advent of powerful computers which has given the impetus to solve the resulting integral equations numerically. The particular peculiarity of the method is the discretisation of the boundary borrowing from the Finite Element Method special kinds of basis functions. Because the integral equations are determined over the boundary, the method is going to be particularly sensitive to singularities on the boundary, and in the case of Neumann problems care has to be taken to ensure suitable conditions are satisfied to ensure uniqueness of the solution. There are clear benefits in using the Boundary Element Method, but these should not hide the fact that it is not always advantageous to use it for every problem.

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