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ON THE MULTI-VALUED METRIC PROJECTION IN NORMED VECTOR SPACES II

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Introduction

The present paper completes the results of our notes [12] and [13] which have appeared as preprints.

Let X be a normed vector space and M an arbitrary subset of X. The *metric projection* on M is the mapping $P_M : X \to 2^M$, defined by:

$$P_M(x) = \{m \in M : \| x - m \| = d(x, M)\},$$

where $d(x, M)$ is the distance from x to M. If card $P_M(x) \geq 2$, for all $x \in X \setminus M$, we say that the metric projection is *totally multi-valued* and in the special case when card $P_M(x) = \aleph_0$, for all $x \in X \setminus M$, we say that the metric projection is *countably multi-valued*. For some results on countably multi-valued metric projections see [11].

The set M is called *proximinal* if $P_M(x) \neq \emptyset$ for all $x \in X \setminus M$. If P_M is a totally multi-valued metric projection, then the set M will be called *strongly proximinal*. It is clear that every proximinal set is closed and every strongly proximinal set is proximinal, hence closed.

§1. Countably multi-valued metric projections.

In this paragraph we shall construct a normed space X containing a bounded strongly proximinal set M with P_M countably multi-valued. First, concerning strongly proximinal sets we have:

PROPOSITION 1. *If M is a strongly proximinal set of the Banach space X, then:*

$$\text{card } P_M(x) \geq \mathrm{c},$$

for all $x \in X \setminus M$.

Proof. We denote by $B(x, r)$ the closed ball of center x and radius r. Let M be a strongly proximinal set of the Banach space X. Then M is a closed set and $P_M(x) = M \cap B(x, d(x, M))$ is a closed set too, as an intersection of two closed sets. We will show that if $x \in X \setminus M$, then $P_M(x)$ does not contain isolated points.

We suppose, on the contrary, that $m_0 \in P_M(x)$ is an isolated point of $P_M(x)$, for a given $x \in X \setminus M$. Then there exists an $\varepsilon \in (0, 1)$ such that $B(m_0, \varepsilon d(x, M)) \cap P_M(x) = \{m_0\}$.

Let $x_0 = (\varepsilon/3)\ x + (1 - \varepsilon/3)\ m_0$. We have:

$$\| x - x_0 \| = \| x - (\varepsilon/3)\ x - (1 - (\varepsilon/3))\ m_0 \| = (1 - (\varepsilon/3)) \| x - m_0 \| = \\ = (1 - (\varepsilon/3))\ d(x, M) < d(x, M).$$

It follows that $x_0 \in X \setminus M$. On the other hand

$$\| x_0 - m_0 \| = \| (\varepsilon/3)\ x + (1 - (\varepsilon/3))\ m_0 - m_0 \| = (\varepsilon/3) \| x - m_0 \| = \\ = (\varepsilon/3)\ d(x, M).$$

From this follows $d(x_0, M) \leqslant (\varepsilon/3)\ d(x, M)$. Let $m \in M$. If $m \notin P_M(x)$ we have:

$$\| x_0 - m \| \geqslant | \ \| x - m \| - \| x_0 - x \| \ | = \| x - m \| - \| x_0 - x \| > \\ > d(x, M) - \| x_0 - x \| = d(x, M) - (1 - (\varepsilon/3))\ d(x, M) = (\varepsilon/3)\ d(x, M).$$

It is clear now that $m \notin P_M(x_0)$. If $m \in P_M(x) \setminus \{m_0\}$ we have:

$$\| x_0 - m \| \geqslant | \| m_0 - m \| - \| m_0 - x_0 \| | > \varepsilon\ d(x, M) - (\varepsilon/3)\ d(x, M) = \\ = (2\varepsilon/3)\ \ d(x, M) \text{ and } m \notin P_M(x_0).$$

Then, for all $m \neq m_0$, $m \in M$, we have $m \notin P_M(x_0)$ and it follows that $P_M(x_0) = \{m_0\}$ and this contradicts the fact that M is a strongly proximinal set. It follows that $P_M(x)$ is a closed set, dense in itself, i.e. a perfect set in X for all $x \in X \setminus M$. But every perfect subset of a complete metric space has the cardinality at least c (theorem 6.65, p. 72, [5]). Hence card $P_M(x) \geqslant c$, for all $x \in X \setminus M$.

In the precedent proposition, the condition „X is a Banach space" cannot be improved by „X is a normed space".

Example. Let X be the space of all real sequences $x = (x_n)_{n=1}^{\infty}$ having only a finite number of nonzero terms. With the norm:

$$\| x \| = \max_{n \in \mathrm{N}} \{ |x_n| \},$$

X is a noncomplete normed vector space. Let M be the set

$$M = \{x \in X : x_n \in \{0, 1/2^n\}\}.$$

We will show that P_M is a countably multi-valued metric projection, i.e. card $P_M(x) = \aleph_0$ for all $x \in X \setminus M$. Let be $x \in X \setminus M$. Then the terms of the sequence $x = (x_n)_{n=1}^{\infty}$ will be of the form:

$$x_n = 1/2^n, \text{ if } n = n_1,\ n_2, \ldots, n_k$$
$$x_n \in \mathbb{R} \setminus \{0,\ \ 1/2^n\} \text{ if } n = n_{k+1}, \ldots, n_r$$
$$x_n = 0 \text{ in rest.}$$

Let $m^0 = (m_n^0)_{n=1}^{\infty}$ the element of M defined by

$$m_n^0 = \begin{cases} 1/2^n \text{ if } n = n_1, \ldots, n_k, \\ \quad \text{or if } n \in \{n_{k+1}, \ldots, n_r\} \text{ and } |x_n| > |x_n - 1/2^n|, \\ 0 \quad \text{in rest.} \end{cases}$$

For every $m = (m_n)_{n=1}^{\infty} \in M$ we have:

$$\| x - m \| = \max_{n \in \mathrm{N}} \{ |x_n - m_n| \} \geqslant \max_{n \in \{n_{k+1}, \ldots, n_r\}} \{ |x_n - m_n| \} \geqslant$$
$$\geqslant \max_{n \in \{n_{k+1}, \ldots, n_r\}} \{ \min(|x_n|,\ |x_n - 1/2^n|) \} =$$
$$= \max_{n \in \{n_{k+1}, \ldots, n_r\}} \{ | x_n - m_n^0 | \} = \max_{n \in \{n_1, \ldots, n_r\}} \{ |x_n - m_n^0| \} =$$
$$= \max_{n \in \mathrm{N}} \{ | x_n - m_n^0 | \} = \| x - m^0 \|.$$

This implies that $m^0 \in P_M(x)$ and then $d(x, M) = \| x - m^0 \| > 0$.

Let $N_0 \in \mathrm{N}$ be such that $1/2^n < \| x - m^0 \|$, for all $n > N_0$. Let $N_1 = \max \{n_1, n_2, \ldots, n_r\}$ and $N_2 = \max \{N_0 + 1, N_1 + 1\}$. Let

$$M_1 = \{m^0 + (1/2^n) \quad e_n\}_{n \geqslant N_2},$$

where $e_n = (\underbrace{0, \ldots, 0, 1}_{n}, 0, \ldots)$.

It is clear that M_1 is a countable subset of M and if $m^1 = (m_n^1)_{n=1}^{\infty} \in M_1$, then:

$$\| x - m^1 \| = \max_{n \in \mathrm{N}} \{ | x_n - m_n^1 | \} = \max \{ \max_{n \geqslant N_2} |x_n - m_n^1|,\ \max_{n < N_2} |x_n - m_n^1| \} \leqslant$$
$$\leqslant \max\{1/2^{N_2}, \| x - m^0 \| \} = \| x - m^0 \|.$$

We have that $\| x - m^1 \| \leqslant \| x - m^0 \|$ and since $m^0 \in P_M(x)$, it follows that $m^1 \in P_M(x)$ for all $m^1 \in M_1$.

Finally, if $x \in X \setminus M$ we have proved that card $P_M(x) \geqslant$ card $M_1 = \aleph_0$ and card $P_M(x) \leqslant$ card $M = \aleph_0$. This implies that P_M is a countably multi-valued metric projection.

§2. Normed spaces with bounded or compact strongly proximinal sets.

As it was shown in § 1, if X is a Banach space and $M \subset X$ is a strongly proximinal set, then card $P_M(x) \geqslant c$ for all $x \in X \setminus M$ and this property is not true in a general normed space. By a result of S. B. STECKIN [15], if M is a subset of a strictly convex normed space then we have card $P_M(x) \leqslant 1$, for x in a dense subset of X. On the other hand, if X does not be a strictly convex normed space, then there exists a hyperplane H in X with card $P_H(x) \geqslant c$, for all $x \in X \setminus H$.

Accordingly, the normed space X contains a strongly proximinal subset M, if and only if, X is not strictly convex.

S. V. KONJAGIN [8] posed the problem of finding the Banach spaces X which contain bounded or compact strongly proximinal sets. He calls a strongly proximinal set, a set with the anti-uniquenness property. S. V. Konjagin has observed that in a finite dimensional space there exist no such sets. For some concrete spaces he have obtained the following results:

a) If A is a complete metric space and A_0 is a closed nowhere dense subset of A, then the Banach space of real bounded and continuous functions f on A with $f/_{A_0} = 0$, endowed with the sup norm,

1) contains a bounded strongly proximinal set if and only if card $A \geqslant \aleph_0$;

2) contains a compact strongly proximinal set if and only if card $A \geqslant \aleph_0$ and $A_0 \neq \emptyset$.

b) If (A, Σ, μ) is a positive measure space and $L^1(A, \Sigma, \mu)$ is the Banach space of integrable (classes of integrable) functions on the space (A, Σ, μ) then $L^1(A, \Sigma, \mu)$ contains a bounded strongly proximinal set if and only if the measure μ is non-atomic.

We shall give in this paragraph necessary and respectively sufficient conditions in order to a normed space contain compact and respectively bounded strongly proximinal sets.

Let X^* be the set of all continuous linear functionals on X. We say that $x^* \in X^*$ is a *support functional (exposing functional)* for a given set $M \subset X$, if there exists an $m_0 \in M$ so as to have either $x^*(m_0) \geqslant x^*(m)$ (respectively $x^*(m_0) > x^*(m)$) for every $m \in M \setminus \{m_0\}$, or $x^*(m_0) \leqslant x^*(m)$ (respectively $x^*(m_0) < x^*(m)$) for every $m \in M \setminus \{m_0\}$.

If $x^* \in X^*$ is a non-zero support functional (exposing functional) for $M \subset X$, it is clear that $\lambda x^*(\lambda \neq 0)$ is a support functional (exposing functional) for M. In the sequel by support functional (exposing functional) we understand such a functional of norm one.

The set of support functionals, respectively exposing functionals (of norm one) for the set M will be denoted by $\mathcal{S}(M)$ respectively by $\mathcal{E}(M)$. If for some $m_0 \in M$ there exists a functional $x^* \in X^*$ such that $x^*(m_0) \geqslant x^*(m)$ (respectively $x^*(m_0) > x^*(m)$), then m_0 is called a *support* (respectively an *exposed*) *point* of M. We denote by $U(X)$ the closed unit ball of X.

LEMMA 1. *If X is a normed space and M is a strongly proximinal subset of X, then:*

$$[\mathcal{S}(U(X)) \cap \mathcal{E}(M)] \cup [\mathcal{S}(M) \cap \mathcal{E}(U(X))] = \emptyset.$$

Proof. Suppose that M is a strongly proximinal subset of X, and $[\mathcal{S}(U(X)) \cap \mathcal{E}(M)] \cup [\mathcal{S}(M) \cap \mathcal{E}(U(X))] \neq \emptyset$. Then either

a) there exists $x^* \in \mathcal{S}(U(X)) \cap \mathcal{E}(M)$ or

b) there exists $x^* \in \mathcal{S}(M) \cap \mathcal{E}(U(X))$.

First, we consider the case a). From a) it follows that there exists $m_0 \in M$ such that either c) $x^*(m_0) > x^*(m)$ for every $m \in M$, $m \neq m_0$ or d) $x^*(m_0) < x^*(m)$ for every $m \in M$, $m \neq m_0$. It is enough to discuss only the case c). It follows from a), that there exists $x_0 \in U(X)$ such that $x^*(x_0) \leqslant x^*(x)$, for all $x \in U(X)$. (If we suppose that $x^*(x_0) \geqslant x^*(x)$ for all $x \in U(X)$, then $x^*(-x_0) \leqslant x^*(x)$, for all $x \in U(X)$, and $-x_0$ will be the element which we need).

Consequently

$$-x^*(x_0) = x^*(-x_0) = \sup_{x \in U(X)} |x^*(x)| = \|x^*\| = 1, \text{ so that } x^*(x_0) = -1.$$

But from $1 = |x^*(x_0)| \leqslant \|x^*\| \cdot \|x_0\| = \|x_0\| \leqslant 1$, it follows $\|x_0\| = 1$. Let now: $x_1 = m_0 - x_0$. We have:

$$\|x_1 - m\| \geqslant |x^*(x_1 - m)| = |x^*(m_0 - m) - x^*(x_0)| = x^*(m_0 - m) + 1 > 1 = \|-x_0\| = \|x_1 - m_0\|,$$

for every $m \in M$, $m \neq m_0$. Then $\|x_1 - m_0\| < \|x_1 - m\|$, for every $m \in M$, $m \neq m_0$. It follows that $P_M(x_1) = \{m_0\}$ and by hypothesis this implies that $x_1 = m_0$, i.e. $x_0 = 0$, in contradiction with $\|x_0\| = 1$.

We consider now the case b). There exists $m_0 \in M$ such that c') $x^*(m_0) \geqslant x^*(m)$ for all $m \in M$. Suppose that there exists $x_0 \in U(X)$ such that $x^*(x_0) < x^*(x)$, for every $x \in U(X) \setminus \{x_0\}$. Then as in the case a) we have $x^*(x_0) = -1$ and $\|x_0\| = 1$.

Let $x_1 = m_0 - x_0$ and $m \in M$, $m \neq m_0$. If $x^*(m_0) > x^*(m)$ then with the same proof as in the case a) we have

$$\|x_1 - m\| > \|x_1 - m_0\|.$$

If $x^*(m) = x^*(m_0)$ then, $x^*(x_1 - m) = x^*(x_1 - m_0) = 1$, and from $x_1 - m \neq x_1 - m_0 = -x_0$, it follows that $x_1 - m \notin U(X)$, since, if contrary, $1 = x^*(-x_0) > x^*(x_1 - m)$, i.e. $x^*(x_1 - m) < 1$.

Hence $\|x_1 - m\| > 1 = \|-x_0\| = \|x_1 - m_0\|$, $m \in M$, $m \neq m_0$. Then for every $m \in M$, $m \neq m_0$, if $x^*(m_0) \geqslant x^*(m)$ we have

$$\|x_1 - m\| > \|x_1 - m_0\|,$$

and $P_M(x_1) = \{m_0\}$. It follows that $x_1 = m_0$ i.e. $x_0 = 0$, in contradiction with $\|x_0\| = 1$. Then, from cases a) and b) it follows that $[\mathcal{S}(M) \cap \mathcal{E}(U(X))] \cup [\mathcal{S}(U(X)) \cap \mathcal{E}(M)] = \emptyset$.

Remarks. 1) If X is a finite dimensional Banach space then X does not contain bounded strongly proximinal sets. Indeed, if M is a bounded strongly proximinal subset of X, M is a closed set, so that it is a compact set. Then $\mathcal{S}(M) = \mathrm{Fr}\ U(X^*)$. But, it is known that in a reflexive Banach space every bounded, closed, convex set has an exposed point and $\mathcal{E}(U(X)) \neq \emptyset$. Then $\mathcal{S}(M) \cap \mathcal{E}(U(X)) = U(X^*) \cap \mathcal{E}(U(X)) = \mathcal{E}(U(X)) \neq \emptyset$, relation contradicting the condition of our lemma.

2) If in a normed space X, there exists a compact strongly proximinal subset M, then $U(X)$ contains no exposed points. If contrary, then $\mathcal{S}(M) \cap \mathcal{E}(U(X)) = U(X^*) \cap \mathcal{E}(U(X)) = \mathcal{E}(U(X)) \neq \emptyset$ and by lemma, M is not a strongly proximinal set.

The next proposition contains a sufficient condition in order to exist a bounded strongly proximinal set in a normed space. As usual, m is an *extremal point* of M means that m is not the midpoint of any segment of positive length contained in M.

PROPOSITION 2. *The unit ball $U(X)$ of a normed space is strongly proximinal if and only if $U(X)$ does not contain extremal points.*

Proof. a) Let $x_0 \in X \setminus U(X)$ and let $y_0 = x_0/\|x_0\|$. It is clear that $\|x_0 - x\| \geqslant \|x_0 - y_0\| = \|x_0\| - 1 = \lambda > 0$, for all $x \in U(X)$. Since $U(X)$ hasn't extremal points, it follows that there exist $y_1, y_2 \in U(X)$, $y_1 \neq y_2$, such that $y_0 = (y_1 + y_2)/2$ and $\|y_1\| = \|y_2\| = \|y_0\| = 1$.

Denote by $\alpha = \|y_1 - y_2\|/2$. Then $\alpha = \|y_1 - y_2\|/2 \leqslant (\|y_1\| + \|y_2\|)/2 \leqslant 1$. If $\beta = \alpha(\min\{1, \lambda\})$ then $0 < \beta \leqslant 1$ and $\beta \leqslant \alpha\lambda \leqslant \lambda$. If $z_0 = y_0 + \beta(y_1 - y_2)/2$ then we have:

$$\|z_0\| = \|y_0 - \beta(y_1 - y_2)/2\| = \|(y_1 + y_2)/2 + \beta(y_1 - y_2)/2\| = \|(1 + \beta)y_1/2 + (1 - \beta)\,y_2/2\| \leqslant 1,$$

hence $z_0 \in U(X)$. On the other hand we have:

$$\|x_0 - z_0\| = \|x_0 - y_0 - \beta(y_1 - y_2)/2\| = \|\lambda y_0 - \beta(y_1 - y_2)/2\| =$$
$$= \|\lambda(y_1 + y_2)/2 - \beta(y_1 - y_2)/2\| = \|(\lambda - \beta)\,y_1/2 + (\lambda + \beta)\,y_2/2\| =$$
$$= \lambda\,\|(\lambda - \beta)\,y_1/2\lambda + (\lambda + \beta)\,y_2/2\lambda\| \leqslant \lambda.$$

From this, it follows that $z_0 = y_0 + \beta(y_1 - y_2)/2 \in P_M(x_0)$. But $\beta \neq 0$, $y_1 - y_2 \neq 0$ implies $z_0 \neq y_0$. Then $P_M(x_0) \supset \{y_0, z_0\}$ and since x_0 was an arbitrary element of $X \setminus U(X)$, it follows that $U(X)$ is a bounded strongly proximinal set in X.

b) If we suppose that $x \in U(X)$ is an extremal point of $U(X)$ then $\|x\| = 1$. Suppose that $U(X)$ is strongly proximinal. Then for $y = 2x$ we have $\|y\| = 2$, $y \notin U(X)$ and for all $z \in U(X)$:

$$\|y - z\| = \|2x - z\| = 2\left\|x - \frac{z}{2}\right\| \geqslant 2\left(\|x\| - \frac{\|z\|}{2}\right) =$$
$$= 2\left(1 - \frac{\|z\|}{2}\right) \geqslant 2\left(1 - \frac{1}{2}\right) = 1 = \|x\| = \|y - x\|.$$

Hence $x \in P_{U(X)}(y)$. Let x_1 be an element of $P_{U(X)}(y)$, $x_1 \neq x$. Then $\|2x - x_1\| = \|2x - x\| = 1$ and it follows that $2x - x_1 \in U(X)$. But $x_1 \in U(X)$, $2x - x_1 \in U(X)$ and $x_1 \neq 2x - x_1$, accordingly $x = (x_1 + 2x - x_1)/2$. Then x isn't an extremal point. It follows that $U(X)$ is not a strongly proximinal set. Q.E.D.

In particular, if we denote by (A, Σ, μ) a positive measure space it is well known (see for instance R. B. HOLMES [6] p. 118) that $U(L^1(A, \Sigma, \mu))$ contains an extremal point if and only if Σ contains at least an atom. If A_0 is an atom of Σ then $f = \pm \chi_{A_0}/\mu(A_0)$, where χ_{A_0} is the characteristic function of A_0, is an extremal point of $U(L^1(A, \Sigma, \mu))$. Then if Σ does not contain atoms, then $U(L^1(A, \Sigma, \mu))$ contains no extremal points and hence by the preceding proposition $U(L^1(A, \Sigma, \mu))$ will be a convex, bounded strongly proximinal set. On the other hand, the unit ball of a dual space is not strongly proximinal. Indeed $U(X^*)$ is in the weak* topology a compact convex set and contains (by the Krein-Milman theorem) an extreme point.

We shall give in the sequel a generalization of the notion of exposed point.

The point $x_0 \in M$ is called a *k-exposed point* of the set M if there exist the linear independent functionals $x_1^*, x_2^*, \ldots, x_k^* \in U(X^*)$ such that:

$$\begin{aligned}
x_1^*(x_0) &\geqslant x_1^*(x) && \text{for all } x \in M\\
x_2^*(x_0) &\geqslant x_2^*(x) && \text{for all } x \in M \cap H_1\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_{k-1}^*(x_0) &\geqslant x_{k-1}^*(x) && \text{for all } x \in M \cap H_1 \cap \ldots \cap H_{k-2}\\
x_k^*(x_0) &> x_k^*(x) && \text{for all } x \in M \cap H_1 \cap \ldots \cap H_{k-1} \setminus \{x_0\},
\end{aligned}$$

where H_i are the hyperplanes of equations $x_i^*(x) = x_i^*(x_0)$, $i = 1, 2, \ldots, k - 1$. If $k = 1$, then $x_0 \in M$ is 1-exposed point if and only if it is an exposed point. If the dimension of X is at least $k + 1$, then a point $x_0 \in M$ which is a k-exposed point is also a $(k + 1)$-exposed point. It is clear that a k-exposed point is an extreme point. On the other hand there exist k-exposed points, which are not $(k - 1)$-exposed points. Such points will be called *effectively k-exposed*. We will give an example of a 2-exposed point which is not an exposed point.

If we consider in $\mathbf{R}^3$ the convex body obtained from a cylinder completed with two semi-spheres it is easy to see that a point p of the circle of contact of the cylinder with a semi-sphere is not an exposed point but it is a 2-exposed point. Here H_1 is the unique supporting plane in the point p, to this convex body and H_2, $(x_2^*(x) = x_2^*(p))$ is for instance, the plane containing the circle of contact passing through p.

For finite dimensional Banach spaces E. ASPLUND [1] has used another notion of k-exposed point. In the finite dimensional case every k-exposed, extreme point in the sense of Asplund is a point which is at most effectively $(k + 1)$-exposed in our sense.

Concerning k-exposed points and compact strongly proximinal sets we have:

PROPOSITION 3. *If the normed space X contains a compact strongly proximinal set, then $U(X)$ contains no k-exposed points, for all $k \in \mathbf{N}$.*

Proof. Let M be a compact strongly proximinal set of X. We suppose that $x_0 \in U(X)$ is a k-exposed point of $U(X)$, for a given $k \in \mathrm{N}$. Then, there exist the functionals $x_i^* \in U(X^*)$, $i = 1, \ldots, k$, such that

$$x_1^*(x_0) \geqslant x_1^*(x) \quad \text{for all } x \in U(X)$$

$$x_2^*(x_0) \geqslant x_2^*(x) \quad \text{for all } x \in U(X) \cap H_1^x$$

$$\cdots\cdots\cdots$$

$$x_k^*(x_0) > x_k^*(x) \quad \text{for all } x \in U(X) \cap H_1^x \cap \ldots \cap H_{k-1}^x \setminus \{x_0\},$$

where H_i^x are the hyperplanes $x_i^*(x) = x_i^*(x_0)$, $i = 1, \ldots, k-1$. We have $\|x_0\| = x_1^*(x_0) = 1$. Since M is a compact set it follows that the sets defined inductively by

$$M_1 = \{m \in M : x_1^*(m) = \inf_{x \in M} x_1^*(x)\},$$

$$M_2 = \{m \in M_1 : x_2^*(m) = \inf_{x \in M_1} x_2^*(x)\},$$

$$\cdots\cdots\cdots$$

$$M_k = \{m \in M_{k-1} : x_k^*(m) = \inf_{x \in M_{k-1}} x_k^*(x)\},$$

are all nonvoid and compact.

Let be $m_0 \in M_k$. We have:

$$x_1^*(m_0) \leqslant x_1^*(m) \quad \text{for all } m \in M$$

$$x_2^*(m_0) \leqslant x_2^*(m) \quad \text{for all } m \in M \cap H_1^m$$

$$\cdots\cdots\cdots$$

$$x_k^*(m_0) \leqslant x_k^*(m) \quad \text{for all } m \in M \cap H_1^m \cap \ldots \cap H_{k-1}^m.$$

Let $x_1 = m_0 - x_0$ and $m \in M$, $m \neq m_0$. If $x_1^*(m_0) < x_1^*(m)$, then $x_1^*(m - x_1) = x_1^*(m - m_0 + x_0) > x_1^*(x_0) = 1$. Hence $\|m - x_1\| \geqslant |x_1^*(m - x_1)| > 1 = \|x_0\| = \|m_0 - x_1\|$. In this case it is clear that $m \notin P_M(x_1)$. If $x_1^*(m_0) = x_1^*(m)$, then $x_1^*(m - x_1) = x_1^*(m_0 - x_1) = x_1^*(x_0)$ and it follows that $m - x_1 \in H_1^x$. If in what follows we suppose that $x_2^*(m_0) < x_2^*(m)$ for all $m \in M \cap H_1^m$ $m \neq m_0$ we have $x_2^*(m - x_1) > x_2^*(m_0 - x_1) = x_2^*(x_0)$ and since $m - x_1 \in H_1^x$, we have again $m - x_1 \notin U(X)$ so that $m \notin P_M(x_1)$. We obtain inductively that $m \notin P_M(x_1)$ if at least one inequality $x_i^*(m_0) \leqslant x_i^*(m)$ is sharp.

Suppose that $x_i^*(m_0) = x_i^*(m)$ for $i = 1, 2, \ldots, k$. This implies that $x_i^*(m - x_1) = x_i^*(m_0 - x_1) = x_i^*(x_0)$, $i = 1, \ldots, k$ i.e. $m - x_1 \in H_1^x \cap H_2^x \cap \ldots \cap H_k^x$. Now, since $m \neq m_0$, it follows that $m - x_1 \neq x_0$. But $x_k^*(m - x_1) = x_k^*(m_0 - x_1) = x_k^*(x_0) > x_k^*(x)$, for all $x \in U(X) \cap H_1^x \cap \ldots \cap H_{k-1}^x \setminus \{x_0\}$. Hence $m - x_1 \notin U(X)$, i.e. $m \notin P_M(x_1)$. Therefore in all of the cases $m \neq m_0$ implies $m \notin P_M(x_1)$. Then $P_M(x_1) = \{m_0\}$ and this contradiction shows that $U(X)$ does not contain k-exposed points.

R e m a r k. If A is a separable Hausdorff topological space and $x_1, x_2, \ldots$ is a dense sequence of points in A, if $C(A)$ is the Banach space of real bounded continuous functions defined on A, normed with the sup norm, then it is clear that the continuous linear functional on $C(A)$ given by

$$x^*(f) = \sum_{i=1}^{\infty} f(x_i)/2^i,$$

has the property that

$$x^*(e) > x^*(f),$$

for all $f \neq e$, with $\|f\| \leqslant 1$. (Here e stands for the function identically 1). This implies that e is an exposed point of $C(A)$ and by the preceding proposition $C(A)$ does not contain compact strongly proximinal sets.

The point $m_0 \in M$ is a *vertex* of M if the set of functionals which attain their supremum on M at m_0 is total over X. Recall that a subset $Y^* \subset X^*$ is *total* over X if $x^*(x) = 0$ for all $x^* \in Y^*$ implies $x = 0$. It is known that a vertex is not always an exposed point and conversely an exposed point is not always a vertex. Every vertex is an extreme point.

PROPOSITION 4. *If X is a normed vector space and $U(X)$ has a vertex, then X does not contain compact strongly proximinal sets.*

Proof. Let x be a vertex of $U(X)$. Let

$$\mathfrak{F} = \{x^* \in U(X^*) : x^*(x) = \|x\| = 1\},$$

be the set of support functionals of $U(X)$ at x. By the well-ordering theorem of Zermelo, there exists a well-ordered set I of indices such that $\mathfrak{F} = \{x_\alpha^*\}_{\alpha \in I}$. Let C be a compact set of X. Define by transfinite induction:

$$C_\alpha = \Big\{x \in \bigcap_{\beta<\alpha} C_\beta : x_\alpha^*(x) \leqslant x_\alpha^*(y),\ \forall y \in \bigcap_{\beta<\alpha} C^\beta \cap C\Big\} \cap C, \text{ for all } \alpha \in I.$$

If α_0 is the first element of I then $C_{\alpha_0} = \{x \in C : x_{\alpha_0}^*(x) \leqslant x_{\alpha_0}^*(y),\ \forall y \in C\} \subset C$ is a non-void compact set of X. If we suppose that for all $\beta < \alpha$, C_β is a non-void compact set, then the compact set $\bigcap_{\beta<\alpha} C_\beta$ is non-void. Indeed, it is clear that $\beta_1 \leqslant \beta_2 \Rightarrow C_{\beta_1} \supseteq C_{\beta_2}$ from the construction of $\{C_\alpha\}_{\alpha \in I}$ and if $\bigcap_{\beta<\alpha} C_\beta = \emptyset$ then from F. Riesz' condition of compactness it follows that there exist $\alpha_1, \alpha_2, \ldots, \alpha_n < \alpha$ such that $\bigcap_{i=1}^{n} C_{\alpha_i} = \emptyset$. But $\bigcap_{i=1}^{n} C_{\alpha_i} = C_{\alpha_j} \neq \emptyset$ where $\alpha_j = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Then, supposing that for every $\beta < \alpha$, C_β is a non-void compact set, it follows that $\bigcap_{\beta<\alpha} C_\beta$ is a non-void compact subset of C, and from the construction of C_α we have that C_α is a non-void closed (hence compact) subset of $\bigcap_{\beta<\alpha} C_\beta$. From the transfinite induction principle it follows that C_α is a non-void and compact set for all $\alpha \in I$. Moreower, if $\alpha < \beta$ then $C_\alpha \supset C_\beta$.

Let C_∞ be the non-void compact set $\bigcap_{\alpha\in I} C_\alpha$. We will show that C_∞ consists of a single element of C. Suppose that there exist $c_1, c_2 \in C_\infty$ and $c_1 \neq c_2$. We have that $x^*_\alpha(c_1) \leqslant x^*_\alpha(c_2)$ and $x^*_\alpha(c_2) \leqslant x^*_\alpha(c_1)$ for all $\alpha \in I$. Hence $x^*_\alpha(c_1 - c_2) = 0$, for all $\alpha \in I$ and since $(x^*_\alpha)_{\alpha\in I}$ is a total subset of X^* over X we have that $c_1 = c_2$.
Then C_∞ consists of a single point $c \in C$. Let be $y = c - x$. We have: $\|y - c\| = \|-x\| = 1$. On the other hand if $c' \in C$, we have:

$$\|y - c'\| \geqslant |x^*_\alpha(y - c')| = |x^*_\alpha(c - c') - x^*_\alpha(x)| =$$
$$= |x^*_\alpha(c - c') - 1|, \text{ for all } \alpha \in I.$$

If there exists an $\alpha \in I$ such that $c' \notin C_\alpha$ then let K be the non-void subset of the well-ordered set I, defined by $\alpha \in K$ if and only if $c' \notin C_\alpha$. Let α_1 be the first element of K. Then we have $c' \in \bigcap_{\beta<\alpha_1} C_\beta$ and $x^*_{\alpha_1}(c - c') < 0$. Then

$$\|y - c'\| \geqslant |x^*_{\alpha_1}(c - c') - 1| = 1 + x^*_{\alpha_1}(c' - c) > 1 = \|y - c\|.$$

Hence $c' \notin P_C(y)$. If, by contrary, $c' \in C_\alpha$ for all $\alpha \in I$ it follows that $c' = c$. Hence, $\|y - c'\| > \|y - c\| = 1$ for all $c' \in C$, $c' \neq c$ and $P_C(y) = \{c\}$; $y \in X \setminus C$. Then C is not a strongly proximinal set and the theorem follows.

Remarks. H. F. BOHNENBLUST and S. KARLIN [2] proved that the identity element of a Banach algebra is a vertex of the unit ball. By Proposition 4 we have that a Banach algebra with unit does not contain compact strongly proximinal sets. Particularly, the Banach algebra $C(T)$ of all real continuous and bounded functions on the Hausdorff topological space T, with the sup norm does not contain compact strongly proximinal sets.

Also, $L(X)$ the space of continuous linear operators on the Banach space X, does not contain compact strongly proximinal sets.

§ 3. Strongly proximinal sets and the Radon-Nikodym property.

The real Banach space X has the *Radon-Nikodym property* (see [4]) if for every countably additive X — valued map F, defined on a sigma-algebra, possessing finite variation $|F|$ there exists a Bochner $|F|$ integrable function f such that $F(A) = \int_A f \, d|F|$ for each A in F's domain.

Let M be a subset of X. A point $x \in M$ is called a *strongly exposed point* of M if there exists an $x^* \in X^*$, such that (*i*) $x^*(x) > x^*(y)$ for all $y \neq x$ in M, (*ii*) for any sequence (x_n) in M with $x^*(x_n) \to x^*(x)$, $x_n \to x$ in norm. We call the above x^* a *strongly exposing functional*.

R. R. PHELPS [10] has shown that X possesses the Radon-Nikodym property if and only if every closed, bounded, convex subset of X is the closed convex hull of its strongly exposed points.
Huff and Morris observed that if X has the Radon-Nidokym property then the set of strongly exposing functionals of a bounded, closed, convex subset is a dense G_δ set in X^*. For a more general result see K. S. LAU [9].

A wide class of Banach spaces possess the Radon-Nikodym property: the reflexive spaces, the separable duals, $L(\mu)$ with μ purely atomic, $l(\Gamma)$ with Γ an arbitrary set etc.

PROPOSITION 5. *If the real Banach space X possesses the Radon-Nikodym property, then X does not contain bounded, convex, strongly proximinal sets.*

Proof. If we suppose that M is a bounded, convex, strongly proximinal set then M is also a closed set. By the result of Huff, Morris and Lau there exist the dense G_δ sets G_1 and G_2, in X^* such that every $x^*_1 \in G_1$ is a strongly exposing functional of M and every $x^*_2 \in G_2$ is a strongly exposing functional of $U(X)$.
We have

$$G_1 \cap G_2 = \left(\bigcap_{n=1}^{\infty} G''_1\right) \cap \left(\bigcap_{m=1}^{\infty} G'''_2\right) = \bigcap_{n,m=1}^{\infty} (G''_1 \cap G'''_2),$$

with G''_1 and G'''_2 open dense subsets of X. From the Baire category theorem it follows that $G_1 \cap G_2$ is a dense subset of X^*. If $x^* \in G_1 \cap G_2$, $x^* \neq 0$ then $x^*/\|x^*\| \in \mathcal{S}(M) \cap \mathcal{S}(U(X)) \subset \mathcal{E}(M) \cap \mathcal{E}(U(X))$. By Lemma 1 it follows that in every Banach space with the Radon-Nikodym property there exists no bounded, convex, strongly proximinal set.

Moreover, there exist Banach spaces without the Radon-Nikodym property which does not contain any bounded, strongly proximinal sets. If (A, Σ, μ) is a positive measure space such that Σ contains at least an atom and μ isn't purely atomic, then $L(A, \Sigma, \mu)$ does not possess the Radon-Nikodym property and it does not contain bounded, strongly proximinal sets, by Konjagin's mentioned result.

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