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A COSEPARATOR IN A CATEGORY OF CONES

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The category we shall refer to in this note will be denoted by Con_f. Let X be a locally convex space over the field $\mathbf{R}$ of real numbers.

DEFINITION 1. *$K \subseteq X$ will be called a cone in X if:*

1. *K is a closed subset of X*
2. $x, y \in K,\ \alpha, \beta \geqslant 0 \Rightarrow \alpha x + \beta y \in K$
3. $K \cap (-K) = \{\theta\}$ *(θ being the zero element of X).*

DEFINITION 2. *The cone $K \subseteq X$ is called full if int $K \neq \emptyset$.*

The objects of Con_f are the pairs (X, K) where X is a locally convex space (over $\mathbf{R}$) and $K \subseteq X$ is a full cone in X. The morphisms from (X_1, K_1) to (X_2, K_2) are those linear and continuous functions $f: X_1 \to X_2$ for which $f(K_1) \subseteq K_2$.
The composition of morphisms and the identities are defined usually.

In order to establish our central theorem, let's recall some well — known results of functional analysis, as they are to be found in [2]. Let X be a topological vector space over $\mathbf{R}$.

PROPOSITION 1. *If $Y \subseteq X$ is a convex set, then for every $x \in Y$ and $y \in$ int Y and every $\alpha \in [0,1[$ we have $\alpha x + (1-\alpha)y \in$ int Y.*

COROLLARY 1. *If $Y \subseteq X$ is a convex set, then int Y is also a convex set.*

THEORME 1. (Hahn—Banach). *If $Y \subseteq X$ is an open, non — empty convex set and $Z \subseteq X$ is a linear subspace with $Y \cap Z = \emptyset$, then there is a linear and continuous functional $x^* : X \to \mathbf{R}$ for which $x^*(z) = 0$, $\forall z \in Z$ and $x^*(y) > 0$, $\forall y \in Y$.*

We now establish two lemmas which will be helpful to the proof of the main theorem.

Let X be a locally convex space over $\mathbf{R}$, $X \neq \{\theta\}$. Then there is $\mathcal{B}(0)$ a basis of the neighbourhoods of the origin of X, each set of $\mathcal{B}(0)$ being convex, absorbent and equilibrated. Then, for every $x \in X$, $\mathcal{B}(x) = \{x + B | B \in \mathcal{B}(0)\}$ is a basis of the neighbourhoods of x, each $B' \in \mathcal{B}(x)$ being convex.

Let's choose arbitrarily $x_0 \in X$, $B' \in \mathcal{B}(x_0)$ and denote $\mathbf{R}_+ = \{\lambda \in \mathbf{R} | \lambda \geqslant 0\}$.

LEMMA 1. $\mathbf{R}_+ B' = \{\lambda u | \lambda \geqslant 0, u \in B'\}$ *is a convex set.*

Proof. We choose $\lambda_1, \lambda_2 \geqslant 0$, $u_1, u_2 \in B'$ and $\alpha \in [0,1]$.

a) If $\alpha\lambda_1 = (1-\alpha)\lambda_2 = 0$, then $\alpha(\lambda_1 u_1) + (1-\alpha)\cdot(\lambda_2 u_2) = \theta \in \mathbf{R}_+ B'$

b) If $\alpha\lambda_1 + (1-\alpha)\lambda_2 > 0$, then we have

$$\alpha(\lambda_1 u_1) + (1-\alpha)(\lambda_2 u_2) =$$
$$= [\alpha\lambda_1 + (1-\alpha)\lambda_2]\left(\frac{\alpha\lambda_1}{\alpha\lambda_1 + (1-\alpha)\lambda_2} u_1 + \frac{(1-\alpha)\lambda_2}{\alpha\lambda_1 + (1-\alpha)\lambda_2} u_2\right)$$

and this element belongs to $\mathbf{R}_+ B'$ because B' is convex.

LEMMA 2. *If* $K \subseteq X$, *is a cone and* $\theta \notin K + B'$, *then* $\theta \notin int\,(K + \mathbf{R}_+ B')$

Proof. Supposing the contrary, $K + \mathbf{R}_+ B'$ would be a neighbourhood of θ, so that there is $B_1 \in \mathcal{B}(0)$ with $B_1 \subseteq K + \mathbf{R}_+ B'$. Let's choose $y \in X \setminus \{\theta\}$. B_1 being absorbent and equilibrated, there is $\alpha > 0$ such that $z = \alpha y \in B_1$ and $-z \in B_1$. But $B_1 \subseteq K + \mathbf{R}_+ B'$, so that there are $x_1, x_2 \in K$, $\lambda_1, \lambda_2 \geqslant 0$, $u_1, u_2 \in B'$ with $z = x_1 + \lambda_1 u_1$ and $-z = x_2 + \lambda_2 u_2$. From here it follows:

$$\theta = (x_1 + x_2) + \lambda_1 u_1 + \lambda_2 u_2. \tag{1}$$

If we should have $\lambda_1 = \lambda_2 = 0$, it would follow that $x_1 + x_2 = \theta$, so $x_2 = -x_1$. As $x_1, x_2 \in K$ and K is a cone, it follows that $x_1 = \theta$, so $z = \theta$, which is not true. Thus $\lambda_1 + \lambda_2 > 0$ and (1) implies:

$$\theta = \frac{1}{\lambda_1 + \lambda_2}(x_1 + x_2) + \frac{\lambda_1}{\lambda_1 + \lambda_2} u_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} u_2. \tag{2}$$

As K is a cone and B' is convex, (2) leeds to $\theta \in K + B'$, which is contrary to our hypothesis.

We are now able to formulate our main result.

THEOREM 2. *The category* Con_f *has a coseparator.*

Proof. We show that the object $(\mathbf{R}, [0, \infty[)$ is a coseparator in Con_f. Let (X', K') and (X, K) be objects of Con_f and $f, g : (X', K') \to (X, K)$ two distinct morphisms of Con_f. Then there is $x'_0 \in X'$ with $x_0 = f(x'_0) - g(x'_0) \neq \theta$. Thus $X \neq \{\theta\}$, so the preceeding lemmas are appliable to X.

Two situations are possible:

a) $x_0 \in K$. Then, as $x_0 \neq \theta$ and K is a cone, we have $x_0 \notin (-K)$. But $(-K)$ is a closed set, so that there is $B' \in \mathcal{B}(x_0)$ with

$$B' \cap (-K) = \emptyset. \tag{3}$$

Supposing $\theta \in K + B'$, it would follow that there are $x \in K$ and $u \in B'$ with $\theta = x + u$, so that $u = -x \in B' \cap (-K)$, which contradicts (3). Thus $\theta \notin K + B'$. Applying lemma 2, we have $\theta \notin int\,(K + \mathbf{R}_+ B')$.

b) $x_0 \notin K$. K being closed, there is $B'' \in \mathcal{B}(x_0)$ with $B'' \cap K = \emptyset$. $(-K)$ being also a cone, we obtain in the same manner as in the a) case that $\theta \notin int\,((-K) + \mathbf{R}_+ B'')$.

As we are in the a) or b) case we denote $Y = int\,(K + \mathbf{R}_+ B')$ or $Y = int\,((-K) + \mathbf{R}_+ B'')$. Due to lemma 1 and corollary 1, Y is convex (K is evidently convex). We saw before that $\theta \notin Y$, so taking $Z = \{\theta\}$ we have $Z \cap Y = \emptyset$.

Denoting $B = B'$ (in the a) case) or $B = B''$ (in the b) case), B is a neighbourhood of x_0, thus $x_0 \in int\,B \subseteq int\,(\mathbf{R}_+ B) \subseteq Y$, the last inclusion following from $\theta \in K$. So $Y \neq \emptyset$ and Y is clearly open.

All the conditions of theorem 1 being satisfied, there is a linear and continuous functional $x^* : X \to \mathbf{R}$ with $x^*(y) > 0$, $\forall y \in Y$.

If we are in the a) case we put $h = x^*$ and we have $h(int\,K) = x^*(int\,K) \subseteq x^*(Y) \subseteq]0, \infty[$.

If the case is b), we take $h = -x^*$ and similarly we have $h\,(int\,K) = -x^*(int\,K) = x^*\,(int\,(-K)) \subseteq x^*(Y) \subseteq]0, \infty[$.

Thus, in both cases $h\,(int\,K) \subseteq]0, \infty[$ and $h : X \to \mathbf{R}$ is linear and continuous.

K being a full cone, there is $z_0 \in int\,K$. We show that $h(x) \geqslant 0$, $\forall x \in K$.

If $x \in K$, proposition 1 implies $\alpha x + (1-\alpha)z_0 \in int\,K$, $\forall \alpha \in [0,1[$. Thus $h(\alpha x + (1-\alpha)z_0) > 0$, or $\alpha h(x) + (1-\alpha)h(z_0) > 0$, $\forall \alpha \in [0,1[$. Taking the limit when $\alpha \to 1$, we obtain $h(x) \geqslant 0$.

As a conclusion, $h \in Hom_{Con_f}((X, K), (\mathbf{R}, [0, \infty))$.

A hittle before we had $x_0 \in Y$, thus $x^*(x_0) > 0$.

So $h(x_0) \neq 0$, that is $h(f(x'_0) - g(x'_0)) \neq 0$, or $h \circ f \neq h \circ g$, which completes the proof.

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