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MONOTONY OF WEIGHT-MEANS OF HIGHER ORDER

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1. Introduction. For $a > 0$ we denote by $L[0, a]$ the space of Lebesgue summable functions on $[0, a]$ and by $L^+[0, a]$ the subset of non-negative functions in $L[0, a]$.

Let g be a real-valued function defined on $[0, a]$, which satisfies the following conditions : (i) g is continuous on $[0, a]$, (ii) g is continuously differentiable on $[0, a]$, (iii) $g(0) = 0$ and (iv) $g'(x) > 0$ for all $x \in [0, a]$. Such a function will be called a *weight-function.*

If $f \in L[0, a]$, we define the *weight-mean* of f, (of weight g), as the function $F = A_g(f)$ given by

$$(1) \qquad F(x) = \frac{1}{g(x)} \int_0^x f(t) g'(t) dt, \text{ for } x \in \,]0, a], \text{ and } F(0) = f(0).$$

The *weight-mean of order* n of f is defined by $F_n = A_g(f) = A_g^n(F_{n-1})$, $F_1 = F$. The main result of this paper states that if $f \in L^+[0, a]$ and f is continuous and strictly increasing on a neighborhood of the origin, then F_n is increasing on $[0, a]$ for sufficiently large n. Some particular cases are considered.

2. Preliminaires. The operator $A_g : L[0, a] \to R^{[0,a]}$ defined by (1) is linear, positive and $A_g(1) = 1$, i.e., it is an *averaging operator.* In the particular case $g(x) = x$, $A_g = A$ is the well-known Cesàro operator. If $g(x) = x^\gamma$, $\gamma > 0$, the operator A_g, denoted by A_γ is the so called generalized Cesàro operator.

Since the weight-function g is strictly increasing on $[0, a]$, we can define the function $\varphi : [0, a] \times [0, 1] \to \mathbf{R}$, by

$$(2) \qquad \varphi(x, u) = g^{-1}[ug(x)], \; x \in [0, a], \; u \in [0, 1].$$

It is easy to check the following properties of this function.

(i) $0 \leqslant \varphi(x, u) \leqslant x$, for all $x \in [0, a]$ and $u \in [0, 1]$;

(ii) $\varphi(., u)$ is continuous and strictly increasing on $[0, a]$, for all $u \in [0, 1]$;

(iii) $\varphi(x, .)$ is continuous and strictly increasing on $[0\ 1]$, for all $x \in [0, a]$;

(iv) $\varphi(0, u) = 0$, $u \in [0, 1]$ and $\varphi(x, 0) = 0$, $x \in [0, a]$;

(v) $\varphi[\varphi(x, u), v] = \varphi(x, uv)$, for all $x \in [0, a]$ and $u, v \in [0, 1]$.

Making the substitution $t = \varphi(x, u)$ in (1) we obtain

$$F(x) = \int_0^1 f[\varphi(x, u)]du, \quad x \in [0, a]. \tag{3}$$

We note that the image by A_g of a function in $L[0, a]$ is not necessarily in $L[0, a]$. In the case $g(x) = x$ a simple counter example is given in [2]. We also remark that F is well-defined even in the case $fg' \in L[0, a]$.

If $fg' \in L[0, a]$ the function $F = A\ (f)$ is absolutely continuous on $[0, a]$ and differentiable a.e. on $[0, a]$. Moreover if f is continuous at 0, i.e., $f(0^+) = f(0)$, then from (3) it follows that F is continuous at 0, hence F is summable on $[0, a]$.

PROPOSITION 1. *If f is increasing on $[0, a]$, then $F = A_g(f)$ is increasing n $[0. a]$. Moreover, if f is continuous and strictly inreasing on $[0, a]$, then F is continuous and strictly increasing on $[0, a]$.*

P r o o f. Let $0 \leqslant x_1 < x_2 \leqslant a$. If f is increasing, from (3) we deduce

$$F(x_1) = \int_0^1 f[\varphi(x_1, u)]du \leqslant \int_0^1 f[\varphi(x_2, u)]du = F(x_2).$$

If f is continuous and strictly on $[0, a]$, then the above inequality is strict.

PROPOSITION 2. *If $f \in L[0, a]$ and $f(0^+) = f(0)$, then $F = A_g(f)$ is increasing on $[0, a]$ if and only if*

$$F \leqslant f, \quad \text{a.e. } [0, a]. \tag{4}$$

Moreover if f is continuous on $[0, a]$ then $F = A\ (f)$ is increasing (strictly increasing) on $[0, a]$ if and only if $F \leqslant f$ on $[0, a]$, ($F < f$ on $[0, a]$).

P r o o f. The function f is differentiable a.e. on $[0, a]$ and

$$F'(x) = \frac{g'(x)}{g(x)}\,[f(x) - F(x)], \quad \text{a.e. on } [0, a]. \tag{5}$$

If F is increasing then $F'(x) \geqslant 0$ a.e. on $[0, a]$ and from (5) we deduce (4).

Suppose that (4) holds and let $0 < x_1 < x_2 \leqslant a$. Since F is absolutely continuous on $[x_1, x_2]$ we get $F(x_2) - F(x_1) = \int_{x_0}^{x_2} F'(t)dt \geqslant 0$, hence F is increasing on $[0, a]$. Since $f(0^+) = f(0)$, from (3) we get $F(0^+) = f(0) = F(0)$ and we deduce that F is increasing on $[0, a]$.

If f is continuous on $[0, a]$, then F is differentiable on $[0, a]$ and (5) holds for all $x \in [0, a]$. The second part of Proposition 2 immediately follows.

In the case $g(x) = x$ some behavior properties of the Cesàro means have been examined in [1], [2] and [3].

3. Weight-means of higher order. Let $f \in L[0, a]$ and let $F_1 = A_g(f)$. Inductively we define the *weight-mean of order n* of f on $[0, a]$ by $F_n = A_g(F_{n-1})$, provided, of course, that F_{n-1} is summable on $[0, a]$.

D e f i n i t i o n. The function $f \in L[0, a]$ is in the class $M_g[0, a]$ provided f possesses weight-means of all orders on $[0, a]$.

We note that if $f \in L[0, a]$ and f is continuous at 0, i.e., $f(0^+) = f(0)$, then $f \in M_g[0, a]$.

If $f \in M_g[0, a]$ and $n > 1$, then $F_n = A_g(f)$ is differentiable on $[0, a]$ and we have

$$F_n'(x) = \frac{g'(x)}{g(x)}\,[F_{n-1}(x) - F_n(x)], \quad \text{for all } x \in [0, a],$$

which shows that F is continuously differentiable on $[0, a]$.

We shall use the following integral representation of the weight-means of higher order, which generalizes that gven in [2] for the case $g(x) = x$.

PROPOSITION 3. *If $f \in M_g[0, a]$, then*

$$F_n(x) = \int_0^1 f[\varphi(x, u)]k_n(u)du, \quad x \in [0, a], \quad n = 1, 2, \ldots, \tag{6}$$

where φ is given by (2) and

$$k_n(u) = \frac{(-1)^{n-1}}{(n-1)!}(\ln u)^{n-1}.$$

P r o o f. For $n = 1$ one obtains formula (3). By induction we have

$$F_{n+1} = \int_0^1 F_n[\varphi(x, u)]du = \int_0^1 du \int_0^1 f[\varphi(\varphi(x, u), v]k_n(v)dv =$$

$$= \int_0^1 du \int_0^1 f[\varphi(x, uv)]k_n(v)dv = \int_0^1 du \int_0^u f[\varphi(x, y)]k_n\left(\frac{y}{u}\right)\frac{dy}{u} =$$

$$= \int_0^1 f[\varphi(x, y)]dy \int_y^1 k_n\left(\frac{y}{u}\right)\frac{du}{u} = \frac{(-1)^n}{n!}\int_0^1 f[\varphi(x, y)](\ln y)^n dy =$$

$$= \int_0^1 f[\varphi(x, u)]k_{n+1}(u)du.$$

PROPOSITION 4. [2]. *The kernel* k_n, $n = 1, 2, \ldots$ *has the following properties:*

(i) $k_n(u) \geqslant 0$, $n = 1, 2, \ldots$

(ii) $\int_0^1 k_n(u)du = 1$, $n = 1, 2, \ldots$

(iii) $\lim_{n\to\infty} k_n(u) = 0$. *The convergence is uniform on every interval* $[\varepsilon, 1]$, $\varepsilon > 0$.

(iv) $k_n(u)$ *is nonincreasing on* $]0, 1[$ *for each* n.

THEOREM 1. *If* $f \in M_g[0, a]$ *and* $f(0^+)$ *exists and is finite, then the sequence* $F_n = A_g^n(f)$ *converges uniformely to* $f(0^+)$ *on* $[0, a]$.

Proof. Let $x \in [0, a]$ and $\varepsilon > 0$. We choose $\delta > 0$ such that $0 < u \leqslant \delta \Rightarrow |f[\varphi(x, u)] - f(0^+)| \leqslant \varepsilon$.

From (6) and Proposition 4 we deduce

$$|F_n(x) - f(0^+)| \leqslant \int_0^1 |f[\varphi(x, u)] - f(0^+)|k_n(u)du \leqslant$$

$$\leqslant \varepsilon \int_0^\delta k_n(u)du + k_n(\delta)\int_\delta^1 |f[\varphi(x, u)] - f(0^+)|du \leqslant$$

$$\leqslant \varepsilon + k_n(\delta)\int_\delta^1 |f[\varphi(x, u)] - f(0^+)|du.$$

Since $\lim_{n\to\infty} k_n(\delta) = 0$ and the last integral is bounded, as a function of x, there exists an $n_0(\varepsilon)$, independent of x, such that

$$x \in [0, a],\ n > n_0 \Rightarrow |F_n(x) - f(0^+)| \leqslant 2\varepsilon$$

and Theorem 1 follows.

For $g(x) = x$ Theorem 1 was proved in [2].

4. Increasing weight-means of higher order. We shall show that if the nonnegative function f is summable on $[0, a]$ and it is continuous

and strictly increasing on a neighborhood of the origin, then $F_n = A_g^n(f)$ is increasing on $[0, a]$, for sufficiently large n. More precisely we have the following result

THEOREM 2. *Let* $f \in L^+[0, a]$ *and suppose there exists an* $\varepsilon \in]0, a[$ *such that* f *is continuous and strictly increasing on* $[0, \varepsilon]$. *Then the weight-mean of order* n *of* f, $F_n = A_g^n(f)$, *is increasing on* $[0, a]$ *for all* $n \geqslant n_0$, *where*

(7) $$n_0 = n_0(\varepsilon, a) = 1 + \frac{M}{m}\ln(1/\delta),$$

(8) $$M = F_1(\varepsilon),\quad m = \min_{x\in[\varepsilon_1, \varepsilon]}[F_1(x) - F_2(x)]$$

$$\delta = \frac{g(\varepsilon)}{g(a)},\quad \varepsilon_1 = \varphi(\varepsilon, \delta).$$

Proof. By Proposition 1 we deduce that $F_1 = A_g(f)$ is continuous and strictly increasing on $[0, \varepsilon]$ and Proposition 2 shows that

$$f(x) - F_1(x) > 0, \text{ for all } x \in]0, \varepsilon[.$$

By induction we get

(9) $F_{n-1}(x) - F_n(x) > 0$, for all $x \in]0, \varepsilon]$ and $n = 1, 2, \ldots$ $(F_0 = f)$.

Since for $n > 1$ the function F_n is continuous on $[0, a]$, by using Proposition 2, we deduce that F_n is increasing on $[0, a]$ if and only if

(10) $$F_{n-1}(x) - F_n(x) \geqslant 0, \text{ for all } x \in]0,a].$$

By noting (9) we see that it suffices to show that (10) holds for all $n \geqslant n_0$, and all $x \in [\varepsilon, a]$.

Let $x \in [\varepsilon, a]$. From (6) we deduce

(11) $$F_{n-1}(x) - F_n(x) = \int_0^1 f[\varphi(x, u)][k_{n-1}(u) - k_n(u)]du$$

Since

$$k_{n-1}(u) - k_n(u) = k_{n-1}(u)\left[1 + \frac{\ln u}{n-1}\right]$$

formula (11) can be written as follows

(12) $$F_{n-1}(x) - F_n(x) = \int_0^1 f[\varphi(x, u)]k_{n-1}(u)\left[1 + \frac{\ln u}{n-1}\right]du.$$

Let $\delta = \delta(\varepsilon) = g(\varepsilon)/g(a)$, i.e., $\varphi(a, \delta) = \varepsilon$. We have

(13) $$x \in [\varepsilon, a],\ u \in [0, \delta] \Rightarrow 0 \leqslant \varphi(x, u) \leqslant \varepsilon.$$

We shall write (11) in the form

(14) $$F_{n-1}(x) - F_n(x) = \int_0^\delta f[\varphi(x,u)][k_{n-1}(x) - k_n(u)]du + $$
$$+ \int_\delta^1 f[\varphi(x, u)][k_{n-1}(u) - k_n(u)]du.$$

On the other hand we have

(15) $$\int_0^u f[\varphi(x, v)dv = u\int_0^1 f[\varphi(x, ut)]dt = uF_1[\varphi(x, u)]$$

and

(16) $$k_n'(u) = -\frac{1}{u} k_n(u).$$

Integrating by parts, and using (15) and (16), we obtain

$$\int_0^\delta f[\varphi(x, u)]k_n(u)du = \delta k_n(\delta)F_1[\varphi(x, \delta)] + \int_0^\delta F_1[\varphi(x, u)]k_{n-1}(u)du,$$

hence

(17) $$\int_0^\delta f[\varphi(x, u)][k_{n-1}(u) - k_n(u)]du =$$
$$= \int_0^\delta f[\varphi(x, u)] - F_1[\varphi(x, u)]k_{n-1}(u)du + \frac{\delta \ln \delta}{n-1} k_{n-1}(\delta)F_1[\varphi(x, \delta)].$$

From (9), by using (13), we deduce

$f[\varphi(x, u)] - F_1[\varphi(x, u)] > 0$, for all $x \in [\varepsilon, a]$ and $u \in]0, \delta]$.

Since $k_{n-1}(u)$ is nonincreasing, using (15), we get

(18) $$\int_0^\delta f[\varphi(x, u)] - F_1[\varphi(x, u)]k_{n-1}(u)du \geqslant$$
$$\geqslant \delta k_{n-1}(\delta)\{F_1[\varphi(x, \delta)] - F_2[\varphi(x, \delta)]\}.$$

Let

$$M = \max_{x\in[\varepsilon, a]} F_1[\varphi(x, \delta)] = \max_{x\in[\varepsilon_1, \varepsilon]} F_1(x) = F_1(\varepsilon)$$

and

$$m = \min_{x\in[\varepsilon, a]} \{F_1[\varphi(x, \delta)] - F_2[\varphi(x, \delta)]\} = \min_{x\in[\varepsilon_1, \varepsilon]} [F_1(x) - F_2(x)].$$

where $\varepsilon_1 = \varphi(\varepsilon, \delta)$, i.e., $g(\varepsilon_1) = \delta g(\varepsilon)$.

Since F_n is continuous, from (9) we obtain $m > 0$. We also have $M > m$.

From (17) and (18) we deduce

$$\int_0^\delta f[\varphi(x, u)][k_{n-1}(u) - k_n(u)]du \geqslant \delta k_{n-1}(\delta)\left[m - \frac{M}{n-1}\ln\frac{1}{\delta}\right].$$

Hence the first integral in (14) is nonnegative if $n \geqslant 1 + \frac{M}{m}\ln\frac{1}{\delta}$. On the other hand, from (12) we obtain

$$\int_\delta^1 f[\varphi(x, u)][k_{n-1}(u) - k_n(u)]du = \int_\delta^1 f[\varphi(x, u)]k_{n-1}(u)\left[1 + \frac{\ln u}{n-1}\right]du \geqslant$$
$$\left[1 - \frac{\ln(1/\delta)}{n-1}\right]\int_\delta^1 f[\varphi(x, u)]k_{n-1}(u)du,$$

which shows that the second integral in (14) is nonnegative if $n \geqslant 1 + \ln(1/\delta)$. Since $M > m$ we deduce that the inequality (10) holds, that is, F_n is increasing, for all $n \geqslant n_0$, where n_0 is given by (7). This completes the proof of Theorem 2.

Remarks. 1) In Theorem 2 condition no f to be strictly increasing on a neighborhood of the origin is essential. Thus, if we take $g(x) = x$, i.e.,

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x \in]0, a], \ F(0) = f(0), \ a > 1$$

and consider the function

$$f(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & x \in]1, a], \end{cases}$$

then

$$F_n(x) = \begin{cases} 1, & x \in [0, 1] \\ \frac{1}{x} S_{n-1}(\ln x), & x \in]1, a], \end{cases}$$

where $S_n(y) = 1 + \frac{y}{1!} + \dots + \frac{y^n}{n!}$.

For $x \in]1, a]$, we have

$$F_n'(x) = -\frac{(\ln x)^{n-1}}{(n-1)!\,x^2} < 0,$$

which shows that F_n is decreasing for all n.

2) If f is bounded below on $[0, a]$ the requirement of being non-negative can be removed by considering the sum of f with a suitably chosen constant.

COROLLARY 1. *Let $f \in L^+[0, a]$ and assume there exists an $e \in]0, a]$ such that $f(x) = [g(x)]^\lambda$, $\lambda > 0$, for $x \in [0, \varepsilon]$. Then $F_n = A_g^n(f)$, is increasing on $[0, a]$ for all $n \geq n_0$, where*

$$(19) \qquad n_0 = n_0(\varepsilon, a) = 1 + \frac{\lambda+1}{\lambda}\frac{1}{\delta^\lambda}\ln\frac{1}{\delta}, \quad \delta = \frac{g(\varepsilon)}{g(a)}.$$

P r o o f. The function g^λ, $\lambda > 0$, is strictly increasing and for $x \in [0, \varepsilon]$ we have

$$F_1(x) = \frac{1}{\lambda+1} g^\lambda(x), \quad F_2(x) = \frac{1}{(\lambda+1)^2} g^\lambda(x).$$

From (8) we deduce

$$M = \frac{1}{\lambda+1} g^\lambda(\varepsilon) \quad \text{and} \quad m = \frac{\lambda}{(\lambda+1)^2} g^\lambda(\varepsilon_1) = \frac{\lambda}{(\lambda+1)^2}\delta^\lambda g^\lambda(\varepsilon),$$

hence

$$\frac{M}{m} = \frac{\lambda+1}{\lambda}\frac{1}{\delta^\lambda}, \quad \delta = \frac{g(\varepsilon)}{g(a)}$$

and (7) becomes (19). Corollary 1 follows immediately from Theorem 2.

COROLLARY 2. *Let $f \in L^+[0, a]$ and assume there exists an $\varepsilon \in]0, a]$ such that g/g' is increasing on $]0, \varepsilon]$ and $f(x) = x + g(x)/g'(x)$, for $x \in]0, \varepsilon]$, $f(0) = 0$. Then $F_n = A_g^n(f)$ is increasing on $[0, a]$ for all $n \geq n_0$, where*

$$(20) \qquad n_0 = \eta_0(\varepsilon, a) = 1 + \frac{\varepsilon g^2(\varepsilon)}{g(a)\int_0^{\varepsilon_1} g(t)\,dt}\ln\frac{g(a)}{g(\varepsilon)}, \quad g(\varepsilon_1) = \frac{g^2(\varepsilon)}{g(a)}.$$

P r o o f. The function f is strictly increasing, on $[0, \varepsilon]$. It is easy to show that $F_1(x) = x$ for $x \in [0, \varepsilon]$. If we let $h = F_1 - F_2$, then

$$h(x) = x - \frac{1}{g(x)}\int_0^x t g'(t)\,dt = \frac{1}{g(x)}\int_0^x g(t)\,dt, \quad \text{for } x \in]0, \varepsilon]$$

By the mean-value theorem of Cauchy there exists a $\xi \in]0, x[$ such that $h(x) = g(\xi)/g'(\xi)$. Since g/g' is increasing, we deduce

$$(21) \qquad h(x) \leq \frac{g(x)}{g'(x)}, \quad \text{for } x \in]0, \varepsilon].$$

On the other hand we have

$$h'(x) = 1 - \frac{g'(x)}{g(x)}h(x)$$

and from (21) we obtain $h'(x) \geq 0$ for $x \in]0, \varepsilon]$, which shows that h is increasing on $]0, \varepsilon]$.

Therefore in (8) we have $M = F_1(\varepsilon) = \varepsilon$ and $m = h(\varepsilon_1) = \frac{1}{g(\varepsilon_1)}\int_0^{\varepsilon_1} g(t)\,dt$, where $g(\varepsilon_1) = \delta g(\varepsilon) = \frac{g^2(\varepsilon)}{g(a)}$. Hence (7) becomes (20) and Corollary 2 follows from Theorem 2.

5. Some particular cases. 1°. Let $g(x) = x^\gamma$, $\gamma > 0$. In this case $\delta = \left(\frac{\varepsilon}{a}\right)^\gamma$ and (19) becomes

$$(22) \qquad n_0 = n_0(\varepsilon, a) = 1 + \frac{\lambda+1}{\lambda}\frac{1}{\delta^\lambda}\ln\frac{1}{\delta} = 1 + \gamma\frac{\lambda+1}{\lambda}\left(\frac{a}{\varepsilon}\right)^{\lambda\gamma}\ln\frac{\varepsilon}{a}.$$

For $\gamma = 1$, we have

$$(23) \qquad n_0 = 1 + \frac{\lambda+1}{\lambda}\frac{1}{\delta^\lambda}\ln\frac{1}{\delta}, \quad \delta = \frac{\varepsilon}{a}.$$

If we take in (23) $\lambda = \frac{1}{2}$, we obtain

$$n_0 = 1 + \frac{3}{\sqrt{\delta}}\ln\frac{1}{\delta}.$$

For $\delta = \frac{1}{2}$ we get $n_0 = 3{,}94\ldots$ and from Corollary 1 we deduce that if $f \in L^+[0, a]$ and $f(x) = \sqrt{x}$, for $x \in \left[0, \frac{a}{2}\right]$ then $A^n(f)$, i.e., the Cesàro mean of order n of f is increasing on $[0, a]$, for all $n \geq 4$.

For $\delta = \frac{2}{3}$ we get $n_0 = 2{,}48\ldots$ and from Corollary 1 we deduce that if $f \in L^+[0, a]$ and $f(x) = \sqrt{x}$, for $x \in [0, 2a/3]$ then the Cesàro mean of order n of f, is increasing on $[0, a]$ for all $n \geq 3$.

For $\delta = 3/4$ we get $n_0 = 1{,}99\ldots$ and from Corollary 1 we deduce that if $f \in L^+[0, a]$ and $f(x) = \sqrt{x}$, for $x \in [0, 3a/4]$ then the Cesàro mean of order n of f, is increasing on $[0, a]$ for all $n \geq 2$.

If we take in (23) $\lambda = 1$, we obtain

$$n_0 = 1 + \frac{2}{\delta} \ln \frac{1}{\delta}.$$

For $\delta = 1/2$, $\delta = 2/3$ and $\delta = 3/4$ we obtain respectively $n_0 = 3{,}77\ldots$, $n_0 = 2{,}21\ldots$ and $n_0 = 1{,}76\ldots$ and the above conclusions remain valid if on the intervals $[0, a/2]$, $[0, 2a/3]$, $[0, 3a/4]$ respectively we take $f(x) = x$.

If $\gamma = 1/2$ and $\lambda = 1$ from (22) we obtain

$$n_0 = 1 + \frac{1}{\sqrt{\delta}} \ln \frac{1}{\delta}$$

For $\delta = 1/2$ we have $n_0 = 1{,}98\ldots$ and from Corollary 1 we deduce that if $f \in L^+[0, a]$ and $f(x) = \sqrt{x}$, for $x \in [0, a/2]$, then $A^n_{1/2}(f)$, i.e., the generalized Cesàro mean of order n of f, with $\gamma = 1/2$, is increasing on $[0, a]$, for all $n \geqslant 2$.

2°. Let $g(x) = x/(1 + x)$, $x \in [0, a]$. In this case (19) becomes

$$n_0 = n_0(\varepsilon, a) = 1 + \frac{\lambda + 1}{\lambda} \left(\frac{a}{\varepsilon}\right)^{\lambda} \left(\frac{1 + \varepsilon}{1 + a}\right)^{\lambda} \ln \frac{a(1 + \varepsilon)}{\varepsilon(1 + a)}.$$

If $\lambda = 1$ and $a/\varepsilon = 2$, from Corollary 1 we deduce that if $f \in L^+[0, a]$ and $f(x) = x/(1 + x)$ for $x \in [0, a/2]$, then the weight — mean of order n, $F_n = A^n_g(f)$, with $g(x) = x/(1 + x)$ is increasing on $[0, a]$ for $n \geqslant 1 + 2\left(\frac{a + 2}{a + 1}\right) \ln \frac{a + 2}{a + 1}$.

3°. Let $g(x) = \sin x$, $x \in [0, a]$, $a \leqslant \pi/2$. In this case we have

$$\int_0^{\varepsilon_1} g(t)dt = 1 - \cos \varepsilon_1 = \frac{\sin^4 \varepsilon}{\sin^2 a \left(1 + \sqrt{1 - \frac{\sin^4 \varepsilon}{\sin^2 a}}\right)}$$

and (20) becomes

$$n_0 = n_0(\varepsilon, a) = 1 + \frac{\varepsilon \sin a}{\sin^2 \varepsilon} \left(1 + \sqrt{1 - \frac{\sin^4 \varepsilon}{\sin^2 a}}\right) \ln \frac{\sin a}{\sin \varepsilon}.$$

If $a = \pi/2$, we have

$$n_0 = 1 + \frac{\varepsilon}{\sin^2 \varepsilon} (1 + \sqrt{1 - \sin^4 \varepsilon}) \ln \frac{1}{\sin \varepsilon}.$$

For $\varepsilon = 0{,}79$ we get $n_0 = 1{,}997\ldots$ and from Corollary 2 we deduce that if $f \in L^+[0, \pi/2]$ and $f(x) = x + \operatorname{tg} x$, for $x \in [0, \varepsilon]$ with $\varepsilon \geqslant 0{,}79$, then $F_n = A^n_g(f)$, with $g(x) = \sin x$, is increasing on $[0, \pi/2]$ for all $n \geqslant 2$.

6. Increasing weight-means of a given order. If n_0 is given and we require F_n to be increasing for $n \geqslant n_0$, provided f satisfies conditions of

Corollary 1, then equation (19) determines ε. We thus obtain the following result.

THEOREM 3. *Let $n_0 \geqslant 2$ and $\lambda > 0$ be given If $f \in L^+[0, a]$ and $f(x) = [g(x)]^{\lambda}$ for $x \in [0, \varepsilon]$, with $\varepsilon \geqslant \varepsilon_0 = \varphi(a, \delta_0)$, where δ_0 is the root of the equation*

(24) $$(n_0 - 1)\lambda\delta^{\lambda} + (\lambda + 1)\ln \delta = 0,$$

then $F_n = A^n_g(f)$ is increasing for all $n \geqslant n_0$.

To illustrate Theorem 3 we consider only the case $n_0 = 2$, $\lambda = 1$ and give some simple examples. In this case equation (24) becomes

$$\delta + 2 \ln \delta = 0$$

and we get $\delta_0 = 0{,}70346\ldots$.

1°. If $g(x) = x^{\gamma}$, $\gamma > 0$, we obtain $\varepsilon_0 = \delta_0^{1/\gamma} a$. For $\gamma = 1/2$ we obtain $\varepsilon_0 = ka$, where $k = \delta_0^2 = 0{,}49866\ldots$.

2°. If $g(x) = \ln(1 + x)$, then $\varepsilon_0 = (1 + a)^{\delta_0} - 1$ and we have the following values of ε_0 for certain values of a:

a	1	2	3	4	5	6
ε_0	0,628 ...	1,165 ...	1,651 ...	2,102 ...	2,526...	2,930...

3°. If $g(x) = x + \alpha x^2$, $\alpha \geqslant 0$, we have $\varphi(x, u) = 2ux(1 + \alpha x)/(\sqrt{1 + 4\alpha ux(1 + \alpha x)} + 1)$, hence

$$\varepsilon_0 = \frac{2\delta_0 a(1 + \alpha a)}{\sqrt{1 + 4\alpha\delta_0 a(1 + \alpha a)} + 1}$$

Putting $\varepsilon_0 = ka$, we remark that $\delta_0 \leqslant k \leqslant \sqrt{\delta_0}$ for all $a > 0$ and $\alpha \geqslant 0$. In our particular case ($n_0 = 2$, $\lambda = 1$) we have $0{,}70346\ldots \leqslant k \leqslant 0{,}83872\ldots$.

4°. If $g(x) = \sin x$, $a \in \,]0, \pi/2]$, then $\varepsilon_0 = \arcsin(\delta_0 \sin a)$ For $a = \pi/2$ we get $\varepsilon_0 = \arcsin \delta_0 = 0{,}78026\ldots$. Since $\varepsilon_0 < \pi/4 = 0{,}785\ldots$, in this particular case Theorem 3 yields the following result: If $f \in L^+[0, \pi/2]$ and $f(x) = \sin x$ for $x \in [0, \pi/4]$, then $F_n = A^n_g(f)$, with $g(x) = \sin x$, is increasing on $[0, \pi/2]$ for all $n \geqslant 2$.

5°. If $g(x) = \operatorname{sh} x$, we have $\varphi(x, u) = \ln[u \operatorname{sh} x + \sqrt{1 + u^2 \operatorname{sh}^2 x}]$ hence $\varepsilon_0 = \ln[\delta_0 \operatorname{sh} a + \sqrt{1 + \delta_0^2 \operatorname{sh}^2 a}] = \varepsilon_0(a)$. We remark that $\lim_{a\to\infty} [a - \varepsilon_0(a)] = ln(1/\delta_0) = 0{,}35173\ldots$.

6°. If $g(x) = x/(1 + x)$, we have $\varphi(x, u) = ux/[1 + (1 - u)x]$, hence

(25) $$\varepsilon_0 = \varepsilon_0(a) = \delta_0 a/[1 + (1 - \delta_0)a].$$

For $a = 1$ we get $\varepsilon_0 = 0{,}54257\ldots$. If we require $\varepsilon_0 \leqslant a/2$, then $a \geqslant (2\delta_0 - 1)/(1 - \delta_0) = 1{,}37231\ldots$.

7°. If $g(x) = \operatorname{th} x$, we have $\varphi(x, u) = \frac{1}{2} \ln \frac{1 + u \operatorname{th} x}{1 - u \operatorname{th} x}$, hence

$$\varepsilon_0 = \varepsilon_0(a) = \frac{1}{2} \ln \frac{1 + \delta_0 \operatorname{th} a}{1 - \delta_0 \operatorname{th} a} \tag{26}$$

and we obtain the following values of ε_0 for certain values of a:

a	1	2	3	4	5
ε_0	0,598...	0,885...	0,867...	0,873...	0,874

8°. If $g(x) = \operatorname{arctg} x$, we have

$$\varepsilon_0 = \varepsilon_0(a) = \operatorname{tg}(\delta_0 \operatorname{arctg} a) \tag{27}$$

and we obtain the following values of ε_0 for certain values of a:

a	1	2	3	4	5
ε_0	0,615...	0,986...	1,206...	1,348...	1,447

We remark that in cases 6°, 7° and 8° the function g has a finite limit as $a \to \infty$ and in (25), (26) and (17) $\varepsilon_0(a)$ goes to a finite limit as $a \to \infty$.

More generally, suppose g is continuous on $[0, \infty[$, continuously differentiable on $]0, \infty[$, $g(0) = 0$ and $g'(x) > 0$ for $x \in], \infty[$. We also assume there exists a finite limit $L = \lim_{x \to \infty} g(x)$. Since $\varepsilon_0 = \varepsilon_0(a) = \varphi(a, \delta_0) = g^{-1}[\delta_0 g(a)]$ we deduce the existence of the finite limit

$$\varepsilon_0(\infty) = \lim_{a \to \infty} \varepsilon_0(a) = g^{-}(\delta_0 L).$$

In this case, from Theorem 3 we obzain the following result.

THEOREM 4. *Let $n_0 \geqslant 2$ and $\lambda > 0$ be given. Suppose f is nonnegative on $[0, \infty[$, summable on $[0, a]$ for each $a > 0$ and $f(x) = g^{\lambda}(x)$ for $x \in [0, \varepsilon]$, where $\varepsilon \geqslant \varepsilon_0(\infty) = g^{-1}(\delta_0 L)$ and δ_0 is the root of the equation (24). Then $F_n = A_g^n(f)$ is increasing on $[0, \infty[$ for all $n \geqslant n_0$.*

If $g(x) = x/(1 + x)$, from (25) we obtain $\varepsilon_0(\infty) = \delta_0/(1 - \delta_0)$ and for $n_0 = 2$, $\lambda = 1$ we get $\varepsilon_0(\infty) = 2{,}37231\ldots$.

If $g(x) = \operatorname{th} x$, from (26) we obtain $\varepsilon_0(\infty) = \frac{1}{2} \ln \frac{1 + \delta_0}{1 - \delta_0}$ and for $n_0 = 2$, $\lambda = 1$ we get $\varepsilon_0(\infty) = 0{,}87413\ldots$.

If $g(x) = \operatorname{arctg} x$, from (27) we obtain $\varepsilon_0(\infty) = \operatorname{tg}(\delta_0\pi/2)$ and for $n_0 = 2$, $\lambda = 1$ we get $\varepsilon_0(\infty) = 1{,}98932\ldots$.

REFERENCES

[1] Beckenbach, E. F., *Superadditivity inequalities*, Pacific J. Math., **14** (1964), 421–438.

[2] Boehme, T. K. *and* Bruckner, A. M., *Functions with convex means*, Pacific J. Math., **14** (1964), 1137–1149.

[3] Bruckner, A. M. *and* Ostrow, E., *Some function classes related to the class of convex functions*, Pacific J. Math., **12** (1962), 1203–1215.

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