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ON THE ORDER OF STARLIKENESS OF CONVEX FUNCTIONS OF ORDER α

by

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Let f be a regular function in the unit disc $U = \{|z| < 1\}$, $f(0) = 0$ and $f'(0) = 1$. If

$$\mathrm{Re}\,\frac{z f''(z)}{f'(z)} + 1 > \alpha, \text{ for } z \in U,$$

then f is called a *convex function of order* α, while if

$$\mathrm{Re}\,\frac{z f'(z)}{f(z)} > \alpha, \text{ for } z \in U,$$

then f is called a *starlike function of order* α. The classes of such functions shall be denoted respectively by $K(\alpha)$ and $S^*(\alpha)$. We note that $\alpha < 1$ and will only be interested in the case $\alpha \geqslant 0$. If $\alpha \in [0,1)$, then $K(\alpha) \subset K(0) = K$, the class of convex functions, while $S^*(\alpha) \subset S^*(0) = S^*$, the class of starlike functions.

For a given $\alpha \in [0,1)$, the *order of starlikeness* of convex funoticns of order α is defined by the largest number $\beta = \beta(\alpha)$ so that $K(\alpha) \subset S^*(\beta)$. It is well-known that $\beta(0) = 1/2$, according to the classical result due to A. MARX [3] and E. STROHHÄCKER [4]. The problem of finding $\beta(\alpha)$, for all $\alpha \in [0, 1)$, was recentely solved by V. A. ZMOROVICH and I. K. KOROBKOVA, by using a variational technique [6].

Another Yay to obtain $\beta(\alpha)$ is to use the following result due to T. H. MACGREGOR [2]: if $f \in K(\alpha)$, $\alpha \in [0, 1)$, then zf'/f is subordinate to q, where

$$(1)\qquad q(z)=\begin{cases}(2\alpha-1)\dfrac{z}{(1+z)^{2-2\alpha}[(1+z)^{2\alpha-1}-1]}, & \alpha\neq\dfrac{1}{2}\\[2mm] \dfrac{z}{(1+z)\log(1+z)}, & \alpha=\dfrac{1}{2}.\end{cases}$$

This result implies

$$\beta(\alpha)=\inf_{|z|<1}\operatorname{Re}q(z)=\min_{|z|=1}\operatorname{Re}q(z).$$

According to I. S. JACK [1], MacGregor conjectured that this infimum occurs at $z=1$, i.e.

$$\beta(\alpha)=\min_{|z|=1}\operatorname{Re}q(z)=q(1)=\begin{cases}\dfrac{2\alpha-1}{2-2^{2(1-\alpha)}}, & \alpha\neq\dfrac{1}{2}\\[2mm] \dfrac{1}{2\log 2}, & \alpha=\dfrac{1}{2}.\end{cases}$$

D. R. WILKEN and J. FENG have recentely proved this conjecture, by using a very ingenious device [5].

In this paper we give a direct and elementary proof of the above conjecture, by showing a little more, namely that $z=1$ is the unique point of minimum. Incidentaly we obtain some interesting trigonometric inequalities.

THEOREM. *If* $\alpha\in(0,1)$ *and* q *is given by* (1) *then* $\min_{|z|=1}\operatorname{Re}q(z)$ *occurs if and only if* $z=1$.

Proof. Let

$$R(t)=\operatorname{Re}q(e^{it}),\qquad t\in(-\pi,\pi).$$

For $\alpha\in(0,1)$, we shall show that

$$(2)\qquad R(t)>R(0)=q(1)=\beta(\alpha),\ \text{for all}\ t\in(-\pi,\pi),\ t\neq 0.$$

Since $q(\bar z)=\overline{q(z)}$, it is sufficient to suppose $t\in(0,\pi)$.

First consider the case $\alpha\neq 1/2$. From (1) we deduce

$$R(t)=\frac{2\alpha-1}{N(t)}\{1+\cos t-\cos\alpha t[2(1+\cos t)]^{1-\alpha}\},$$

where

$$N(t)=2(1+\cos t)-2[\cos\alpha t+\cos(1-\alpha)t][2(1+\cos t)]^{1-\alpha}+$$
$$+[2(1+\cos t)]^{2(1-\alpha)}$$

Therefore we have

$$(3)\qquad R(t)-R(0)=\frac{(2\alpha-1)M(t)}{[2-2^{2(1-\alpha)}]N(t)}$$

where

$$M(t)=-2^{2(1-\alpha)}(1+\cos t)-[2(1+\cos t)]^{2(1-\alpha)}+[2^{2(1-\alpha)}\cos\alpha t+$$
$$+2\cos(1-\alpha)t][2(1+\cos t)]^{1-\alpha}.$$

We can write

$$N(t)=\{1+\cos t-[2(1+\cos t)]^{1-\alpha}\cos(1-\alpha)t\}^2+$$
$$+\{\sin t-[2(1+\cos t)]^{1-\alpha}\sin(1-\alpha)t\}^2.$$

If $N(t)=0$ for some $t\in(0,\pi)$, then $1+\cos t-[2(1+\cos t)]^{1-\alpha}=$ $=\sin t-[2(1+\cos t)]^{1-\alpha}\sin(1-\alpha)t=0$ and we get $\cos\frac{t}{2}\sin\left(\alpha-\frac{1}{2}\right)t=0$. Since this last equality is not possible, we deduce $N(t)>0$, for all $t\in(0,\pi)$. Since $(2\alpha-1)/[2-2^{2(1-\alpha)}]>0$, it remains to show that $M(t)>0$, for $t\in(0,\pi)$. To that end, we shall write

$$(4)\qquad M(t)=[4(1+\cos t)]^{1-\alpha}P(t)$$

where

$$(5)\qquad P(t)=2^{1-\alpha}\cos\alpha t+2^{\alpha}\cos(1-\alpha)t-(1+\cos t)^{1-\alpha}-(1+\cos t)^{\alpha}$$

and we shall show that $P(t)>0$, for all $t\in(0,\pi)$.

Since P is invariant when α is replaced by $1-\alpha$, we can suppose $1/2<\alpha<1$. If we let $x=2\alpha-1$, then we have $x\in(0,1)$ and $\cos[(1-x)t/2]>\cos(t/2)>[\cos(t/2)]^{1+x}$. Hence the inequality $P(t)>0$ is equivalent to

$$(6)\qquad F(t)=2^{x}-\frac{\left(\cos\frac{t}{2}\right)^{1-x}t-\cos\frac{1+x}{2}t}{\cos\frac{1-x}{2}t-\left(\cos\frac{t}{2}\right)^{1+x}}>0,\ t\in(0,\pi),\ x\in(0,1).$$

A simple calculation yields

$$(7)\qquad F'(t)=\frac{\sin(xt/2)(1-[\cos(t/2)]^{2x})\,G(t)}{2[\cos(t/2)]^{x}(\cos[(1-x)t/2]-[\cos(t/2)]^{1+x})^2},$$

where

$$(8)\qquad G(t)=\frac{1+[\cos(t/2)]^{2x}-2\cos(xt/2)[\cos(t/2)]^{x}}{1-[\cos(t/2)]^{2x}}-x$$

and

(9)

$$G'(t)=\frac{x\sin\frac{1-x}{2}t}{\left(\cos\frac{t}{2}\right)^{1-x}\left(1-\left[\cos\frac{t}{2}\right]^{2x}\right)^2}\left[\left(\cos\frac{t}{2}\right)^{x}-\frac{\sin\frac{1+x}{4}t}{\sin\frac{1-x}{4}t}\right]\left[\left(\cos\frac{t}{2}\right)^{x}-\frac{\cos\frac{1+x}{4}t}{\cos\frac{1-x}{4}t}\right].$$

If we set

$$H(x) = x \log [\cos (t/2)] - \log\left[\cos \frac{1+x}{4} t\right] + \log\left[\cos \frac{1-x}{4} t\right],$$

for a fixed $t \in (0, \pi)$, then

$$H'(x) = \log [\cos (t/2)] + \frac{t}{4}\left(\operatorname{tg} \frac{1+x}{4} t - \operatorname{tg} \frac{1-x}{4} t\right)$$

and

$$H''(x) = \frac{t^2}{16} \frac{\left(\cos \frac{1-x}{4} t\right)^2 - \left(\cos \frac{1+x}{4} t\right)^2}{\left(\cos \frac{1-x}{4} t\right)^2 \left(\cos \frac{1+x}{4} t\right)^2} > 0, \ x \in (0, 1).$$

Since $H(0) = H(1) = 0$ and $H(x)$ is convex in $(0,1)$, we conclude that $H(x) < 0$ for $x \in (0,1)$, which yields

$$\left(\cos \frac{t}{2}\right)^x < \frac{\cos \frac{1+x}{4} t}{\cos \frac{1-x}{4} t}.$$

Since

$$\left(\cos \frac{t}{2}\right)^x < 1 < \frac{\sin \frac{1+x}{4} t}{\sin \frac{1-x}{4} t},$$

from (9) we deduce $G'(t) > 0$, for $t \in (0, \pi)$. By using $\cos u = 1 - u^2/2 + + \ldots$ in (8), we easily obtain $G(0+) = 0$, so that $G(t) > 0$, for $t \in (0, \pi)$. Therefore from (7) we deduce $F'(t) > 0$, which shows that $F(t)$ given by (6) is an increasing function in $(0, \pi)$, for all $x \in (0,1)$. From (6) we get $F(0+) = \frac{3+x}{3-x} L(x)$, where $L(x) = \frac{3-x}{3+x} 2^x - 1$. Since $L'(x) = 2^x [\log 8 - - 6 - x^2 \log 2](3 + x)^{-2}$, the equation $L'(x) = 0$ has a unique root $x_0 \in \in (0,1)$. Moreover we have $L'(x) > 0$, for $x \in (0, x_0)$ and $L'(x) < 0$, for $x \in (x_0, 1)$. Since $L(0) = L(1) = 0$, we deduce $L(x) > 0$, for $x \in (0,1)$, which implies $F(0+) > 0$. Therefore $F(t) > 0$ and we conclude that $P(t) > 0$, for all $t \in (0, \pi)$, where P is given by (5). From (4) we get $M(t) > 0$ and from (3) we obtain $R(t) - R(0) > 0$, for all $t \in (0, \pi)$ and $\alpha \in (0,1)$, $\alpha \neq 1/2$, which implies (2).

Let $\alpha = 1/2$. From (1) we get

$$R(t) = \operatorname{Re} q(e^{it}) = \frac{1}{2} \frac{u \operatorname{tg} u + \log (2 \cos u)}{u^2 + \log^2(2 \cos u)}, \text{ where } u = \frac{t}{2} \in \left(0, \frac{\pi}{2}\right),$$

so that

$$R(t) - R(0) = \frac{I(u)}{(\log 4)[u^2 + \log^2 (2 \cos u)]}$$

where

$$I(u) = (\log 2) u \operatorname{tg} u - u^2 + (\log 2) \log (2 \cos u) - \log^2(2 \cos u)$$

and

$$I'(u) = J(u) \operatorname{tg} u$$

where

$$J(u) = u \left(\frac{\log 2}{\sin u \cos u} - 2 \operatorname{cotg} u\right) + 2 \log (2 \cos u).$$

Hence

$$J'(u) = 2 \frac{1 + (1 - \log 2) \cos t}{\sin^2 t} K(t)$$

where

$$K(t) = t - (2 - \log 2) \frac{\sin t}{1 + (1 - \log 2) \cos t}$$

and

$$K'(t) = (1 - \cos t) \frac{-1 + 3 \log 2 - \log^2 2 - (1 - \log 2)^2 \cos t}{[1 + (1 - \log 2) \cos t]^2}$$

Since $-1 + 3 \log 2 - \log^2 2 - (1 - \log 2)^2 \cos t \geqslant -1 + 3 \log 2 - - (1 - \log 2)^2 = (2 - \log 2)(2 \log 2 - 1)$, we successively deduce $K'(t) > 0$, $K(t) > K(0) = 0$, $J'(u) > 0$, $J(u) > J(0) = \log 8 - 2 > 0$, $I'(u) > 0$, $I(u) > > I(0) = 0$, for $u \in \left(0, \frac{\pi}{2}\right)$, hence $R(t) > R(0)$, for all $t \in (0, \pi)$, which implies (2).

It is easy to show that $\lim_{t \to \pi} R(t) = \infty$, for all $\alpha \in (0,1)$. This completes the proof of our theorem.

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