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ON INTERPOLATION OPERATORS — III
(A proof of Telyakovskii-Gopangauz's theorem for differentiable functions)

by

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1. Introduction: So far we studied interpolation operators giving elegant proof of Jackson's and Timan's theorem for differentiable functions defined on $[-1, 1]$ in this series of papers [3,4]. We formally established certain identities useful in decision of the form of polynomial operators and other respects too. There we also noticed that the theorems on simultaneous approximation hold good in case of [4]. R. M. TRIGUB'S [7] inequality for the derivative of polynomial was also obtained in this case as well as in case [3].

The present paper aims at giving a new and practical proof of Telyakovskii-Gopengauz's theorem for differentiable functions belonging to $C'[-1, 1]$. It also includes a theorem on simultaneous approximation. Mean while we establish a series of identities in the shape of lemmas leading to an identity relevant for present investigations as well as for further applications. Our operators $G_{np}(f, x)(p = 0,1)$ are, therefore, unique in having these properties as no such operator for practical purposes has so far been obtained.

2. Definition of the opeartor $G_{np}(f, x)$ $(p = 0,1)$

Let $|x| \leqslant 1$, $\cos t = x$ and $\cos t_k = x_k$ with

(2.1) $$t_k = \frac{k\pi}{n},\ k = \overline{0, n};\ n = 1, 2, 3, \ldots.$$

Further, for $k = \overline{0, 2n - 1}$

$$(2.2)\quad l_k(t) = \frac{\sin n(t-t_k)\cos\frac{1}{2}(t-t_k)}{2n\sin\frac{1}{2}(t-t_k)} = \frac{1}{2n}\left[1 + 2\sum_{j=1}^{n-1}\cos j(t-t_k) + \cos n(t-t_k)\right]$$

and

$$(2.3)\quad s_k(t) = a\,l_k^5(t) + b\,l_k^6(t) + c\,l_k^7(t) + d\,l_k^8(t)$$

$$a = \frac{1008}{43},\quad b = \frac{-1820}{43},\quad c = \frac{960}{43},\quad d = \frac{-105}{43}.$$

Then for any arbitrary function $f(x)$ given on $[-1, 1]$, we define the operators

$$(2.4)\quad G_{np}(f, x) = g(x) + \sum_{k=0}^{n}\left\{\sum_{q=0}^{p}(x - x_k)^q f^{(q)}(x_k) - g(x)\right\} r_k(x)$$

where

$$(2.5)\quad \begin{cases} r_0(x) = s_0(t) \\ r_n(x) = s_n(t) \\ r_k(x) = s_k(t) + S_{2n-k}(t), \quad k = \overline{1, n-1} \end{cases}$$

and

$$(2.6)\quad g(x) = \frac{1+x}{2}f(1) + \frac{1-x}{2}f(-1).$$

The structural properties of $G_{np}(f, x)$ are given in Theorem 1: For $p = 0,1$

a) $G_{np}(f, x_k) = f(x_k)$, $G'_{np}(f, x_k) = f'(x_k)$

b) $G_{np}(f, x)$ is an albebraic polynomial of degree $\leqslant 8n + 1$.

The proof of this theorem is easy and can be done on the basis of [3].

3. Some identities

LEMMA 1: Let

$$S_m \overset{\text{def}}{=} \sum_{k=0}^{2n-1} l_k^m(t)$$

then, we have

$$(3.1)\quad S_3 = \frac{1}{(2n)^2}\left[\left(3n^2 - \frac{1}{2}\right) + \left(n^2 + \frac{1}{2}\right)\cos 2nt\right]$$

$$(3.2)\quad S_4 = \frac{1}{(2n)^3}\left[\left(\frac{16}{3}n^3 - \frac{4}{3}n + \frac{3}{8}\right) + \left(\frac{8}{3}n^3 + \frac{4}{3}n - \frac{1}{2}\right)\times \cos 2nt + \frac{1}{8}\cos 4nt\right]$$

$$(3.3)\quad S_5 = \frac{1}{(2n)^4}\left[\frac{1}{24}(230n^4 - 50n^2 + 4) + \left(\frac{19}{3}n^4 + \frac{5}{3}n^2 - \frac{1}{2}\right)\cos 2nt + \left(\frac{n^4}{12} + \frac{5n^2}{12} + \frac{1}{8}\right)\cos 4nt\right]$$

$$(3.4)\quad S_6 = \frac{1}{(2n)^5}\left[\left(\frac{88}{5}n^5 - 4n^3 + \frac{23}{20}n - \frac{5}{16}\right) + \left(\frac{208}{15}n^5 + \frac{8}{3}n^3 - \frac{23}{15}n + \frac{15}{32}\right)\times \cos 2nt + \left(\frac{8}{15}n^5 + \frac{4}{3}n^3 + \frac{23}{60}n - \frac{3}{16}\right)\cos 4nt + \frac{1}{32}\cos 6nt\right]$$

$$(3.5)\quad S_7 = \frac{1}{(2n)^6}\left[\frac{1}{720}(23548n^6 - 5390n^4 + 1372n^2 - 275) + \frac{1}{720}\left(21086n^6 + 2765n^4 - 1666n^2 + \frac{675}{2}\right)\cos 2nt + \frac{1}{720}(1444n^6 + 2590n^4 + 196n^2 - 135)\cos 4nt + \frac{1}{720}\left(2n^6 + 35n^4 + 98n^2 + \frac{45}{2}\right)\cos 6nt\right]$$

$$(3.6)\quad S_8 = \frac{1}{71(2n)^7}\left[\left(309248n^7 - 71680n^5 + 1912n^2 - 5280n + \frac{11025}{8}\right) + (304896n^7 + 26880n^5 - 22176n^3 + 7920n + 10395)\cos 2nt + \left(30720n^7 + 43008n^5 - 3168n + \frac{2205}{2}\right)\cos 4nt + (256n^7 + 1792n^5 + 2496n^3 + 528n - 315)\cos 6nt + \frac{1}{128}\cos 8nt\right]$$

For the sums S_3 and S_4, we have the identity

$$(3.7)\quad 4S_3 - 3S_4 = 1 - \frac{3\sin^2 nt}{8n^3}.$$

For the sums of higher order, we have

$$(3.8)\quad aS_5 + bS_6 + cS_7 + dS_8 = 1 - \frac{1}{43\times 8n^6}(112n^4 + 420n^2 + 98 - 455n)\sin^6 nt - \frac{315\sin^8 nt}{512\cdot 43n^7}.$$

The identity (3.7) has been proved in [5]. We establish (3.8) only. For this end, we make use of the technique employed in [5], thus obtaining.

$$(3.9)\quad S_5 = C_{0,5} + 2C_{2n,5}\cos 2nt + 2C_{4n,5}\cos 4nt - 5\cos nt[2C_{n,4}\cos nt + 2C_{3n,4}\cos 3nt] +$$

$$+ 10\cos^2 nt\,[C_{0,3}+2C_{2n,3}\cos 2nt] - 10\cos^3 nt\, 2C_{n,2}\cos nt + 5\cos^4 nt =$$
$$= \Big[C_{0,6} - 5C_{n,4} + 5C_{0,3} + C_{2n,3} - \frac{15}{2}C_{n,2} + \frac{15}{8}\Big] +$$
$$+ [2C_{2n,5} - 5C_{n,4} + 5C_{3n,4} + 5C_{0,3} + 10C_{2n,3} - 5C_{n,2} + 5/4]\cos 2nt$$
$$+ \Big[2C_{4n,5} - 5C_{3n,4} + 5C_{2n,3} - 5C_{n,2} + \frac{5}{8}\Big]\cos 4nt$$

In the similar fashion, we have,

(3.10) $$S_6 = C_{0,6} + 2[C_{2n,6}\cos 2nt + C_{4n,6}\cos 4nt + \cos 6nt] -$$
$$- 6\cos nt\,[2C_{n,5}\cos nt + 2C_{3n,5}\cos 3nt + 2\cos 5nt] +$$
$$+ 15\cos^2 nt\,[C_{0,4} + 2_{2n,4}\cos 2nt + 2\cos 4nt] -$$
$$- 20\cos^3 nt\,[2C_{n,3}\cos nt + 2\cos 3nt] +$$
$$+ 15\cos^4 nt\,[C_{0,2} + 2\cos 2nt] - 6\cos^5 nt\cdot 2\cos nt + \cos^6 nt =$$
$$= \Big[C_{0,6} - 6C_{n,5} + \frac{15}{2}C_{0,4} + \frac{15}{2}C_{2n,4} - 15C_{n,3} + \frac{45}{8}C_{0,2} - \frac{15}{16}\Big] +$$
$$+ \Big[2C_{2n,6} - 6C_{n,5} - 6C_{3n,5} + \frac{15}{2}C_{0,4} + 15C_{2n,4} - 20C_{n,3} +$$
$$+ \frac{15}{2}C_{0,2} - \frac{75}{32}\Big]\cos 2nt + \Big[2C_{4n,6} - 6C_{3n,5} + \frac{15}{2}C_{2n,4} - 5C_{n,3} +$$
$$+ \frac{15}{8}C_{0,2} - \frac{7}{16}\Big]\cos 4nt + \frac{1}{32}\cos 6nt$$

(3.11) $$S_7 = C_{0,7} + 2[C_{2n,7}\cos 2nt + C_{4n,7}\cos 4nt + 2C_{6n,7}\cos 6nt] -$$
$$- 7\cos nt\,[2C_{n,6}\cos nt + 2C_{3n,6}\cos 3nt + 2C_{5n,6}\cos 5nt] +$$
$$+ 21\cos^2 nt\,[C_{0,5} + 2C_{2n,5}\cos 2nt + 2C_{4n,5}\cos 4nt] -$$
$$- 35\cos^3 nt\,[2C_{n,4}\cos nt + C_{3n,4}\cos 3nt] +$$
$$+ 35\cos^4 nt\,[C_{0,3} + C_{2n,3}\cos 2nt] - 21C_{n,2}\cos^5 nt\cdot 2\cos nt +$$
$$+ 7\cos^6 nt = \Big[C_{0,7} - 7C_{n,6} + \frac{21}{2}C_{0,5} + \frac{105}{4}C_{n,4} - \frac{35}{4}C_{3n,4} +$$
$$+ \frac{105}{9}C_{0,3} + \frac{35}{2}C_{2n,3} - \frac{420}{32}C_{n,2} + \frac{35}{16}\Big] + \Big[2C_{2n,7} - 7(C_{n,6} + C_{3n,6}) +$$
$$+ \frac{21}{2}C_{0,5} + 21C_{2n,5} + \frac{21}{2}C_{4n,5} - 35C_{n,4} - \frac{105}{4}C_{3n,4} + \frac{35}{2}C_{0,3} +$$
$$+ \frac{245}{8}C_{2n,3} + \frac{15}{32}(7 - 42C_{n,2})\Big]\cos 2nt +$$

$$+ \Big[2C_{4n,7} - 7(C_{3n,6} + C_{5n,6}) + \frac{21}{2}C_{2n,5} + 21C_{4n,5} - \frac{35}{4}C_{n,4} -$$
$$- \frac{105}{4}C_{3n,4} + \frac{35}{8}C_{4n,7} - 7(C_{3n,6} + C_{5n,6}) + \frac{21}{2}C_{2n,5} + 21C_{4n,5} -$$
$$- \frac{35}{4}C_{n,4} - \frac{105}{4}C_{3n,4} + \frac{35}{8}C_{0,3} - \frac{3}{16}(7 - 42C_{n,2})\Big]\cos 4nt +$$
$$+ \Big[2C_{6n,7} - 7C_{5n,6} + \frac{21}{2}C_{4n,5} - \frac{35}{4}C_{3n,4} + \frac{35}{8}C_{2n,3} +$$
$$+ \frac{1}{32}(4 - 42C_{n,2})\Big]\cos 6nt$$

(3.12) $$S_8 = [C_{0,8} + 2C_{2n,8}\cos 2nt + 2C_{4n,8}\cos 4nt + C_{6n,8} + 2\mathrm{Cos}\, 8nt$$
$$- 8\cos nt\,[2C_{n,7}\cos nt + 2C_{3n,7}\cos 3nt + 2C_{5n,7}\cos 5nt + 2\cos 7nt] +$$
$$+ 28\cos^2 nt\,[C_{0,6} + 2C_{2n,6}\cos 2nt + 2C_{4n,6}\cos 4nt + 2\cos 6nt]$$
$$- 56\cos^3 nt\,[2C_{n,5}\cos nt + 2C_{3n,5}\cos 3nt + 2\cos 5nt] +$$
$$+ 70\cos^4 nt\,[C_{0,4} + 2C_{2n,4}\cos 2nt + 2\cos 4nt] -$$
$$- 56\cos^5 nt\,[2C_{n,3}\cos nt + 2\cos 3nt] + 28\cos^6 nt\,[C_{0,2} + 2\cos 2nt] -$$
$$- 15\cos^8 nt =$$
$$= \Big[C_{0,8} - 8C_{n,7} + 14C_{0,6} + 14C_{2n,6} - 42C_{n,5} - 14C_{3n,5} + \frac{105}{4}C_{0,4} +$$
$$+ 35C_{2n,4}\frac{35}{4} - 35C_{n,3} + \frac{35}{2} + \frac{35}{2} + \frac{35}{4}C_{0,2} + \frac{108}{8}\frac{525}{128}\Big] +$$
$$+ \Big[2C_{4n,8} - 8C_{n,7} - 8C_{3n,7} + 14C_{0,6} + 28C_{2n,6} + 14C_{4n,6} - 56C_{n,5} -$$
$$- 42C_{3n,5} + 35C_{0,4} + \frac{245}{4}C_{2n,4} + \frac{105}{2}C_{n,3} + \frac{105}{8}C_{0,2} - \frac{583}{16}\Big]\cos 2nt$$
$$+ \Big[2C_{4n,8} - 8C_{3n,7} - 8C_{5n,7} + 14C_{2n,6} + 28C_{4n,6} - 14C_{n,5} -$$
$$- 42C_{3n,5} + \frac{35}{4}C_{0,4} + 35C_{2n,4} - 21C_{n,3} + \frac{21}{4}C_{0,2} + \frac{7}{32}\Big]\cos 4nt +$$
$$+ \Big[2C_{6n,8} - 8C_{2n,7} + 14C_{4n,6} + 28 - 14C_{3n,5} + \frac{35}{4}C_{2n,4} - \frac{7}{2}C_{n,3} +$$
$$+ \frac{7}{8}C_{0,2} - \frac{75}{16}\Big]\cos 6nt + \frac{1}{128}\cos 8nt$$

The numbers $C_{0,8}$, $C_{2n,8}$, $C_{n,8}$, $C_{6n,8}$; $C_{r,7}$, $r=\overline{0,6}$; $C_{r,6}$, $r=\overline{0,5}$; $C_{r,5}$, $r=\overline{0,4}$; $C_{r,4}$, $r=\overline{0,3}$; $C_{r,3}$, $r=0,1,2$; $C_{r,2}$, $r=0,1$; are respectively the coefficients of Z^0, Z^{2n}, Z^{4n}, Z^{6n}; Z^{rn}, $r=\overline{0,6}$; Z^{rn}, $r=\overline{0,5}$; Z^{rn}, $r=\overline{0,4}$; Z^{rn}, $r=\overline{0,3}$; Z^{rn}, $r=0,1,2$; Z^{rn}, $r=0,1$ in the expansion

$$\left[\sum_{j=n}^{\infty} Z^j\right]^m = Z^{2n+1}(1-Z)^m \sum_{j=0}^{\infty} \frac{(j+1)_{m-1}}{(m-1)\,!} Z^j \tag{3.13}$$

for $m = 8, 7, 6, 5, 3$ and 2, where

$$(j+1)_{m-1} = (j+1)(j+2)\ldots(j+m-1).$$

Explicity, we have from (3.13)

$$C_{0,8} = \frac{1}{7\,!}\,[(8n+1)_7 - 8(6n)_7 + 28(4n-1)_7 - 56(2n-2)_7] =$$

$$= \frac{1}{7\,!}\,[302 \cdot 2^{10} \cdot n^7 + 7 \cdot 302 \cdot 2^9 n^6 + 1659392n^5 + 1442560n^4 +$$

$$+ 772352n^3 + 257152n^2 + 51168n + 5040]$$

$$C_{2n,8} = \frac{1}{7\,!}\,[(10n+1)_7 - 8(8n)_7 + 28(6n-1)_7 - 56(4n-2)_7 + 70(2n-3)_7] =$$

$$= \frac{1}{7\,!}\,[1191 \cdot (2n)^7 + 359 \cdot 28(2n)^6 + (35406)(2n)^5 + 68600(2n)^4 +$$

$$+ 80199 \cdot (2n)^3 + 57428(2n)^2 + 48648n + 5040]$$

$$C_{4n,8} = \frac{1}{7\,!}\,[(12n+1)_7 - 8(10n)_7 + 28(8n-1)_7 - 56(6n-2)_7 +$$

$$+ 70(4n-3)_7 - 56(2n-4)_7]$$

$$= \frac{1}{7\,!}\,[120 \cdot (2n)^7 + 1624(2n)^6 + 8904 \cdot (2n)^5 + 25480(2n)^4 +$$

$$+ 41160(2n)^3 + 38416(2n)^2 + 20376(2n) + 5040]$$

$$C_{6n,8} = \frac{1}{7\,!}\,[(14n+1)_7 - 8(12n)_7 + 28(10n-1)_7 - 56(8n-2)_7 +$$

$$+ 70(6n-3)_7 + 56(4n-4)_7 + 28(2n-5)_7]$$

$$= \frac{1}{7\,!}\,[(2n)^7 + 28 \cdot (2n)^6 + 322 \cdot (2n)^5 + 1960(2n)^4 + (769)(2n)^3 +$$

$$+ 13132 \cdot (2n)^2 + 13068(2n) + 5040]$$

$$C_{0,7} = \frac{1}{6\,!}\,[(7n+1)_6 - 7(5n)_6 + 2n(3n-1)_6 - 35(n-2)_6]$$

$$= \frac{1}{6\,!}\,[23548n^6 + 70644n^5 + 91000n^4 + 64260n^3 + 26572n^2 +$$

$$+ 6216n + 720]$$

$$C_{n,7} = \frac{1}{6\,!}\,[(8n+1)_6 - 7(6n)_6 + 21(4n-1)_6 - 35(2n-2)_6]$$

$$= \frac{1}{6\,!}\,[302 \cdot (2n)^6 + 1932 \cdot (2n)^5 + 5180 \cdot (2n)^4 + 7560(2n)^3 +$$

$$+ 6398 \cdot (2n)^2 + 3108(2n) + 720]$$

$$C_{2n,7} = \frac{1}{6\,!}\,[(9n+1)_6 - 7(7n)_6 + 21(5n-1)_6 - 35(3n-2)_6 + 35(n-3)_6]$$

$$= \frac{1}{6\,!}\,[10543n^6 + 40299n^5 + 61705n^3 + 22792n^2 + 6936n + 720]$$

$$C_{3n,7} = \frac{1}{6\,!}\,[(10+1)_6 - 7(8n)_6 + 21(6n-1)_6 - 35(4n-2)_6 + 35(2n-3)_6]$$

$$= \frac{1}{6\,!}\,[57 \cdot (2n)^6 + 567(2n)^5 + 2205(2n)^4 + 4305(2n)^3 + 4578(2n)^2 +$$

$$+ 5376n + 720]$$

$$C_{4n,7} = \frac{1}{6\,!}\,[(11n+1)_6 - 7(9n)_6 + 21(7n-1)_6 - 35(5n-2)_6 +$$

$$+ 35(3n-3)_6 - 21(n-4)_6] = \frac{1}{6\,!}\,[722n^6 + 4998n^5 + 7580n^4 +$$

$$+ 18270n^3 + 12698n^2 + 4452n + 720]$$

$$C_{6n,7} = \frac{1}{6\,!}\,[(13n+1)_6 - (11n)_6 + 21(9n-1)_6 - 35(7n-2)_6 +$$

$$+ 35(5n-3)_6(3n) + 7(n-5)_6] =$$

$$= \frac{1}{6\,!}\,[n^6 + 21n^5 + 175n^4 + 735n^3 + 1624n^2 + 1764n + 720]$$

$$C_{0,6} = \frac{1}{5\,!}\,[(6n+1)_5 - 6(4n)_5 + 15(2n-1)_5] =$$

$$= \frac{1}{5\,!}\,[2112n^5 + 5280n^4 + 5520n^3 + 3000n^2 + 8887n + 120]$$

$$C_{n,6} = \frac{1}{5\,!}\,[(7n+1)_5 - 6(5n)_5 + 15(3n-1)_5 - 20(2-n)_5]$$

$$= \frac{1}{5\,!}\,[1682n^5 + 4590n^4 + 5030n^3 + 2850n^2 + 848n + 120]$$

$$C_{2n,6} = \frac{1}{5!}\left[(8n-1)_5 - 6(6n)_5 + 15(4n-1)_5 - 20(2n-2)_5\right]$$

$$= \frac{1}{5!}\left[832n^5 + 2880n^4 + 3760n^3 + 2400n^2 + 808n + 120\right]$$

$$C_{3n,6} = \frac{1}{5!}\left[(9n+1)_5 - 6(7n)_5 + 15(5n-1)_5 - 20(3n-2)_5 + {}\right.$$
$$\left. + 15(n-3)_5\right] =$$

$$= \frac{1}{5!}\left[(9n+1)_5 - 6(7n)_5 + 15(5n-1)_5 - 20(3n-2)_5 + 15(n-3)_5\right]$$

$$= \frac{1}{5!}\left[237n^5 + 1155n^4 + 2085n^3 + 1725n^2 + 678n + 120\right]$$

$$= \frac{1}{5!}\left[(10n+1)_5 - 6(8n)_5 + 15(6n-1)_5 - 20(4n-2)_5 + {}\right.$$
$$\left. + 15(2n+3)_5\right] =$$

$$= \frac{1}{5!}\left[32n^5 + 240n^4 + 680n^3 + 900n^2 + 548n + 120\right]$$

$$C_{5n,6} = \frac{1}{5!}\left[(11n+1)_5 - 6(9n)_5 + 15(7n-1)_5 - 20(5n-2)_5 + {}\right.$$
$$\left. + 15(3n-3)_5 - 6(n-4)_5\right] =$$

$$= \frac{1}{5!}\left[n^5 + 15n^4 + 85n^3 + 225n^2 + 274n + 120\right]$$

$$C_{0,5} = \frac{1}{4!}\left[(5n+1)_4 - 5(3n)_4 + 10(n-1)_4\right]$$

$$= \frac{1}{4!}\left[230n^4 + 460n^3 + 370n^2 + 140n + 24\right]$$

$$C_{n,5} = \frac{1}{4!}\left[(6n+1)_4 - 5(4n)_4 + 10(2n-1)_4\right] =$$

$$= \frac{1}{4!}\left[176n^4 + 400n^3 + 3400n^2 + 140n + 24\right]$$

$${}_{2n,5} = \frac{1}{4!}\left[(7n+1)_4 - 5(5n)_4 + 10(3n-1)_4 - 10(n-2)_4\right]$$

$$= \frac{1}{4!}\left[76n^4 + 240n^3 + 260n^2 + 120n + 24\right]$$

$$C_{3n,5} = \frac{1}{4!}\left[(8n+1)_4 - 5(6n)_4 + 10(4n-1)_4 - 10(2n-2)_4\right]$$

$$= \frac{1}{4!}\left[16n^4 + 80n^3 + 140n^2 + 100n + 24\right]$$

$$C_{4n,5} = \frac{1}{4!}\left[(9n+1)_4 - 5(7n)_4 + 10(5n-1)_4 - 10(3n-2)_4 - 5(n-3)_4\right] =$$

$$= \frac{1}{4!}\left[n^4 + 10n^3 + 35n^2 + 50n + 24\right]$$

The, numbers $C_{r,4}$, $r = \overline{0,3}$; $C_{r,3}$ $r = 0,1,2$; $C_{0,2}$ and C_{n-2} are the coefficients of various powers of Z in the expansion (3.13) for $m = 4$, 3 and 2 respectively. They have already been computed in [5]. Therefore we shall not mention them here.

After the substitution of suitable values of these constants in (3.9) — (3.12) we obtain the indentities (3.1)—(3.6).

To obtain (3.8) we multiply (3.3) by $a = \frac{1008}{43}$, (3.4) by $b = -\frac{1820}{43}$, (3.5) by $c = \frac{960}{43}$ and (3.6) by $d = -\frac{105}{43}$ and perform the intermediate calculations involved. Therefore we obtain after some manipulations

$$(2n)^7 \sum_{k=0}^{2n-1} \{a\, l_k^5(t) + b\, l_k^6(t) + c\, l_k^7(t) + d\, l_k^8(t)\} =$$

$$= 128n^7 - \frac{16}{43}(112n^4 + 420n^2 - 455n + 98)\sin^6 nt - \frac{315}{43\cdot 4}\sin^8 nt$$

$$\sum_{k=0}^{2n-1} s_k(t) = 1 - \frac{(112n^4 + 420n^2 - 455n + 98)\sin^6 nt}{43\cdot 8n} - \frac{315\sin^8 nt}{43\cdot 512n^7}$$

which is essentially the same as (3.8).

LEMMA 2: For $-1 \leqslant x \leqslant 1$, we have,

a) $\left|\sum_{K=0}^{n} r_k(x) - 1\right| \leqslant \frac{4}{n^2}$

b) $\left|\sum_{K=0}^{n} r_k'(x)\right| \leqslant 20$ and $\left|\sqrt{1-x^2}\sum_{k=0}^{n} r_k'(x)\right| \leqslant \frac{20}{n}$.

The proof of this lemma makes use of (3.8), (2.5), (2.3), (2.2) and the differentiated form of (3.8).

LEMMA 3: Fot $t \in [0, \pi]$, we have

a) $|l_k^5(t) + l_{k+1}^5(t)| \leqslant \frac{5\pi}{v_k^6}$

and

b) $\left|\frac{d}{dt}\{l_k^5(t) + l_{k+1}^5(t)\}\right| \leqslant \frac{50\pi\sin^2 nt}{v_k^6}$

where $v_k = 2n\sin\frac{1}{2}(t - t_k)$.

The proof of this lemma depends upon a similar lemma given in [4]

LEMMA 4: For $t \in [0, \pi]$, we have, for $p = 0, 1, 2$

a) $\sum_{k=0}^{2n-1} \sin^p \frac{|t-t_k|}{2} |s_k(t)| \leqslant 120 \frac{|\sin^p nt|}{n^p}$

b) $\sum_{k=0}^{2n-1} \sin^p \frac{|t-t_k|}{2} |s'_k(t)| \leqslant 1707 \frac{|\sin^p nt|}{n^{p-1}}$

c) $\sum_{k=0}^{2n-1} \sin^p \frac{|t-t_k|}{2} \frac{|s'_k(t)|}{\sin t} \leqslant 1707 \frac{|\sin^p nt|}{n^{p-2}}$.

Again the proof of the lemma is obtained on the same lines as in [4]

THEOREM 2: Let $f \in C^{(q)}[-1, 1]$, then for the operators $G_{np}(f, x)$, we have, for $p = 0, 1$

$$|G_{np}^{(q)}(f, x) - f^{(q)}(x)| \leqslant C_p \left(\frac{\sqrt{1-x^2}}{n}\right)^{p-q} w_{f^{(p)}}\left(\frac{\sqrt{1-x^2}}{n}\right). \tag{4.1}$$

The proof of the theorem for $p = 0$ and $q = 0$ is contained in the proof of theorem for $p = 1$ and $q = 0$ and is therefore omitted. Still for $p = 1$ and $q = 0$ we need not present here the proof inasmuchas similar theorem has been done in [4]. However the proof is outlined as follows:

$$G_{n1}(f, x) - f(x) =$$

$$= \left[\frac{1+x}{2}(f(1) - f(x)) + \frac{1-x}{2}(f(-1) - f(x))\right] \times \left[1 - \sum_{k=0}^{n} r_k(x)\right] \tag{4.2}$$

$$\sum_{k=0}^{n} [f(x) - f(x_k) - (x - x_k) f'(x_k)] r_k(x) = S_1 + S_2.$$

Now making use of the identity

$$f(u) - f(v) f'(v) = o(|u - v|) w_{f'}(|u - v|) \tag{4.3}$$

the lemma 2 and the properties of modulus of continuity, we obtain

$$|S_1| \leqslant 6C_1 \frac{4}{n^2} [\omega_{f'}(1 - x) + \omega_{f'}(1 + x)] \frac{1-x^2}{2} \sin^2 nt \leqslant \tag{4.4}$$

$$\leqslant 12 C_1 \left(\frac{\sqrt{1-x^2}}{n}\right) \omega_{f'}\left(\frac{\sqrt{1-x^2}}{n}\right).$$

For S_2, using (2.5), we get

$$S_2 = \sum_{k=0}^{n-1} g_k s_k(t),$$

$$= \frac{1008}{43} \sum_{k=0}^{2n-1} g_k l_k^5(t) +$$

$$+ \frac{1}{43} \sum_{k=0}^{2n-1} g_k \{-1820 l_k^6(t) + 960 l_k^7(t) - 105 l_k^8(t)\} = S_2^{(1)} + S_2^{(2)}$$

where

$$g_k = f(\cos t) - f(\cos t_k) - (\cos t - \cos t_k) f'(\cos t_k). \tag{4.6}$$

Now on account of (4.3) and the inequality

$$|l_k(t)| \leqslant 1$$

We have,

$$|S_2^{(2)}| \leqslant 68 C_1 \sum_{k=0}^{2n-1} |\cos t - \cos t_k| \omega_{f'}(|\cos t - \cos t_k|) \times l_k^6(t). \tag{4.7}$$

It is easily verified that

$$\sum_{k=0}^{2n-1} \sin^p \frac{|t-t_k|}{2} l_k^6(t) \leqslant \frac{|\sin^p nt|}{n^p};\ p = 2, 3, 4. \tag{4.8}$$

Therefore from (4.7) we have, on account of (4.8)

$$|S_2^{(2)}| \leqslant 68 C_1 \sum_{k=0}^{2n-1} \left[\frac{n}{\sin t}(\cos t - \cos t_k)^2 + \right. \tag{4.9}$$

$$\left. + |\cos t - \cos t_k|\right] l_k^6(t) \times \omega_{f'}\left(\frac{\sin t}{n}\right) \leqslant 544 C_1 \left(\frac{\sin t}{n}\right) \omega_{f'}\left(\frac{\sin t}{n}\right).$$

For the estimation of $S_2^{(1)}$ we shall use the idea of Vertesi-Kis [2] for estimating such sums pairwise. We write

$$S_2^{(1)} = \frac{1008}{43} g_j l_j^5(t) + \frac{1008}{43} \sum_{k=0}^{j-1} g_k l_k^5(t) + \frac{1008}{43} \sum_{k=j+1}^{2n-1} g_k l_k^5(t) \tag{4.10}$$

where j_j is defined by

$$|t - t_j| \leqslant \frac{\pi}{2n}. \tag{4.11}$$

We denote these constituent sums respectively by T_1, T_2 and T_3 and show thateach has the order

$$O\left\{\left(\frac{\sqrt{1-x^2}}{n}\right) \omega_{f'}\left(\frac{\sqrt{1-x^2}}{n}\right)\right\}.$$

The actual calculation of these sums are omitted here because they are simply done on the basis of the proofs of the constituent sums already shown in [4]. For example, T_1 can be calculated as under

$$|T_1| \leqslant 24 C_1 \omega_{f'}\left(\frac{\sin t}{n}\right) \left[\frac{n}{\sin t}(\cos t - \cos t_j)^2 + \right. \tag{4.12}$$

$$\left. + |\cos t - \cos t_j|\right] |l_j^5(t)| \leqslant 24 C_1 \omega_{f'}\left(\frac{\sin t}{n}\right) \left[\frac{4n}{\sin t}\left\{\sin^2 t \sin^2 \frac{1}{2}(t - t_j) + \right.\right.$$

$$+ 2\sin t \sin^3 \frac{1}{2}|t - t_j| + \sin^4 \frac{1}{2}(t - t_j)\Big\} +$$

$$+ 2\sin t \sin \frac{1}{2}|t - t_j| + \sin^2 \frac{1}{2}(t - t_j)\Big]|l_j^5(t)| \leqslant$$

$$\leqslant 96C_1\left(\frac{\sqrt{1-x^2}}{n}\right)\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n}\right)$$

5. Proof of the theorem for $p = 1$, $q = 1$.

From (2.4) after differentiation we get the identity

$$G'_{n1}(f, x) - f'(x) = G_{n0}(f', x) - f'(x) + \tag{5.1}$$

$$+\left[\frac{f(1) - f(-1)}{2} - \frac{1+x}{2}f'(1) + \frac{1+x}{2}f'(-1)\right]\left[1 - \sum_{k=0}^{n} r_k(x)\right] -$$

$$-\sum_{k=0}^{n}[f(x) - f'(x_k) - (x - x_k)f'(x_k)]r(x_k') +$$

$$+\left[\frac{1+x}{2}(f(1) - f(x)) + \frac{1-x}{2}(f(-1) - f(x)\right]\sum_{k=0}^{n} r_k'(x) =$$

$$= B_1 + B_2 + B_3 + B_4$$

The sum B_1 is estimated exactly in the same way as is done in the case of $p = 0$. Thus, we, easily have,

$$|B_1| \leqslant C_2\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n}\right). \tag{5.2}$$

For B_2 and B_4 using the suitable forms of lemma 4 and the equality (4.3). We, thus, obtain

$$|B_4| \leqslant C_1\frac{1-x^2}{2}\{\omega_{f'}(1-x) + \omega_{f'}(1+x)\}\left|\sum_{k=0}^{n} r_k'(x)\right|.$$

Using the properties of modulus of continuity, we have,

$$|B_4| \leqslant 6C_1\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n}\right) \tag{5.3}$$

and similarly owing to the lemma 4 and (4.3) we have

$$|B_2| \leqslant \frac{4\sin^2 nt}{n^2}C_4\omega_{f'}(1) \leqslant 8C_4\omega_{f'}\left(\frac{\sqrt{1-x^2}}{n}\right) \tag{5.4}$$

For the sum B_3, we transform it by putting $x = \cos t$, $x_k = \cos t_k$, and using (4.11) and (4.6) thus obtaining

$$B_3 = \frac{1}{\sin t}\sum_{k=0}^{2n-1} g_k \frac{d}{dt}\{s_k(t)\} = \frac{g_j}{\sin t}\frac{d}{dt}\{s_j(t)\} + \tag{5.5}$$

$$+\frac{1008}{43\sin t}\sum_{\substack{k=0\\k\neq j}}^{2n-1} g_k \frac{d}{dt}\{l_k^5(t)\} + \frac{1}{43\sin t}\sum_{\substack{k=0\\k\neq j}}^{2n-1} g_j \frac{d}{dt}\{-1820l_k^6(t) + 960l_k^7(t) - 105l_k^8(t)\} =$$

$$= B_{31} + B_{32} + B_{33}.$$

The sum B_{33} can be estimated by making use of

$$|\cos t - \cos t_k| \leqslant 2\sin t \cdot \sin\frac{1}{2}|t - t_k| + 2\sin^2\frac{1}{2}(t - t_k) \tag{5.6}$$

and

$$-|l_k'(t)| \leqslant \frac{2n}{|v_k|}. \tag{5.7}$$

Thus we obtain

$$|B_{33}| \leqslant \frac{860\, C_1 n \sin^2 nt}{\sin t}\sum_{\substack{k=0\\k\neq j}}^{2n-1}|\cos t - \cos t_k|\omega_{f'}(|\cos t - \cos t_k|) \leqslant \tag{5.8}$$

$$\leqslant \frac{860C_1\, n\, \sin^2 nt}{\sin t}\omega_{f'}\frac{(\sin t)}{n}\sum_{k=0}^{2n-1}\left[\frac{n}{\sin t}(\cos t - \cos t_k)^2 + |\cos t - \cos t_k|\right]\frac{1}{v_k^6} \leqslant$$

$$\leqslant 3.440C_1\omega_{f'}\left(\frac{\sin t}{n}\right)\sum\frac{1}{v_k^2} = 0\left\{\omega_{f'}\left(\frac{\sin t}{n}\right)\right\}.$$

B_{31} is to be estimated as a single term of B_{33} and hence, we have,

$$B_{31} = 0\left\{\omega_{f'}\left(\frac{\sin t}{n}\right)\right\}. \tag{5.9}$$

We, now, use exactly the same technique of Breaking the sum B_{32} employed in [4] in order to retain the estimate in dessired form. Thus, we have,

$$B_{32} = 0\left\{\omega_{f'}\left(\frac{\sin t}{n}\right)\right\}. \tag{5.10}$$

Combining (5.10), (5.9) and (5.8) we obtain our theorem. Lastly we announce a theorem on the derivative of the polynomial which is as follows:

THEOREM 3: Let $f^{(b)} \in C[-1, 1]$, then for the operators $G_{np}(f, x)$, we have

$$|G'_{np}(f, x)| \leqslant C\frac{\omega_{f^{(b)}}\left(\frac{\sqrt{1-x^2}}{n}\right)}{(\sqrt{1-x^2}/n)}.$$

The proof of this theorem is based upon the proof of the theorem given in [4]. Therefore we need not present it here. However, for the sake of completeness we adjoin the proof of the theorem for $p = 0$.

We have from (2.5) after differentiation

$$G_{n0}(f, x) = \left[\frac{f(1) - f(-1)}{2}\right]\left[1 - \sum_{k=0}^{n} r_k(x)\right] +$$

$$+ \sum_{k=0}^{n} [f(x_k) - f(x)] r_k'(x) -$$

$$- \left[\frac{1+x}{2}(f(1) - f(x)) + \frac{1+x}{2}(f(-1) - f(x))\right] \sum_{k=0}^{n} r_k'(x) =$$

$$= R_1 + R_2 + R_3$$

For R_1, we make use of Lemma 2(a) and the properties of modulus of continuity and obtain

$$|R_1| \leqslant \frac{1}{2} \frac{4 \sin^2 nt}{n^2} \omega_f(2)$$

which in both the cases of $\sqrt{1 - x^2} \leqslant \frac{1}{n}$ and $\sqrt{1 - x^2} > \frac{1}{n}$ gives

$$(5.11) \qquad |R_1| \leqslant 8 \frac{\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right)}{\left(\frac{\sqrt{1-x^2}}{n}\right)}.$$

Now for R_3 using

$$(1 + x)\omega_f(1 - x) + (1 - x)\omega_f(1 + x) \leqslant 6\omega_f(1 - x^2),$$

and lemma 2(b) first part, we get,

$$(5.10) \qquad |R_3| \leqslant 60\omega_f(1 - x^2) \leqslant 60\{n\sqrt{1 - x^2} + 1\}\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right) \leqslant$$

$$\leqslant 60\{n\sqrt{1 - x^2} + 1\}\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right) \leqslant 60 \frac{n}{\sqrt{1-x^2}}\left\{1 - x^2 + \right.$$

$$\left. + \frac{\sqrt{1-x^2}}{n}\right\}\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right) \leqslant 120 \frac{\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right)}{\left(\frac{\sqrt{1-x^2}}{n}\right)}.$$

Now we transform R_2 by putting $x = \cos t$, $x_k = \cos t_k$ thus obtaining

$$(5.13) \qquad R_2 = \frac{1}{\sin t} \sum_{k=0}^{2n-1} (f(\cos t) - f(\cos t_k))\, s_k(t).$$

We break the sum R_2 as follows

$$R_2 = \frac{1}{\sin t}(f(\cos t) - f(\cos t_j)) \frac{d}{dt}\{s_j(t)\} +$$

$$+ \frac{1008}{43 \sin t} \sum_{\substack{k=0 \\ k \neq j}}^{2n-1} (f(\cos t) - f(\cos t_k)) \frac{d}{dt}\{l_k^5(t)\} +$$

$$+ \frac{1}{43 \sin t} \sum_{\substack{k=0 \\ k \neq j}}^{2n-1} [f(\cos t) - f(\cos t_k)] \frac{d}{dt}\{-1820 l_k^6 + 960 l_k^7 - 105 l_k^8\} =$$

$$= R_{21} + R_{22} + R_{23}$$

where j is given by (4.11).

The constituent sum R_{21} is estimated as under

$$(5.14) \qquad |R_{21}| \leqslant \frac{526n}{\sin t} \omega_f\left(\frac{\sin t}{n}\right)\left[\frac{n}{\sin t}|\cos t - \cos t_j| + 1\right] \frac{\sin^4 nt}{v_j^4} \leqslant$$

$$\leqslant \frac{2626n}{\sin t} \omega_{f'}\left(\frac{\sin t}{n}\right)$$

For the sum R_{23} we argue similarly and get

$$(5.15) \qquad |R_{23}| \leqslant \frac{433n}{\sin t} \omega_f\left(\frac{\sin t}{n}\right) \sum_{\substack{k=0 \\ k \neq j}}^{2n-1} \left[\frac{n}{\sin t}|\cos t - \cos t_k| + 1\right] \frac{\sin^4 nt}{v_k^6} \leqslant$$

$$\leqslant \frac{433n}{\sin t} \omega_f\left(\frac{\sin t}{n}\right) \sum_{\substack{k=0 \\ k \neq j}}^{2n-1} \left[\frac{n}{\sin t}\left\{2 \sin t \sin \frac{1}{2}|t - t_k| + 2 \sin^2 \frac{1}{2}(t - t_k)\right\} + 1\right] \times$$

$$\times \frac{\sin^4 nt}{v_k^6} \leqslant \frac{433u}{\sin t} \omega_f\left(\frac{\sin t}{n}\right)\left[\frac{1}{|v_k^5|} + \frac{\sin nt}{2n \sin t v_k^4} + \frac{1}{v_k^6}\right] \leqslant$$

$$\leqslant \frac{433n}{\sin t} \omega_f\left(\frac{\sin t}{n}\right) \cdot \frac{5}{2} \sum \frac{1}{v_k^5} \leqslant \frac{109n}{\sin t} \omega_f\left(\frac{\sin t}{n}\right)$$

Again we break the sum R_{22} in two parts viz.

$$R_{22} = R_{22}^{(1)} + R_{22}^{(2)}$$

where

$$R_{22}^{(1)} = \frac{1008}{43 \sin t} \sum_{k=0}^{j-1} [f(\cos t) - f(\cos t_k)] \frac{d}{dt}\{l_k^5(t)\}$$

and

$$R_{22}^{(2)} = \frac{1008}{43 \sin t} \sum_{k=j+1}^{2n-1} [f(\cos t) - f(\cos t_k)] \frac{d}{dt}\{l_k^5(t)\}.$$

As earlier in [4] these two sums are estimated in the same way and therefore we obtain only one of them e.g.

$$R_{22}^{(1)} = \frac{1008}{43 \sin t} \sum [f(\cos t) - f(\cos t_k)] \frac{d}{dt} \{l_k^5(t) + l_{k+1}^5(t)\} +$$
$$+ \frac{1008}{43 \sin t} \sum [f(\cos t_{k+1}) - f(\cos t_k)] \frac{d}{dt} \{l_{k+1}^5(t)\} = R_{22}^{(1)'} + R_{22}^{(1)''}.$$

Now using lemma 3(b), we have

$$(5.16) \quad |R_{22}^{(1)'}| \leqslant \frac{108 \cdot 50n}{43 \sin t} \omega_f \left(\frac{\sin t}{n}\right) \sum \left[\frac{n}{\sin t} |\cos t - \cos t_k| + 1\right] \frac{\sin^2 nt}{v_k^6} \leqslant$$
$$\leqslant \frac{1200n\pi}{\sin t} \omega_f \left(\frac{\sin t}{n}\right) \sum \left[\frac{1}{|v_k^5|} + \frac{\sin^2 nt}{2n \sin t v_k^4} + \frac{1}{v_k^6}\right] \leqslant 300\pi \frac{\omega_f\left(\frac{\sin t}{n}\right)}{\left(\frac{\sin t}{n}\right)}.$$

similarly we have

$$(5.17) \quad | R_{22}^{(1)''} | \leqslant 60\pi \frac{\omega_f\left(\frac{\sin t}{n}\right)}{\left(\frac{\sin t}{n}\right)}.$$

Combining (5.17), (5.16), (5.14) and (5.13) we get the theorem.

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