MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
**L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION**
**Tome 11, N° 1—2, 1982, pp. 139—145**

# ESTIMATES OF THE DEGREE OF COMONOTONE INTERPOLATING POLYNOMIALS

by

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**1.** Let there be given $m \in \mathbf{N}$, $m \geqslant 1$ and the points:

(x) $$0 = x_0 < x_1 < \ldots < x_m = 1,$$

(y) $$0 = y_0, y_1, \ldots, y_m \in \mathbf{R}.$$

We suppose that $\Delta_i(y) = y_i - y_{i-1} \neq 0$, $i = 1, 2, \ldots, m$.

S. W. YOUNG [1] has proved the existence of a polynomial $P$ satisfying the following two conditions:

(1) $$P(x_i) = y_i, \; i = 0, 1, \ldots, m$$

(2) $$P'(x)\Delta_i(y) > 0, \; x \in \,]x_{i-1}, x_i[, \; i = 1, 2, \ldots, m.$$

Such a polynomial is named comonotone interpolating polynomial.

An estimate of the degree of comonotone interpolating polynomials can be found in [2]. Other estimates have been established [3], [4], [5], [6], [7], only for the particular cases:

(3) $y_i < y_{i+1}$; $i = 0, 1, \ldots, m - 1$; or $y_i > y_{i+1}$, $i = 0, 1, \ldots, m - 1$

and

(4) $$0 = x_0 < \ldots < x_k = 1/2 < x_{k+1} < \ldots < x_m = 1,$$
$$1 \geqslant y_0 > \ldots > y_k = 0 < y_{k+1} < \ldots < y_m \leqslant 1.$$

The purpose of this note is to show how we can continue, from a constructive viewpoint, the proof of Young's existence theorem, for to obtain estimates of the degree of comonotone interpolating polynomials.

After, we show that Nikolčeva's estimate [4], [5] for the particular case (3) can be adapted for the general case.

**2.** To prove the existence of a polynomial which satisfies (1) and (2), Young constructs the following convex cone of polynomials:

(5) $$D = \{Q \in \mathfrak{P} \mid Q(x) \cdot \Delta_i(y) > 0,\ x \in ]x_{i-1},\ x_i[,\ i = 1, 2, \ldots, m\}$$

and he considers the function $F$ defined on the $[0, 1]$ — integrable rea functions set, with values in $\mathbf{R}^m$:

(6) $$F(f) = \Big(\int_0^{x_i} f(x)\mathrm{d}x\Big)_{1\le i \le m} \in \mathbf{R}^m.$$

It is obvious that $F(D) \subset \mathbf{R}^m$ is a convex cone.

After, he shows: $y = (y_1, y_2, \ldots, y_m)^T \in F(D)$.

For, he observes that

(7) $$y = \sum_{j=1}^{m} |\Delta_j(y)|\lambda_j$$

where $\lambda_j = (0, 0, \ldots, 0, \sigma_j, \ldots, \sigma_j)^T \in \mathbf{R}^m$ has the first $j-1$ components null, and $\sigma_j = \operatorname{sign} \Delta_j(y)$.
Because $\lambda_1, \lambda_2, \ldots, \lambda_m$ is a base for $\mathbf{R}^m$, $a_j = |\Delta_j(y)| > 0, j = 1, 2, \ldots, m$, and $\lambda_j \in \overline{F(D)}$, $j = 1, 2, \ldots, m$, a new base for $\mathbf{R}^m$: $\bar\lambda_1, \bar\lambda_2, \ldots, \bar\lambda_m \in F(D)$ can be found such that

(8) $$y = \sum_{j=1}^{m} (a_j + \bar a_j)\bar\lambda_j,\quad a_j + \bar a_j > 0,\quad j = 1, 2, \ldots, m,$$

and consequently $y \in F(D)$. Let $Q \in D$ be such that $y = F(Q)$, then $P(x) = \int_0^x Q(x)\mathrm{d}x$ is the required polynomial.

Let us denote by $A$ and $A + \bar A$ the matrices having $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively $\bar\lambda_1, \bar\lambda_2, \ldots, \bar\lambda_m$, as colums. If we put

$$a = (|\Delta_1(y)|, \ldots, |\Delta_m(y)|)^T,\quad a + \bar a = (a_1 + \bar a_1, \ldots, a_m + \bar a_m)^T$$

then the systems (7) and (8) can be written under the following form:

(7′) $$y = Aa,\quad a > 0$$

(8′) $$y = (A + \bar A)(a + \bar a).$$



A first problem on the systems (7′) and (8′) is that of determining the delimitation for the norm of the perturbation $\bar A$ under which (8′) has a positive solution $a + \bar a$.

To solve this problem we observe that

$$a = A^{-1}(A + \bar A)(a + \bar a)$$

or, equivalently $a = (I + A^{-1}\bar A)(a + \bar a)$, and we infer from this that

(9) $$\bar a = -(I + A^{-1}\bar A)^{-1}(A^{-1}\bar A a).$$

The existence of the inverse $(I + A^{-1}\bar A)^{-1}$ follows from a well known C. NEUMANN's theorem (see [10], theorem 2, p. 69) if the following condition holds:

(10) $$\|A^{-1}\bar A\| < 1.$$

Moreover, in this case, we have

(11) $$\|(I + A^{-1}\bar A)^{-1}\| \le \frac{1}{1 - \|A^{-1}\bar A\|}.$$

From (9) and (11) we obtain

(12) $$\|\bar a\| = \max_{1\le j\le m} |\bar a_j| \le \frac{\|A^{-1}\bar A\|}{1 - \|A^{-1}\bar A\|}\, \|a\|.$$

If we put $\alpha = \|a\| = \max_{1\le j\le m} a_j$ and $\beta = \min_{1\le j\le m} a_j > 0$, then a sufficient condition in order to have $a + \bar a > 0$ is that $\|\bar a\| < \beta$. In virtue of (12) the last relation takes place if:

$$\frac{\|A^{-1}\bar A\|}{1 - \|A^{-1}\bar A\|}\,\alpha < \beta,$$

from where we finally get

(13) $$\|A^{-1}\bar A\| < \frac{\beta}{\alpha + \beta}.$$

We remark that the relation (13) assures (10).

*Thus (13) is a sufficient condition in order that (8′) has a positive solution $a + \bar a$ (whenever (7′) has an unique positive solution $a$).*

R e m a r k. The anterior result has been directely proved, without using C. NEUMANN's theorem, in [7] (see also [4]).

**3.** The second problem on the systems (7′) and (8′) is that of constructing approximations $\bar\lambda_j \in F(D)$, of the vectors $\lambda_j \in \overline{F(D)}$, $j = 1, 2, \ldots, m$, in a such way that (13) will be satisfied.

If $Q \in D$ is a fixed polynomial and $s_j$ are the functions:

$$s_j(x) = \begin{cases} 0, & x \notin [x_{j-1}, x_j] \\ c_j, & x \in [x_{j-1}, x_j], \end{cases} \tag{14}$$

where $c_j = 1 / \int_{x_{j-1}}^{x_j} |Q|(x)\mathrm{d}x$, $j = 1, 2, \ldots, m$,

then we see that

$$\lambda_j = F(s_j Q), \quad j = 1, 2, \ldots, m. \tag{15}$$

S. W. YOUNG [1] indicates the Weierstrass's approximation theorem for to approximate „uniformly" the functions $s_j$, by positive polynomials $\bar{s}_j$; thus his approximations of the vectors $\lambda_j$ are $\bar{\lambda}_j = F(\bar{s}_j Q)$.

Clearly to construct $\bar{\lambda}_j$ it is not neccesary to approximate "uniformly" $s_j$ it suffices to approximate uniformly the functions $\int_0^x s_j(t)Q(t)\mathrm{d}t$ by comonotone polynomials. This is done by M. NIKOLCĚVA and G. ILIEV [4], [5], [6], [7] under the assumption (3) (in [6], [7] the case (4) is also studied). In the general case this idea permits to M. IVAN [8] to avoid the use of Weierstrass's theorem in the proof of the existence Young's theorem, but no estimates are made.

Next we show that the Nikolcěva's estimates on the degree of monotone interpolating polynomials can be adapted for the general case.

LEMMA (M. Nikolcěva). *If $n, k \in \mathbf{N}$, $n > 0$ are such that $1 \leqslant k \leqslant n/(21\mathrm{n}\ n)$ then there exists a polynomial, $A_{n,k}$ of degree $\leqslant 2n$ which satisfies the following three conditions:*

$$A_{n,k}(x) \geqslant 0, \quad x \in [-1, 1], \tag{16}$$

$$A_{n,k}(x) \leqslant 2e^4 n^{-2k+1}, \quad \lambda_{k,n} \leqslant |x| \leqslant 1, \quad \lambda_{k,n} = kn^{-1}\ln n. \tag{17}$$

$$\int_{-\lambda_{k,n}}^{\lambda_{k,n}} A_{n,k}(x)\mathrm{d}x \geqslant 1. \tag{18}$$

Let $\delta$ be such that

$$\lambda_{k,n} \leqslant \delta/2 < \min_{1 \leqslant j \leqslant n} \Delta_j(x)/2 \tag{19}$$

and let us denote

$$d_j = 1 / \int_{-\delta/2}^{\delta/2} A_{n,k}(t) |Q| \left(t + \frac{x_{j-1} + x_j}{2}\right)\mathrm{d}t. \tag{20}$$





We look for the comonotone interpolating polynomial in the following form:

$$P_{n,k}(x) = \sum_{j=1}^{m} (a_j + \bar{a}_j) \int_0^x d_j A_{n,k}\left(t - \frac{x_{j-1} + x_j}{2}\right) Q(t)\mathrm{d}t. \tag{21}$$

Therefore, the elements of $A + \bar{A} = (\alpha_{ij})_{1 \leqslant i, j \leqslant m}$ are

$$\alpha_{ij} = \int_0^{x_i} d_j A_{n,k}\left(t - \frac{x_{j-1} + x_j}{2}\right) Q(t)\mathrm{d}t, \quad 1 \leqslant i, j \leqslant m. \tag{22}$$

Now, if we observe that $A^{-1} = (\sigma'_{ij})_{1 \leqslant i, j \leqslant m}$, where

$$\sigma'_{ij} = \begin{cases} 0, & \text{if } i \neq j \text{ and } i \neq j + 1 \\ \sigma_i, & \text{if } i = j \\ -\sigma_i, & \text{if } i = j + 1, \end{cases}$$

then by (22) it follows that the elements of $I + A^{-1}\bar{A} = (\beta_{ij})_{1 \leqslant i, j \leqslant m}$ are

$$\beta_{ij} = \int_{x_{i-1}}^{x_i} d_j A_{n,k}\left(t - \frac{x_{j-1} + x_j}{2}\right) |Q|(t)\mathrm{d}t, \quad 1 \leqslant i, j \leqslant m. \tag{23}$$

We shall determine $n$ such that (13) will be satisfied. More precisely we shall establish the following:

THEOREM. *Let $Q \in D$, $n, k \in N$, $n > 0$. If $k, \delta, q, n$ are such that:*

$$1 \leqslant k \leqslant n/(21\mathrm{n}\ n) \tag{24}$$

$$2kn^{-1}\ln n \leqslant \delta < \min_{1 \leqslant j \leqslant m} \Delta_j(x) \tag{25}$$

$$0 < q \leqslant |Q|(x) \textit{ for any } x \in \bigcup_{i=1}^{m} \left[\frac{x_{i-1} + x_i}{2} - \frac{\delta}{2}, \frac{x_{i-1} + x_i}{2} + \frac{\delta}{2}\right] \tag{26}$$

$$\ln n > (2k - 1)^{-1} \ln\left(2e^4 \frac{\alpha + \beta}{\beta} \cdot \frac{\|Q\|}{q} \cdot m\right), \tag{27}$$

*then there are $a_j + \bar{a}_j > 0$, $j = 1, 2, \ldots, m$, such that the polynomial $P_{n,k}$ given by (21) satisfies (1) and (2), and*

$$\textit{degree } P_{n,k} \leqslant 2n + 1 + \textit{degree } Q. \tag{28}$$

Proof. Let us give estimates for $\beta_{ij}$, $1 \leqslant i, j \leqslant m$. We begin with the case $i = j$; we have

$$\beta_{ii} = \int_{-\Delta_i(x)/2}^{\Delta_i(x)/2} d_i A_{n,k}(t)|Q|\left(t + \frac{x_{i-1}+x_i}{2}\right) dt =$$

$$= \int_{-\Delta_i(x)/2}^{-\delta/2} d_i A_{n,k}(t)|Q|\left(t + \frac{x_{i-1}+x_i}{2}\right) dt +$$

$$+ \int_{-\delta/2}^{\delta/2} d_i A_{n,k}(t)|Q|\left(t + \frac{x_{i-1}+x_i}{2}\right) dt + \int_{\delta/2}^{\Delta_i(x)/2} d_i A_{n,k}(t)|Q|\left(t + \frac{x_{i-1}+x_i}{2}\right) dt$$

and hence, since by (20) the second term is equal to 1, we have on using lemma and (25),

$$|\beta_{ii} - 1| \leqslant \|Q\| \, d_i(\Delta_i(x) - \delta) 2e^4 n^{-2k+1}.$$

From (20), (26) (25) and (18) we see that $d_i \leqslant 1/q$ and so

(29) $$|\beta_{ii} - 1| \leqslant \frac{\|Q\|}{q} 2e^4 n^{-2k+1}.$$

To give such estimates for $\beta_{ij}$ when $i \neq j$ let write $\beta_{ij}$ in the following form:

(30) $$\beta_{ij} = \int_{x_{i-1}-(x_{j-1}+x_j)/2}^{x_i-(x_{j-1}+x_j)/2} d_j A_{n,k}(t)|Q|\left(t + \frac{x_{j-1}+x_j}{2}\right) dt.$$

If $i > j$ we have

$$\delta/2 < \Delta_{i-1}(x)/2 = x_{i-1} - \frac{x_{i-2}+x_{i-1}}{2} \leqslant x_{i-1} - \frac{x_{j-1}+x_j}{2} =$$

$$= x_i - \frac{x_{j-1}+x_j}{2} - \Delta_i(x).$$

Therefore by (30)

(31) $$\beta_{ij} \leqslant \frac{\|Q\|}{q} 2e^4 n^{-2k+1}.$$



If $i < j$ we have

$$-\delta/2 > -\Delta_{i+1}(x)/2 = x_i - \frac{x_i + x_{i+1}}{2} \geqslant x_i - \frac{x_{j-1}+x_j}{2} =$$

$$= x_{i-1} - \frac{x_{j-1}+x_j}{2} + \Delta_i(x)$$

and consequently the estimates (31) are also true.

Now (29) and (31) yield

(32) $$\|A^{-1}\bar{A}\| \leqslant \frac{\|Q\|}{q} \cdot 2e^4 n^{-2k+1} \cdot m.$$

Lastly let us observe by (32) that to have (13) it sufices to have (27). This completes the proof.

Remarks. It is obvious that we can take $Q \in D$ such that degree $Q$ will be equal to the number of monotony changes of (y).

We obtain Nikolčeva's results when (y) satisfies (3) and therefore $Q \equiv 1$.

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Received 16.XII.1981.

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