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ON A METHOD FOR SOLVING THE DECISION PROBLEM

by

M. BALÀZS and I. FABIAN
(Cluj-Napoca)

In this paper we give a mathematical formulation of the decision problem. We also give a method for solving this problem.

To give a rigorous solving we shall take a minimal number of notions for primary ones. For this purpose we have chosen notions which are (at least we think so) intuitively clear enaugh to the reader. These notions are: *the activity, the goal of an activity* and *the resolver*. These notions are also used in [2] and [3] as well as in many other books and papers dealing with the decision problem.

DEFINITION 1. *By action we understand an activity together with its goal.*

Thus we can consider an action as an ordered pair with an activity as its first component and a goal as second component.

DEFINITION 2. *A class of actions is a set of actions having the same goal as second component.*

Two elements of a class of actions are distinct from each other if and only if their activities are distinct. We can identify each activity by some specific features, for instance the time it takes, the place where it takes place, the money which must be invested etc. A set of specific features which allows us to distinguish any two elements of a given class of activities and which is minimal with respect to the inclusion relation on the set of all sets is called a set of attributes for that class of activities.

From now on we shall call the elements of a class of actions alternatives and we shall note by $\mathcal{V}$ the set of all alternatives of a given class of actions.

Every activity leads to some changes of the environment. We consider the environment composed of objects on which the activities of a given class of actions can cause at most one type of transformation. Let

$\mathcal{O}$ denote the set of those objects of the environment which are actually changed by the activities of the elements of $\mathcal{V}$.

Let's attach to each object $o \in \mathcal{O}$ a totally ordered set $(C, \leqslant)$, the elements of which are the possible transformations of the object o.

DEFINITION 3. *By criterion we understand an ordered pair composed of an object and the attached set C.*

Actually the descomposition of the environment into objects is made such that to each attribute corresponds one criterion and only one.

Let's take a criterion (o, C) and an alternative $v \in \mathcal{V}$. One can not measure the result of the transformation on o before the activity has taken place and so it must be estimated. The result of the estimation (which is an element in C) is called the level attached to the alternative v and the criterion (o, C).

A criterion (o, C), where $C \subseteq \mathbf{R}_+$ is called a quantitative criterion. All the other criterions are called qualitative ones.

Although we consider the resolver to be a primary notion, we shall explain here what we understand by it. For us a resolver is, intuitively, an object (a person, a group of persons, a machine etc.) which has to make a choice from a set of alternatives. It must choose that alternative which is the most „convenient" to it.

Let (o, C) be a criterion and $m_1, m_2 \in C$ such that $m_1 \leqslant m_2$. If the resolver prefers the transformation having the result m_2, then we call (o, C) a maximum criterion, if not we call it a minimum criterion.

DEFINITION 4. *By unicriterial decision problem we understand a problem in which there are given a set of alternatives, a criterion and the levels attached to the considered alternatives and criterion and the choice of the most convenable alternative is demanded.*

REMARK. Solving a unicriterial decision problem is the same with solving a maximum problem, in the case of a maximum criterion, or a minimum problem, in the case of a minimum criterion.

In practice the unicriterial decision problems are very uncommon, the resolver being obliged to make its choice considering more than one criterion. These types of problems are called multicriterial ones. For these the choice of the most convenable alternative means to take in consideration, with different weights, all the levels corresponding to each alternative. Because of the heterogenous nature of the criterions it is needed a way to measure the consequence of the choice of an alternative on the whole environment. For this purpose the notion of utility has been suggested. The first mathematically rigurous definition of the utility was given by von Neumann and Morgenstern in [6].

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ denote the set of alternatives given in the decision problem. On this set we introduce two relations called the preference relation and the indifference relation noted respectively by $\mathfrak{P}$ and $\mathfrak{I}$. These relations have the following properties:

(i) *For all v_i, $v_j \in \mathcal{V}$ one and only one of the following relations holds:*
a) $v_i \mathfrak{P} v_j$ b) $v_j \mathfrak{P} v_i$ c) $v_i \mathfrak{I} v_j$;

(ii) *The preference relation is transitive, that is for all v_i, v_j, $v_k \in \mathcal{V}$*

$$(v_i \mathfrak{P} v_j) \wedge (v_j \mathfrak{P} v_k) \Rightarrow (v_i \mathfrak{P} v_k);$$

The indifference relation is transitive and symmetrical, that if sor all v_i, v_j, $v_k \in \mathcal{V}$

$$(v_i \mathfrak{I} v_j) \wedge (v_j \mathfrak{I} v_k) \Rightarrow (v_i \mathfrak{I} v_k)$$

and

$$(v_i \mathfrak{I} v_j) \Rightarrow (v_j \mathfrak{I} v_i);$$

(iii) *Besides the set of simple alternatives $\mathcal{V}$, the resolver can consider probabilistic mixtures of two simple alternatives. These mixtures are of the following form: $v' = [pv_i, (1-p)v_j]$, where p is the probability of the accomplishment of the alternative v_i and $1-p$ that of the alternative v_j;*

(iv) *For v_i, v_j, $v_k \in \mathcal{V}$ such that $v_i \mathfrak{P} v_j \mathfrak{P} v_k$, there exists the mixture $v' = [p'v_i, (1-p')v_k]$ and the mixture $v'' = [p''v_i, (1-p'')v_k]$ so that $v' \mathfrak{P} v_j$ and $v_j \mathfrak{P} v''$;*

(v) *For v_i, v_j, $v_k \in \mathcal{V}$ such that $v_i \mathfrak{P} v_j$ it results that the mixture $[pv_i, (1-p)v_k]$ is prefered to the mixture $[pv_j, (1-p)v_k]$.*

In the theory built up in the quoted book these properties are considered to be axioms. On the base of these is given the definition of the utility.

DEFINITION 5. ([3]). *Let's consider the set of alternatives $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ on which we have a preference relation ($\mathfrak{P}$) and an indiference relation ($\mathfrak{I}$). By utility we understand a mapping $u: \mathcal{V} \to \mathbf{R}_+$ for which we have:*

a) *For any v_i, $v_j \in \mathcal{V}$ from $v_i \mathfrak{P} v_j$ results $u(v_i) \geqslant u(v_j)$;*

b) *If the considered criterions are $K_j = (o_j, C_j)$, $j = \overline{1, m}$, then $u(v_i) = \sum_{j=1}^{m} u_j(v_i)$, where u_j denotes the utility mapping for the decision problem with $\mathcal{V}$ as set of alternatives and the unique criterion K_j.*

Starting from this definition the first methods for computing utilities have been worked out. These, together with the experience accumulated in this field of research led to new methods. Simultanously a new problem raised, that concerning how close the results of these methods are to the reality. One can find a collection of the best known methods for computing utilities in Fishburn's paper [4].

In many papers dealing with decision problems the authors criticise the above mentioned methods, saying that they are subjective. An explanation to this is that there hasn't been found a method of specification for the utility mappings universally valid for all decision problems. Because of this there have been worked out many methods for estimating utilities but these are mostly valid for a limited class of applications, for which they are close enaugh to the relality.

In this paper we present our opinion concerning the reasons why the above mentioned methods are subjective. Certainly the formulation of a decision problem can introduce a great amount of subjectivism because the estimation of the levels can be made in various ways and the estimation method used depends only on the person who puts the problem. We don't want to eliminate this kind of subjectivism. We tried to work out a method for specifying the utility mappings which doesn't introduce any more subjectivism.

Trying to surprise the reasons of the subjective character of estimating utilities we started from the idea that the greatest difficulty holds in determinating the dependence between the levels and their "use" (see the St. Petersburg paradox [3]). We must mention here that generally speaking there is no proportionality between them.

The starting point for our method was the following observation: every resolver estimates the "use" of an alternative not only by the corresponding levels but also by the effort needed to reach these levels. This observation led us to the conclusion that the needed effort must have an important weight in the estimation of the utilities. In this paper by effort we understand the cost of the alternative.

The method we are going to present has two stages. In the first stage a homogenization of the expressions of the levels, using the concept of proximity degree of a level to the best level*, is made. This homogenization is needed because of the existence of various measures for the different criterions (that is because $C_1, C_2, \ldots, C_n$ are generally different from each orther).

DEFINITION 6. *Let $a_1, a_2, \ldots, a_n$ be the levels corresponding to the criterion K_j and let $a^j (a^j \in \{a_1, a_2, \ldots, a_n\})$ be the best level. If the criterion K_j is a maximum one, then the proximity degree X_i of the level a_i to the best level is given by*

$$X_i = \frac{a_i}{a^j}$$

and if it is a minimum criterion it is given by

$$X_i = \frac{a^j}{a_i} \cdot$$

Evidently this way of computing the proximity degrees is possible only in the case of the quantitative criterions. This means that for the qualitative criterions another method is needed.

It is rather difficult to express how close is "good" to "very good" or "very good" to "excellent", if these are levels for a qualitative criterion. In our attempts to find a numerical expression of a proximity degree analogous to that in the case of the quantitative criterions we started from the following observations:

— The qualitative criterions can allways be considered to be maximum ones (not numerically spoken) because the levels of such a criterion are symbolic names and they can be suitably redefined;

* By best level we understand the highest level in the case of a maximum criterion and the lowest level in the case of a minimum criterion.

— Generally a resolver gives up easier "very good" for "good" than "good" for "satisfactory". The conclusion to which we came through this observation is that the method we need must give such proximity degrees that the difference between two proximity degrees corresponding to consecutive levels diminishes when the proximity degrees grow. Mathematically we could formulate this observation as in the followings.

Let $C = \{a_1, a_2, \ldots, a_n\}$ be the set of all possible levels for a qualitative criterion so that $a_i \leqslant a_{i+1}$, $i = \overline{1, n-1}$. If X is the mapping which attaches to each level its proximity degree to the best level a_n, then for all the triples $a_{i-1}, a_i, a_{i+1} \in C$, $i = \overline{2, n-1}$, the following relation must hold:

$$(*) \qquad X(a_i) - X(a_{i-1}) \geqslant X(a_{i+1}) - X(a_i).$$

We shall represent the levels $a_1, a_2, \ldots, a_n$ by points uniformly distributed on [0, 1], with o the first one (a_1) and 1 the last one (a_n). From now on, when not specified, a_i will denote the number associated to the level a_i, for $i = \overline{1, n}$.

From the mathematical analysis it is known that a mapping $X: [a, b] \rightarrow \mathbf{R}$, which is continuous and for which relation (*) holds is concave (see [1]). This leads us to search for the "proximity degree mapping" among the concave mappings.

About the mapping X we also suppose that $X(a_1) = 0$ and $X(a_n) = 1$, that is a_1 is the worst approximation of the best level and a_n is the best one.

Summarizing, the "proximity degree mapping" must be a solution of the following problem:

$$(\mathrm{P}) \qquad \begin{cases} X'(a) \leqslant 0, \text{ for all } a \in (0, 1) \\ X''(a) \geqslant 0, \text{ for all } a \in (0, 1) \\ X(0) = 0 \\ X(1) = 1 \end{cases}$$

The problem (P) has an infinitely large number of solutions. A class of soluitions it for instance that of the mappings of the form $X_p(a) = p \cdot a^2 + (1-p)a$, $p \in [-1, 0]$. If we want to take the proximity degree mapping from this class we have to choose a value for p such that the obtained mapping X_p is as close as possible to the statistical reality.

To establish p we suggest the following method:

— for each level a_i, $i = \overline{2, n-1}$ we consider as statistical proximity degree the meanvalue of k proximity degrees estimated by k specialists, that is

$$\overline{X}_i = \frac{\sum_{j=1}^{k} X_{ij}}{k};$$

— we choose p such that the sum

$$\sum_{i=2}^{n-1} (p \cdot a_i^2 + (1-p) \cdot a_i - \overline{X}_i)^2$$

has the lowest value.

If p_0 is the value obtained, then X_{p_0} may be used as proximity degree mapping. The problem remains open in the direction of establishing the class of solutions of (P) which are the closest to the reality.

After computing the proximity degree for each level we have a uniform expression of the possible consequences for each alternative.

In the second step the utilities are computed. For this purpose we first compute the proximity degree of the effort needed by each alternative to the best (minimum) effort. The utilities are computed as products between the proximity degree of each level and the proximity degree of the effort for each alternative.

In conclusion the formula for the utility of the level a_{ij} is

$$u(a_{ij}) = \frac{e_m}{e_i} \cdot X_{ij},$$

where

e_m — is the minimum effort;

e_i — is the effort needed for the alternative v_i;

X_{ij} — is the proximity degree of the level a_{ij}.

REMARKS 1. When $e_i = e_m = 0$ and $X_{ij} = 0$ we have by convention $u(a_{ij}) = 1$ (this would suit a situation in which no effort is needed to obtain nothing);

2. When $e_i = e_m = 0$ and $X_{ij} \neq 0$ we agree to take $u(a_{ij}) = \infty$ (this would suit a situation in which something is obtained without any effort, situation which would be extremely "useful" for anyone).

3. When $e_m \neq 0$ and $X_{ij} = 0$ we have $u(a_{ij}) = 0$ (that is although an effort has been made no result is obtained, a situation which is not at all desirable).

4. In the manner we treated this problem a criterion "cost" has no sens.

In the followings we shall present the mathematical model which results from the presentation made above.

Let $\{X_i\}_{i=\overline{1,n}}$ be a family of sets considered to be attributes in the sens we specified at the beginning of the paper. An element of the set X_i will be called a value of this attribute.

DEFINITION 7. *For the given attributes* $X_1, X_2, \ldots, X_n$, *by set of alternatives we understand a subset* $\mathcal{V}$ *of the cartesian product* $X_1 \times X_2 \times \ldots \times X_n$.

Let $\{K_1, K_2, \ldots, K_m\}$ be the set of the considered criterions, where $K_i = (o_i, C_i)$, $i = \overline{1, m}$ and let's note $\mathcal{C} = C_1 \times C_2 \times \ldots \times C_m$. The set $\mathcal{N} = \{N_1, N_2, \ldots, N_s\}$ denotes the states of nature [2], [3].

DEFINITION 8. *By estimation mapping of the levels corresponding to the state* N_i *of nature we understand a mapping* $E_i : \mathcal{V} \to \mathcal{C} \times [0, 1]$, *which satisfies the following conditions:*

(i) $\sum_{i=1}^{s} \Pi_2(E_i(v)) = 1$ *for all* $v \in \mathcal{V}$;

(ii) $\Pi_2(E_i(v)) = \Pi_2(E_i(w))$ *for all* $v, w \in \mathcal{V}$,

where Π_2 *denotes the projection mapping on the second component of the cartesian product* $\mathcal{C} \times [0, 1]$.

REMARKS 1. $\Pi_2(E_i(v))$ is the probability of the accomplishment of the state N_i of nature.

2. The estimation mappings can be specified through tables of formu las. If the estimation mappings don't have second components they are called partial estimation mappings.

The estimation of the needed effort for each alternative and state of nature is made by the means of a mapping $V_j : \mathcal{V} \to \mathbf{R}_+$, called effort mapping.

DEFINITION 9. *The coeficient of importance is a mapping* $K : \{K_i\}_{i=\overline{1,m}} \to [0, 1]$ *such that* $\sum_{i=1}^{m} K(K_i) = 1$ *and which shows the weight of each criterion in the given decision problem.*

The solving of the decision problem stands in solving the following maximum problem

$$\max_{v_i \in \mathcal{V}} \sum_{l=1}^{s} \left(\sum_{j=1}^{m} u_j(v_i) \cdot K(K_j) \right) \cdot \Pi_2(E_l(v_i)),$$

Where

$$u_j(v_i) = \frac{\min_{v_i \in \mathcal{V}} V_k(v_i)}{V_k(v_i)} \cdot X_{ij}$$

for thes state N_k of nature. X_{ij} represents the proximity degree.

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Universitatea Babeş-Bolyai
Str. Kogălniceanu nr. 1 Cluj-Napoca