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ON THE INEQUALITIES OF POPOVICIU AND RADO

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1. Let $0 < a < b$ and let n be an integer, $n \geqslant 2$. Let $x = (x_1, \ldots, x_n)$ be such that $x_i \in [a, b]$, $i = 1, \ldots, n$. We use the following notations:

$$A_n = (x_1 + \ldots + x_n)/n, \quad G_n = (x_1 \ldots x_n)^{1/n}, \quad S_n(x) = \sum_{i<j} (x_j - x_i)^2,$$

$$\ln(x) = (\ln(x_1), \ldots, \ln(x_n)), \quad \sqrt{x} = (\sqrt{x_1}, \ldots, \sqrt{x_n}),$$

$$f_1(t) = \ln(t) + \frac{t^2}{2a^2}, \quad f_2(t) = -\ln(t) - \frac{t^2}{2b^2}, \quad f_3(t) = \exp(t) - \frac{t^2}{2}\exp(a),$$

$$f_4(t) = \frac{t^2}{2}\exp(b) - \exp(t) \quad \text{for } a \leqslant t \leqslant b.$$

The functions $f_1, \ldots, f_4$ are convex on $[a, b]$.

2. Applying the Jensen inequality to f_1 and f_2 we obtain

$$\exp\frac{S_n}{2n^2b^2} \leqslant \frac{A_n}{G_n} \leqslant \exp\frac{S_n}{2n^2a^2} \tag{1}$$

Applying the same inequality to f_3 and f_4, and substituting a by $\ln(a)$, x_i by $\ln(x_i)$, b by $\ln(b)$, we obtain

$$\frac{a}{2n^2}S_n(\ln(x)) \leqslant A_n - G_n \leqslant \frac{b}{2n^2}S_n(\ln(x)) \tag{2}$$

Cartwright and Field have proved the following inequalities (see [1]):

$$\frac{1}{2bn^2}S_n(x) \leqslant A_n - G_n \leqslant \frac{1}{2an^2}S_n(x) \tag{3}$$

Let us note that the inequality

$$\frac{a}{2n^2} S_n(\ln(x)) \leqslant \frac{1}{2bn^2} S_n(x)$$

is equivalent to

$$\frac{b}{2n^2} S_n(\ln(x)) \leqslant \frac{1}{2an^2} S_n(x)$$

Thus, if one delimitation is better in (3), then the other is better in (2), and conversely.

From (3) it follows

$$(4) \qquad 1 + \frac{S_n}{2b^2n^2} \leqslant \frac{A_n}{G_n} \leqslant 1 + \frac{S_n}{2a^2n^2}$$

From (1) and (4) we deduce

$$(5) \qquad \exp \frac{S_n}{2b^2n^2} \leqslant \frac{A_n}{G_n} \leqslant 1 + \frac{S_n}{2a^2n^2}$$

On the other hand, from

$$\frac{2}{\sqrt{b}} \leqslant \frac{\ln(x_j) - \ln(x_i)}{\sqrt{x_j} - \sqrt{x_i}} \leqslant \frac{2}{\sqrt{a}}$$

it follows

$$(6) \qquad \frac{4}{b} S_n(\sqrt{x}) \leqslant S_n(\ln(x)) \leqslant \frac{4}{a} S_n(\sqrt{x})$$

Combining (2) and (6) we obtain

$$(7) \qquad \frac{2a}{bn^2} S_n(\sqrt{x}) \leqslant A_n - G_n \leqslant \frac{2b}{an^2} S_n(\sqrt{x})$$

Concerning these inequalities, we recall here the inequalities of Kober (see [2]):

$$(8) \qquad \frac{1}{n(n-1)} S_n(\sqrt{x}) \leqslant A_n - G_n \leqslant \frac{1}{n} S_n(\sqrt{x})$$

3. Let f a be convex function on $[a, b]$, and let

$$F_n(f) = nf(A_n) - f(x_1) - \ldots - f(x_n)$$

According to Jensen inequality we have $F(f) \leqslant 0$. Vasić and Mijalković have obtained an improvement of this inequality (see [3]):

$$(9) \qquad F_n(f) \leqslant F_{n-1}(f).$$

Applying (9) to f_1 and f_2 we have

(10)

$$\left(\frac{A_{n-1}}{G_{n-1}}\right)^{n-1} \exp \frac{n-1}{2nb^2}(x_n - A_{n-1})^2 \leqslant \left(\frac{A_n}{G_n}\right)^n \leqslant \left(\frac{A_{n-1}}{G_{n-1}}\right)^{n-1} \exp \frac{n-1}{2na^2}(x_n - A_{n-1})^2$$

This is an improvement of Popoviciu's inequality (see [2]):

$$\left(\frac{A_{n-1}}{G_{n-1}}\right)^{n-1} \leqslant \left(\frac{A_n}{G_n}\right)^n$$

Applying (9) to f_3 and f_4, and substituting a by $\ln(a)$, x_i by $\ln(x_i)$, b by $\ln(b)$, we obtain

$$(11) \qquad (n-1)(A_{n-1} - G_{n-1}) + \frac{a(n-1)}{2n} \ln^2 \frac{x_n}{G_{n-1}} \leqslant n(A_n - G_n) \leqslant$$

$$\leqslant (n-1)(A_{n-1} - G_{n-1}) + \frac{b(n-1)}{2n} \ln^2 \frac{x_n}{G_{n-1}}$$

This is an improvement of Rado's inequality (see [2]):

$$(n-1)(A_{n-1} - G_{n-1}) \leqslant n(A_n - G_n)$$

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