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SOME PROPERTIES OF SOLUTIONS OF EQUATION $\Delta^4 u = 0$ (II)

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1. In the paper [3] it was studied some properties of the functions which are solutions of equation

(1) $$\Delta^4 u = 0,$$

where Δ is Laplace's operator.

The purpose of this note is to give another alike properties, as aprioric estimates.

The notations and notions are those used in the paper [3].

2. In this section we deduce some triharmonic, biharmonic and harmonic functions in the domain D from the solutions of equation (1).

It holds the following

THEOREM 1. *If $w = w(x, y, z)$ is a solution of equation* (1) *then the function*

(2) $$W_1 = \Delta w_x - \frac{1}{4}(x - a)\Delta^2 w$$

is biharmonic, for any $(a, y, z) \in D$.

Proof. Through direct calculation one finds that

$$\Delta^2 W_1 = \Delta^2\left(\Delta w_x - \frac{1}{4}(x - a)\Delta^2 w\right) = \Delta^3 w_x - \frac{1}{4}\Delta^2[(x - a)\Delta^2 w] = 0,$$

which shows that W_1 is biharmonic function.

Thus, it holds the following

THEOREM 2. *If w is a solution of equation* (1) *then the function*

$$W_2 = \Delta w_{xx} - \frac{1}{4}\Delta^2 w - \frac{1}{2}(x-a)\,\Delta^2 w_x + \frac{1}{8}(x-a)^2\Delta^3 w \tag{3}$$

is harmonic, for any $(a, y, z) \in D$.

Proof. Through direct calculation we have

$$\Delta W_2 = \Delta^2 w_{xx} - \frac{1}{4}\Delta^3 w - \frac{1}{2}\Delta[(x-a)\,\Delta^2 w_x] + \frac{1}{8}\Delta[(x-a)^2\Delta^3 w] = 0,$$

therefore W_2 is harmonic function.

Also, we have

THEOREM 3 *If w is a solution of equation* (1) *then the function*

$$W_3 = \left|\nabla W_1 - \frac{1}{2} r \cdot \Delta W_1\right| \tag{4}$$

is subharmonic, for any $(a, y, z) \in D$.

Proof. The Theorem 5 of [1] must be applied to the biharmonic function W_1.

THEOREM 4 *If the function w is solution of equation* (1) *then the function*

$$W_4 = W_1 - 2r \cdot \nabla W_1 + \frac{1}{2} r^2 \cdot \Delta W_1 \tag{5}$$

is harmonic, for any $(a, y, z) \in D$.

Proof. The Theorem 6 of [1] can be used to the biharmonic function W_1.

THEOREM 5. *If the function w is solution of equation* (1) *then the function*

$$W_5 = \left|\nabla W_0 - \frac{1}{2} r \cdot \Delta W_0\right| \tag{6}$$

is a subharmonic function, for any $(a, y, z) \in D$, where W_0 is the function

$$W_0 = w_{xx} - \frac{1}{6}\Delta w - \frac{1}{3}(x-a)\,\Delta W_x + \frac{1}{24}(x-a)^2\Delta^2 w. \tag{7}$$

Proof. Because W_0 is the biharmonic function of Theorem 2 of [3], we can apply the Theorem 5 of [1] to it.

THEOREM 6. *If the function w is solution of equation* (1) *then the function*

$$W_6 = W_0 - 2r \cdot \nabla W_0 + \frac{1}{2} r^2 \Delta W_0 \tag{8}$$

is a harmonic function, for any $(a, y, z) \in D$.

Proof. We can apply the Theorem 6 of [1] to the biharmonic function W_0.

3. In this section we give some aprioric estimates for the functions which are solutions of equation (1).

Thus, it holds the

THEOREM 7. *Let w be a solution of equation* (1), *which have all partial derivatives until the 6-th order continuous on D and on the boundary Γ of domain D. Then, for any point $(a, b, c) \in D$ the estimate*

$$\Delta w_{xx}(a, b, c) - \frac{1}{4}\Delta^2 w\,(a, b, c) \leqslant \max_{\Gamma} W_2 \tag{9}$$

is true.

Proof. We can apply the common maximum principle for the harmonic function W_2 of Theorem 2.

Also, it holds the

THEOREM 8. *Under the assumtions of Theorem 7 the estimate*

$$|\nabla \Delta w_x(a, b, c)| \leqslant \max_{\Gamma} W_3 \tag{10}$$

is true.

Proof. We can apply the maximum principle for the subharmonic function ($\Delta u \geqslant 0$) in the case of subharmonic function W_3 of the Theorem 3.

THEOREM 9. *Under the assumptions of Theorem 7 the estimate*

$$\Delta w_x(a, b, c) \leqslant \max_{\Gamma} W_4 \tag{11}$$

is true.

Proof. We can apply the common maximum principle in the case of harmonic function W_4 of the Theorem 4.

THEOREM 10 *Under the assumptions of Theorem 7 the estimate*

$$\left|\nabla\left[w_{xx}(a, b, c) - \frac{1}{6}\Delta w(a, b, c)\right]\right| \leqslant \max_{\Gamma} W_5 \tag{12}$$

is true.

Proof. We can apply the maximum principle for subharmonic functions in the case of subharmonic function W_5 of the Theorem 5.

THEOREM 11 *Under the assumptions of Theorem 7 the estimate*

$$w_{xx}(a, b, c) - \frac{1}{6}\Delta w(a, b, c) \leqslant \max_{\Gamma} W_6 \tag{13}$$

is true.

Proof. We can apply the common maximum principle in the case of harmonic function W_6 of the Theorem 6.

REMARK. The Theorem 11 is just the Theorem 8 of [3] in another words.

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