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CONTINUITY OF GENERALIZED CONVEX MAPPINGS TAKING VALUES IN AN ORDERED TOPOLOGICAL LINEAR SPACE

by

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Criteria on the continuity of real-valued convex functions defined on non-empty convex subsets of topological linear spaces have been well known for a long time (see, for instance, BOURBAKI N. [1, p. 60], LAURENT P.-J. [7, pp. 333—336], COBZAŞ Ş. and MUNTEAN I. [5]), but so far very little has been done in order to extend these results to convex mappings taking values in an ordered topological linear space. This is surprising, all the more since during the last years there has been a considerable expansion in the field of convex analysis in general, and in the study of optimization problems concerning mappings with values in ordered topological linear spaces in particular.

BRECKNER W. W. [2] has dealt with continuity properties of real-valued rationally *s*-convex (respectively *s*-convex) functions defined on non-empty convex subsets of topological linear spaces. These two classes of functions arise in problems of functional analysis and are wider than the class of convex functions. The main purpose of the present paper is to make a similar study for rationally *s*-convex (respectively *s*-convex) mappings with values in an ordered topological linear space. In this way not only the want of a systematic study of the continuity properties of convex mappings with values in ordered topological linear spaces will be supplied, but also a more general theory will be obtained.

The paper is divided into three chapters. In chapter 1 we summarize the terminology used concerning ordered topological linear spaces and introduce the rationally *s*-convex mappings as well as the *s*-convex mappings. Some properties of these mappings are also pointed out here. Chapter 2 devoted to mappings which are locally majorized, precontinuous and upper semi-continuous, respectively, at an interior point of their domain.

In chapter 3 the continuity of rationally s-convex mappings is characterized by means of these three properties. Two kinds of criteria are obtained: on the one hand, local criteria for the continuity of a rationally s-convex mapping at an interior point of its domain, and, on the other hand, global criteria for the continuity of a rationally s-convex mapping on the whole interior of its domain. When the range space is taken to be the field of real numbers, the results stated by BRECKNER W. W. [2] are recovered.

Throughout the paper the following notations are used: N denotes the set of natural numbers, R denotes the set of real numbers, K denotes either the set of real numbers or the set of complex numbers, int M denotes the interior of a set M, cl M denotes the closure of a set M, and s is any real number belonging to the interval $]0,1]$.

CHAPTER 1

Rationally s-Convex Mappings

In this introductory chapter we summarize the terminology used with respect to ordered topological linear spaces and introduce the rationally s-convex mappings with values in an ordered linear space.

1.1. Fundamentals of Ordered Topological Linear Spaces. In this section some basic definitions and results from the theory of ordered topological linear spaces are recalled. For detailed information on ordered topological linear spaces we refer the reader to JAMESON G. [6] and to WONG Y.-C. and NG K.-F. [10].

By an *ordered linear space* Y we mean a real linear space Y on which there is defined a binary relation $\leqslant$ such that for all $x, y, z \in Y$ the following conditions are satisfied:

(i) $x \leqslant x$;

(ii) $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$;

(iii) $x \leqslant y$ implies $x + z \leqslant y + z$;

(iv) $x \leqslant y$ implies $ax \leqslant ay$ for all real numbers $a > 0$.

The conditions (i) and (ii) express that $\leqslant$ is an ordering, while (iii) and (iv) express the compatibility of this ordering with the linear structure of Y. Sometimes we write $y \geqslant x$ instead of $x \leqslant y$.

The simplest examples of ordered linear spaces are function spaces with the natural pointwise ordering. If Y is a real linear space of real-valued functions defined on a non-empty set T, and the linear operations are the usual pointwise ones, then the pointwise ordering of Y is defined by

$$x \leqslant y \text{ if } x(t) \leqslant y(t) \text{ for all } t \in T.$$

The *positive wedge* of an ordered linear space Y is the set Y_+ of all elements $x \in Y$ such that $o \leqslant x$, where o denotes the zero-element of Y.

It is easily seen that Y_+ is a wedge, i.e. a non-empty convex set closed under multiplication by non-negative real numbers.

If Y is an ordered linear space and x and y are elements of Y, the set

$$[x, y] = \{z \in Y : x \leqslant z \text{ and } z \leqslant y\}$$

is called the *order-interval* between x and y. Clearly, $[x, y]$ is non-empty if and only if $x \leqslant y$.

Let M be a subset of an ordered linear space Y. If there exists $y \in Y$ such that $x \leqslant y$ for all $x \in M$, then M is called *majorized*. M is said to be *order-bounded* if it is contained in some order-interval. M is said to be *full* (or *order-convex*) if $[x, y] \subseteq M$ for all $x, y \in M$.

An *ordered topological linear space* (respectively an *ordered normed linear space*) is defined to be an ordered linear space which is also a real topological linear space (respectively a real normed linear space). It should be noted that no relation is postulated between topology and ordering except that arising indirectly through their mutual relationship with the linear structure of the space. In most naturally arising examples of ordered topological linear spaces there are stronger direct relations. For instance, frequently the positive wedge is sequentially closed or has a non-empty interior. Ordered topological linear spaces with these properties will be also used in our considerations. Two other types of ordered topological linear spaces whose topology and ordering are strongly linked and which will be of frequent occurrence in our paper are those which are locally full or have the boundedness property.

An ordered topological linear space is said to be *locally full* (or *locally order-convex*) if it admits a neighbourhood base at o consisting of full sets. It can be proved that an ordered normed linear space Y is locally full if and only if there is an equivalent norm $||\cdot||$ on Y which is monotone, i.e. for which $||x|| \leqslant ||y||$ whenever $o \leqslant x \leqslant y$.

We shall say that an ordered topological linear space Y has the *boundedness property* if every order-bounded subset M of Y is bounded, i.e. for each neighbourhood V of o there exists a real number $a > 0$ such that $M \subseteq aV$. Obviously, an ordered topological linear space has the boundedness property if and only if every order-interval is bounded. In particular, every locally full ordered topological linear space has the boundedness property. The converse of this assertion is not true (see BRECKNER W. W. and ORBÁN G. [3]). However, any ordered topological linear space Y with the boundedness property and with int $Y_+ \neq \emptyset$ is locally full.

1.2. Rationally s-Convex Mappings. Throughout this section we denote by X a linear space over K, by M a non-empty subset of X, and by Y an ordered linear space.

Definition 1.2.1. *Let M be a convex set. A mapping $f: M \to Y$ is said to be rationally s-convex (respectively s-convex) if, for all rational* *(respectively real) numbers* $a > 0$ *and* $b > 0$ *with* $a + b = 1$, *the following inequality holds*

$$f(ax + by) \leqslant a^s f(x) + b^s f(y) \tag{1.2.1}$$

whenever $x, y \in M$.

For $s = 1$, the s-convexity of a mapping $f: M \to Y$ coincides with the usual property of f to be convex. Any additive mapping $f: X \to Y$ is rationally 1-convex.

It is obvious that every s-convex mapping is rationally s-convex. The converse of this property fails to be true. However, under some additional assumptions rational s-convexity implies s-convexity as the following theorem shows.

THEOREM 1.2.1. *Let* X *be a topological linear space,* M *a convex set, and* Y *an ordered topological linear space with* Y_+ *sequentially closed. Then any continuous rationally* s*-convex mapping* $f: M \to Y$ *is* s*-convex.*

Proof. Let $a > 0$ and $b > 0$ be real numbers with $a + b = 1$, and let x, y be elements of M. Choose a sequence $(a_n)_{n \in N}$ of rational numbers in the interval $]0, 1[$ which converges to a. Since

$$f(a_n x + (1 - a_n)y) \leqslant a_n^s f(x) + (1 - a_n)^s f(y) \text{ for all } n \in N,$$

we conclude from the continuity and closedness assumption that (1.2.1) holds. Hence f is s-convex. ∎

Some useful properties of the rationally s-convex mappings are given by the following propositions, which are analogous to results stated for s-convex mappings by BRECKNER W. W. and ORBÁN G. [4].

PROPOSITION 1.2.2. *If* M *is a convex set and if* $f: M \to Y$ *is a rationally* s*-convex mapping with* $s \in]0, 1[$, *then* $f(x) \geqslant o$ *for all* $x \in M$.

Proof. If x is in M, then

$$f(x) = f\left(\frac{1}{2}x + \frac{1}{2}x\right) \leqslant \frac{1}{2^s}f(x) + \frac{1}{2^s}f(x) = 2^{1-s}f(x)$$

implies $o \leqslant (2^{1-s} - 1)f(x)$. Since $2^{1-s} - 1 > 0$, we obtain $f(x) \geqslant o$. ∎

PROPOSITION 1.2.3. *Let* M *be a convex set, and let* $f: M \to Y$ *be a rationally* s*-convex (respectively* s*-convex) mapping. If* x_0 *and* x *are elements of* X *such that* $x_0 - x$ *and* $x_0 + x$ *belong to* M, *then we have*

$$\begin{aligned} -a^s[f(x_0 - x) - \theta(s)f(x_0)] &\leqslant \\ \leqslant f(x_0 + ax) - f(x_0) &\leqslant \\ \leqslant a^s[f(x_0 + x) - \theta(s)f(x_0)]& \end{aligned} \tag{1.2.2}$$

for every rational (respectively real) number $a \in [0, 1]$, *where*

$$\theta(s) = \begin{cases} 0 & \text{if } s \in]0, 1[\\ 1 & \text{if } s = 1. \end{cases} \tag{1.2.3}$$

Proof. Suppose f is rationally s-convex and a is any rational number belonging to $[0, 1]$. Taking into consideration that

$$x_0 + ax = a(x_0 + x) + (1 - a)x_0,$$

we have

$$f(x_0 + ax) - f(x_0) \leqslant a^s f(x_0 + x) + [(1 - a)^s - 1]f(x_0). \tag{1.2.4}$$

Since

$$[(1 - a)^s - 1]f(x_0) \leqslant -a^s\theta(s)f(x_0),$$

the inequality (1.2.4) implies

$$f(x_0 + ax) - f(x_0) \leqslant a^s[f(x_0 + x) - \theta(s)f(x_0)]. \tag{1.2.5}$$

On the other hand, because of the representation

$$x_0 = \frac{a}{1 + a}(x_0 - x) + \frac{1}{1 + a}(x_0 + ax),$$

it follows that

$$f(x_0) \leqslant \left(\frac{a}{1 + a}\right)^s f(x_0 - x) + \left(\frac{1}{1 + a}\right)^s f(x_0 + ax).$$

This inequality yields

$$-a^s f(x_0 - x) + [(1 + a)^s - 1]f(x_0) \leqslant f(x_0 + ax) - f(x_0). \tag{1.2.6}$$

Since

$$a^s\theta(s)f(x_0) \leqslant [(1 + a)^s - 1]f(x_0),$$

we get from (1.2.6)

$$-a^s[f(x_0 - x) - \theta(s)f(x_0)] \leqslant f(x_0 + ax) - f(x_0). \tag{1.2.7}$$

The relations (1.2.5) and (1.2.7) give together (1.2.2).

If f is s-convex, then the above considerations are obviously valid for any real number $a \in [0, 1]$. Consequently, in this case (1.2.2) holds for all real numbers $a \in [0, 1]$. ∎

CHAPTER 2

Locally Majorized Mappings, Precontinuous Mappings and Upper Semi-Continuous Mappings

In this chapter we present three classes of mappings taking values in an ordered linear (respectively ordered topological linear) space, each of them being defined by a local condition. Furthermore, we investigate whether a rationally s-convex mapping satisfying one of these conditions at an interior point of its domain satisfies the same condition at every interior point of its domain.

Throughout the chapter X denotes a topological linear space over K, M a non-empty subset of X, and Y an ordered linear space.

2.1. Locally Majorized Mappings. We start with the following definition.

Definition 2.1.1. *A mapping $f: M \to Y$ is said to be locally majorized (respectively locally order-bounded) at $x_0 \in M$ if there exists a neighbourhood U of x_0 contained in M and such that the set $\{f(x) : x \in U\}$ is majorized (respectively order-bounded).*

It is easy to verify that a mapping $f: M \to Y$ is at $x_0 \in M$:

1° locally majorized if and only if there exist a neighbourhood U of x_0 and an element $\alpha \in Y$, such that

$$U \subseteq M \text{ and } f(x) - f(x_0) \leqslant \alpha \text{ for all } x \in U;$$

2° locally order-bounded if and only if there exist a neighbourhood U of x_0 and an element $\alpha \in Y$, such that

$$U \subseteq M \text{ and } f(x) - f(x_0) \in [-\alpha, \alpha] \text{ for all } x \in U.$$

The following result demonstrates the equivalence of these concepts for rationally s-convex mappings.

THEOREM 2.1.1. *Let M be a convex set. Then for any rationally s-convex (respectively s-convex) mapping $f: M \to Y$ and any $x_0 \in M$ the following statements are equivalent:*

1° *f is locally majorized at x_0.*

2° *There exist a neighbourhood W of the origin of X and an element $\alpha \in Y$, such that $x_0 + W \subseteq M$ and*

$$f(x_0 + ax) - f(x_0) \in a^s[-\alpha, \alpha] \tag{2.1.1}$$

for every rational (respectively real) number $a \in [0, 1]$ and every $x \in W$.

3° *f is locally order-bounded at x_0.*

Proof. Assume f is locally majorized at x_0. Then there exist a neighbourhood U of x_0 and an element $\alpha_0 \in Y$, such that

$$U \subseteq M \text{ and } f(x) \leqslant \alpha_0 \text{ for all } x \in U.$$

Choose a balanced neighbourhood W of the origin of X, such that $x_0 + W \subseteq U$, and put $\alpha = \alpha_0 - \theta(s)f(x_0)$, where $\theta(s)$ is the number defined by (1.2.3). We have then $x_0 + W \subseteq M$ and

$$f(x_0 - x) - \theta(s)f(x_0) \leqslant \alpha, \qquad f(x_0 + x) - \theta(s)f(x_0) \leqslant \alpha$$

for all $x \in W$. According to (1.2.2) it follows that (2.1.1) is valid for every rational (respectively real) number $a \in [0, 1]$ and every $x \in W$. Thus 1° implies 2°.

The implications 2° $\Rightarrow$ 3° and 3° $\Rightarrow$ 1° are trivial. So our proof is complete. ■

The next theorem will be useful in our further investigation on the continuity of rationally s-convex mappings.

THEOREM 2.1.2. *Suppose M is a convex set and $f: M \to Y$ is a rationally s-convex mapping locally majorized at a point $x_0 \in M$. Then f is locally majorized at every interior point of M.*

Proof. Let y_0 be any interior point of M. We shall show that f is locally majorized at y_0.

Since f is locally majorized at x_0, there exist a neighbourhood U_0 of x_0 and an element $\alpha \in Y$, such that

$$U_0 \subseteq M \text{ and } f(x) \leqslant \alpha \text{ for all } x \in U_0.$$

Notice that

$$\lim_{n\to\infty} \left[y_0 + \frac{1}{n}(y_0 - x_0)\right] = y_0. \tag{2.1.2}$$

Hence there exists $n \in N$, such that

$$y = y_0 + \frac{1}{n}(y_0 - x_0)$$

belongs to M. Put

$$U = \frac{n}{n+1}y + \frac{1}{n+1}U_0.$$

By the convexity of M we conclude that U is contained in M. Furthermore, the set $\{f(x) : x \in U\}$ is majorized.

Indeed, if x is in U, then there exists $u \in U_0$, such that

$$x = \frac{n}{n+1}y + \frac{1}{n+1}u.$$

Therefore we have

$$f(x) \leqslant \left(\frac{n}{n+1}\right)^s f(y) + \left(\frac{1}{n+1}\right)^s f(u) \leqslant \left(\frac{1}{n+1}\right)^s [n^s f(y) + \alpha].$$

Hence $\{f(x) : x \in U\}$ is majorized, as claimed.

On the other hand, U is a neighbourhood of y_0, because we have

$$U = y_0 + \frac{1}{n+1}(U_0 - x_0).$$

Hence f is locally majorized at y_0. ■

2.2. Precontinuous Mappings. In this section we assume that Y is an ordered topological linear space.

Definition 2.2.1. *A mapping $f: M \to Y$ is said to be precontinuous at $x_0 \in M$ if, for each neighbourhood V of the origin of Y, there exist a neighbourhood U of x_0 and a real number $a > 0$, such that $U \subseteq M$ and*

$$f(x) - f(x_0) \in aV \text{ for every } x \in U. \tag{2.2.1}$$

PROPOSITION 2.2.1. *Let x_0 be an interior point of M, and let $f: M \to Y$ be a mapping for which there exists a neighbourhood U of x_0 contained in M and such that the set $\{f(x) : x \in U\}$ is bounded. Then f is precontinuous at x_0.*

Proof. Since $\{f(x) : x \in U\}$ is bounded, $\{f(x) - f(x_0) : x \in U\}$ is also bounded. Thus, for each neighbourhood V of the origin of Y, there exists a real number $a > 0$, such that (2.2.1) holds. Consequently, f is precontinuous at x_0. ■

Remark. The converse of proposition 2.2.1 is not true, not even for linear mappings. Indeed, let Y be the real linear space of all real-valued continuous functions $x : [0, +\infty[\to R$, equipped with the topology induced by the family $(p_n)_{n \in N}$, where $p_n : Y \to R$ is the semi-norm defined by

$$p_n(x) = \max \{|x(t)| : t \in [0, n]\},$$

and with the pointwise ordering. The linear mapping $f : Y \to Y$ defined by $f(x) = x$ for all $x \in Y$ is obviously continuous, and therefore precontinuous at all points of Y. However, there exists no neighbourhood U of the origin of Y such that $\{f(x) : x \in U\}$ is bounded, because the origin of the space Y possesses no bounded neighbourhoods.

THEOREM 2.2.2. *Let M be a convex set, Y a locally full ordered topological linear space, and $f: M \to Y$ a rationally s-convex mapping which is precontinuous at a point $x_0 \in M$. Then f is precontinuous at every interior point of M.*

Proof. Let y_0 be any interior point of M. We shall show that f is precontinuous at y_0.

Let V be any neighbourhood of the origin of Y. Since Y is locally full, there exists a full neighbourhood V_0 of the origin of Y, such that $V_0 \subseteq V$. Take now a balanced neighbourhood V_1 of the origin of Y with the property $V_1 + V_1 \subseteq V_0$. Since f is precontinuous at x_0, there correspond to V_1 a balanced neighbourhood W of the origin of X and a real number $a > 0$, such that $x_0 + W \subseteq M$ and

(2.2.2) $$f(x_0 + x) - f(x_0) \in aV_1 \text{ for every } x \in W.$$

On the other hand, there exists in view of (2.1.2) a natural number n such that

$$z = y_0 + \frac{1}{n}(y_0 - x_0)$$

belongs to M. Put $b_0 = 1/(n + 1)$, and choose a rational number b in the interval $]0, b_0[$ such that

$$\alpha = b^s \left[f(x_0) + \left(\frac{1 - b_0}{b_0}\right)^s f(z) - \frac{\theta(s)}{b_0^s} f(y_0) \right]$$

belongs to aV_1, where $\theta(s)$ is the number defined by (1.2.3). The set $U = y_0 + bW$ is a neighbourhood of y_0, and, by the convexity of M, we have

$$U \subseteq y_0 + b_0 W = \frac{n}{n+1} z + \frac{1}{n+1}(x_0 + W) \subseteq M.$$

We wish to show that

(2.2.3) $$f(y) - f(y_0) \in aV \text{ for every } y \in U.$$

If y is in U, it must be of the form $y = y_0 + bx$ for a suitable $x \in W$. By proposition 1.2.3, the following relation holds

(2.2.4) $$-\left(\frac{b}{b_0}\right)^s [f(y_0 - b_0 x) - \theta(s) f(y_0)] \leqslant f(y) - f(y_0) \leqslant$$
$$\leqslant \left(\frac{b}{b_0}\right)^s [f(y_0 + b_0 x) - \theta(s) f(y_0)].$$

But, from the equalities

$$y_0 - b_0 x = (1 - b_0) z + b_0 (x_0 - x),$$
$$y_0 + b_0 x = (1 - b_0) z + b_0 (x_0 + x),$$

it follows that

$$f(y_0 - b_0 x) \leqslant (1 - b_0)^s f(z) + b_0^s f(x_0 - x),$$
$$f(y_0 + b_0 x) \leqslant (1 - b_0)^s f(z) + b_0^s f(x_0 + x).$$

Hence (2.2.4) implies

(2.2.5) $$-b^s [f(x_0 - x) - f(x_0)] - \alpha \leqslant f(y) - f(y_0) \leqslant$$
$$\leqslant b^s [f(x_0 + x) - f(x_0)] + \alpha.$$

However, according to (2.2.2) and to the choice of b, we have

$$-b^s [f(x_0 - x) - f(x_0)] - \alpha \in -ab^s V_1 - aV_1 \subseteq a(V_1 + V_1) \subseteq aV_0,$$
$$b^s [f(x_0 + x) - f(x_0)] + \alpha \in ab^s V_1 + aV_1 \subseteq a(V_1 + V_1) \subseteq aV_0.$$

Since aV_0 is full, (2.2.5) implies $f(y) - f(y_0) \in aV_0 \subseteq aV$. Hence (2.2.3) is proved. Therefore f is precontinuous at y_0. ■

2.3. Upper Semi-Continuous Mappings. Throughout this section we assume that Y is an ordered topological linear space whose positive wedge has interior points. It should be noted that then the following equality holds

(2.3.1) $$Y_+ + \text{int } Y_+ = \text{int } Y_+.$$

Another property which will be used in the sequel and which is easily proved asserts that $\alpha \in \text{int}\, Y_+$ if and only if the order-interval $[-\alpha,\alpha]$ is a neighbourhood of the origin of Y.

Definition 2.3.1. *A mapping $f: M \to Y$ is said to be upper semi-continuous at $x_0 \in M$ if for each $\alpha \in \text{int}\, Y_+$ there exists a neighbourhood U of x_0 contained in M and such that*

(2.3.2) $$f(x) - f(x_0) \leqslant \alpha \text{ for every } x \in U.$$

PROPOSITION 2.3.1. *For any $x_0 \in M$ and any mapping $f: M \to Y$ the following statements are equivalent:*

1° *f is upper semi-continuous at x_0.*

2° *For each $\alpha \in \text{int}\, Y_+$ there exists a neighbourhood U of x_0 contained in M and such that*

(2.3.3) $$\alpha + f(x_0) - f(x) \in \text{int}\, Y_+ \text{ for every } x \in U.$$

3° *$(x_0,\alpha_0) \in \text{int}\, E(f)$ for each $\alpha_0 \in f(x_0) + \text{int}\, Y_+$, where $E(f)$ is defined by*

(2.3.4) $$E(f) = \{(x,\alpha) \in X \times Y : x \in \text{int}\, M,\ \alpha \in f(x) + \text{int}\, Y_+\}.$$

Proof. Suppose that statement 1° is true. Let α be any interior point of Y_+. Since $(1/2)\alpha$ belongs also to $\text{int}\, Y_+$ and f is upper semi-continuous at x_0, there exists a neighbourhood U of x_0 contained in M and such that

$$\frac{1}{2}\alpha + f(x_0) - f(x) \in Y_+ \text{ for every } x \in U.$$

Hence we obtain

$$\alpha + f(x_0) - f(x) = \left[\frac{1}{2}\alpha + f(x_0) - f(x)\right] + \frac{1}{2}\alpha \in Y_+ + \text{int}\, Y_+$$

for all $x \in U$. Thus (2.3.3) holds, in view of (2.3.1). Hence 1° implies 2°.

Suppose now that 2° is true. If α_0 lies in $f(x_0) + \text{int}\, Y_+$, then $\alpha = (1/2)\,[\alpha_0 - f(x_0)]$ belongs to $\text{int}\, Y_+$. Therefore there exists, by the hypothesis, an open neighbourhood U of x_0 contained in M and such that (2.3.3) holds. Since $\alpha + f(x_0) = \alpha_0 - \alpha$, (2.3.3) implies

(2.3.5) $$\alpha_0 - \alpha - f(x) \in \text{int}\, Y_+ \text{ for every } x \in U.$$

We show now that

(2.3.6) $$U \times [\alpha_0 - \alpha,\ \alpha_0 + \alpha] \subseteq E(f).$$

If (x,β) is in $U \times [\alpha_0 - \alpha,\ \alpha_0 + \alpha]$, then we have $x \in \text{int}\, M$ and $\beta - \alpha_0 + \alpha \in Y_+$. By (2.3.5) it follows that

$$\beta - f(x) = (\beta - \alpha_0 + \alpha) + [\alpha_0 - \alpha - f(x)] \in Y_+ + \text{int}\, Y_+ .$$

According to (2.3.1), $\beta - f(x)$ is then an interior point of Y_+. Hence (x, β) belongs to $E(f)$. So (2.3.6) is proved.

Notice that $[\alpha_0 - \alpha,\ \alpha_0+\alpha] = \alpha_0 + [-\alpha,\alpha]$. Since $[-\alpha,\alpha]$ is a neighbourhood of the origin of Y, it follows that $[\alpha_0 - \alpha,\ \alpha_0 + \alpha]$ is a neighbourhood of α_0. Therefore (2.3.6) implies $(x_0, \alpha_0) \in \text{int}\, E(f)$. Hence 2° implies 3°.

Assume 3° is true, and let α be any element of $\text{int}\, Y_+$. By our hypothesis, we have then $(x_0, \alpha_0) \in \text{int}\, E(f)$, where $\alpha_0 = \alpha + f(x_0)$. Thus there exists a neighbourhood U of x_0 such that $U \times \{\alpha_0\} \subseteq E(f)$. Consequently, we have $U \subseteq M$ and

$$\alpha_0 \in f(x) + \text{int}\, Y_+ \text{ for all } x \in U.$$

The latter relation yields (2.3.2). Hence f is upper semi-continuous at x_0. So the implication 3° ⇒ 1° is proved. ■

Obviously, any mapping $f: M \to Y$ which is upper semi-continuous at a point $x_0 \in M$ must be locally majorized at that point. This remark enables us to establish the following theorem concerning the upper semi-continuity of a rationally s-convex mapping $f: M \to Y$ on the whole interior of M.

THEOREM 2.3.2. *Let M be a convex set, Y an ordered topological linear space with the boundedness property, and $f: M \to Y$ a rationally s-convex mapping which is upper semi-continuous at a point $x_0 \in M$. Then f is upper semi-continuous at every interior point of M.*

Proof. Let y_0 be any interior point of M. We shall show that f is upper semi-continuous at y_0.

Let α be any interior point of Y_+. Since f is locally majorized at x_0, it follows by applying theorem 2.1.2, that f is locally majorized at y_0 too. In view of theorem 2.1.1 there exist then a neighbourhood W of the origin of X and an element $\alpha_0 \in Y$ such that $y_0 + W \subseteq M$ and

(2.3.7) $$f(y_0 + ay) - f(y_0) \in a^s[-\alpha_0,\ \alpha_0]$$

for every rational number $a \in [0, 1]$ and every $y \in W$.

On the other hand, $[-\alpha,\ \alpha]$ is a neighbourhood of the origin of Y. Since Y has the boundedness property, there exists a natural number n such that $(1/n)^s[-\alpha_0,\ \alpha_0] \subseteq [-\alpha,\ \alpha]$. From (2.3.7) we conclude then

(2.3.8) $$f\left(y_0 + \frac{1}{n}y\right) - f(y_0) \in [-\alpha,\ \alpha] \text{ for all } y \in W.$$

The neighbourhood $U = y_0 + (1/n)W$ of y_0 is contained in M, and one has, according to (2.3.8)

$$f(y) - f(y_0) \leqslant \alpha \text{ for all } y \in U.$$

In other words, f is upper semi-continuous at y_0. ■

CHAPTER 3

Continuity of Rationally s-Convex Mappings

In this chapter we give results dealing with the relationship between the property of a rationally s-convex mapping of being locally majorized, precontinuous, and upper semi-continuous, respectively, and its continuity.

As in chapter 2, we denote by X a topological linear space over K, and by M a non-empty subset of X.

3.1. Continuity of Locally Majorized Rationally s-Convex Mappings. In this section we continue the discussion of locally majorized rationally s-convex mappings which was begun in section 2.1.

THEOREM 3.1.1. *Let M be a convex set, x_0 a point of M, Y an ordered topological linear space with the boundedness property, and $f: M \to Y$ a rationally s-convex mapping which is locally majorized at x_0. Then f is continuous at x_0.*

Proof. By theorem 2.1.1 there exist a neighbourhood W of the origin of X and an element $\alpha \in Y$ such that $x_0 + W \subseteq M$ and also such that (2.1.1) holds for every rational number $a \in [0, 1]$ and every $x \in W$.

If V is any neighbourhood of the origin of Y, we can find a natural number n such that $(1/n)^s[-\alpha, \alpha] \subseteq V$, since the order-interval $[-\alpha, \alpha]$ is ubonded. From (2.1.1) we conclude

$$f\left(x_0 + \frac{1}{n}x\right) - f(x_0) \in V \text{ for all } x \in W. \tag{3.1.1}$$

The neighbourhood $U = x_0 + (1/n)W$ of x_0 is contained in M, and one has $f(x) \in f(x_0) + V$ for all $x \in U$, in view of (3.1.1). Hence f is continuous at x_0. ∎

Remark. In theorem 3.1.1 the hypothesis that Y has the boundedness property cannot be dropped as shown by the following example. Take $X = R$ and $Y = C^2([1/2, 1])$, where $C^2([1/2, 1])$ is the ordered normed linear space of all twice continuously differentiable functions $\alpha: [1/2, 1] \to R$ with the norm

$$\|\alpha\| = \sum_{i=0}^{2} \max\{|\alpha^{(i)}(t)| : t \in [1/2, 1]\}$$

and with the pointwise ordering. Y does not have the boundedness property, since the order-interval between the origin and the function $\alpha_0: [1/2, 1] \to R$, defined by $\alpha_0(t) = 1$ for all $t \in [1/2, 1]$, is not bounded. Put $M = [-1, 1]$. The mapping $f: M \to Y$, defined by

$$(f(x))(t) = \begin{cases} x^4 t^{\left(\frac{1}{x}\right)^2} & \text{if } x \in M \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

for all $t \in [1/2, 1]$, is convex. Since $f(x) \leqslant \alpha_0$ whenever $x \in M$, it follows that f is locally majorized at $x_0 = 0$. But f is not continuous at this point. This assertion is a simple consequence of the inequality

$$\|f(x) - f(x_0)\| \geqslant \left| \frac{d^2 f(x)}{dt^2}(1) - \frac{d^2 f(x_0)}{dt^2}(1) \right| = 1 - x^2 \geqslant \frac{3}{4}$$

which is valid for all $x \in [-1/2, 1/2] \setminus \{0\}$.

The following corollary generalizes results of LOPES PINTO A. J. B. [8, Theorem 3.1] and VALADIER M. [9, Proposition 9] on the continuity of convex mappings.

Corollary 3.1.2. *Let M be a convex set, Y an ordered topological linear space with the boundedness property, and $f: M \to Y$ a rationally s-convex mapping which is locally majorized at a point $x_0 \in M$. Then f is continuous at every interior point of M.*

Proof. Apply theorem 2.1.2 and theorem 3.1.1. ∎

We give now an example which shows that the converse of theorem 3.1.1 is not true. Let Y be the ordered normed linear space $C_0([0, 1])$ of all continuous functions $x: [0, 1] \to R$ that vanish at $t = 0$, with the norm

$$\|x\| = \max\{|x(t)| : t \in [0, 1]\} \tag{3.1.2}$$

and with the pointwise ordering. Since the norm (3.1.2) is monotone, Y is locally full, and consequently has the boundedness property. The mapping $f: Y \to Y$ defined by $f(x) = x$ for all $x \in Y$ is linear and continuous. We prove that f is not locally majorized at the origin of Y. Suppose, on the contrary, that it is locally majorized at the origin of Y. Then there exist a neighbourhood U of the origin of Y and an element $\alpha \in Y$ such that $x \leqslant \alpha$ for all $x \in U$. This means that α is an interior point of Y_+. But, on the other hand, it is easy to verify that the positive wedge of Y has no interior points. Hence f is not locally majorized at the origin of Y. Moreover, by theorem 2.1.2, f is nowhere locally majorized.

The following result implies, however, that continuous mappings whose range space is ordered by a wedge with interior points are always locally majorized.

THEOREM 3.1.3. *Let x_0 be an interior point of M, Y an ordered topological linear space with $\operatorname{int} Y_+ \neq \emptyset$, and $f: M \to Y$ a mapping which is continuous at x_0. Then f is locally order-bounded at x_0.*

Proof. Let α be an interior point of Y_+. Since f is continuous at x_0, we can find a neighbourhood U of x_0 contained in M and such that

$$\{f(x) : x \in U\} \subseteq f(x_0) + [-\alpha, \alpha] = [f(x_0) - \alpha, f(x_0) + \alpha].$$

Hence f is locally order-bounded at x_0. ∎

Corollary 3.1.4. *Let M be a convex set, x_0 an interior point of M, Y an ordered topological linear space with the boundedness property and with $\operatorname{int} Y_+ \neq \emptyset$, and $f: M \to Y$ a rationally s-convex mapping. Then, f is continuous at x_0 if and only if it is locally majorized at this point.*

Proof. Apply theorem 3.1.3 and theorem 3.1.1. ■

3.2. Continuity of Precontinuous Rationally *s*-Convex Mappings. The following theorem is the main result of this section.

THEOREM 3.2.1. *Let M be a convex set, x_0 a point of M, Y a locally full ordered topological linear space, and $f: M \to Y$ a rationally s-convex mapping which is precontinuous at x_0. Then f is continuous at x_0.*

Proof. Let V be any neighbourhood of the origin of Y. Since Y is locally full, there exists a full neighbourhood V_0 of the origin of Y such that $V_0 \subseteq V$. Take now a balanced neighbourhood V_1 of the origin of Y which satisfies $V_1 + V_1 \subseteq V_0$. In view of the precontinuity of f at x_0, we can find a balanced neighbourhood W of the origin of X and a real number $a > 0$, such that $x_0 + W \subseteq M$ and also such that

$$f(x) - f(x_0) \in aV_1 \text{ for all } x \in x_0 + W. \tag{3.2.1}$$

Choose a rational number b in the open interval $]0, 1[$ satisfying $ab^s < 1$ and $b^s[1 - \theta(s)]f(x_0) \in V_1$, where $\theta(s)$ is the number defined by (1.2.3). Put $U = x_0 + bW$. This is a neighbourhood of x_0 contained in M. We claim that

$$f(x) - f(x_0) \in V \text{ for every } x \in U. \tag{3.2.2}$$

If $x \in U$, it must be of the form $x = x_0 + by$ for a suitable $y \in W$. Then, by proposition 1.2.3, the following relation holds

$$-b^s[f(x_0 - y) - \theta(s)f(x_0)] \leqslant f(x) - f(x_0) \leqslant \leqslant b^s[f(x_0 + y) - \theta(s)f(x_0)]. \tag{3.2.3}$$

But $x_0 - y$ and $x_0 + y$ belong to $x_0 + W$. In view of (3.2.1), we conclude that

$$-b^s[f(x_0 - y) - \theta(s)f(x_0)] = -b^s[f(x_0 - y) - f(x_0)] - b^s[1 - \theta(s)]f(x_0) \in -ab^sV_1 - V_1 \subseteq V_1 + V_1 \subseteq V_0,$$

$$b^s[f(x_0 + y) - \theta(s)f(x_0)] = b^s[f(x_0 + y) - f(x_0)] + b^s[1 - \theta(s)]f(x_0) \in ab^sV_1 + V_1 \subseteq V_1 + V_1 \subseteq V_0.$$

Taking now into consideration that V_0 is full, (3.2.3) implies

$$f(x) - f(x_0) \in V_0 \subseteq V.$$

Consequently (3.2.2) holds as claimed.

Since V was an arbitrary neighbourhood of the origin of Y, (3.2.2) shows the continuity of f at x_0. ■

Remark. In theorem 3.2.1 the hypothesis that Y is locally full cannot be dropped. Indeed, the mapping f considered in the remark after theorem 3.1.1 is precontinuous at $x_0 = 0$, since

$$\| f(x) \| \leqslant x^4 + 1 \leqslant 2 \text{ for all } x \in [-\sqrt{2}/2, \sqrt{2}/2].$$

But, as it has been shown, it fails to be continuous at x_0.

Corollary 3.2.2. *Let M be a convex set, Y a locally full ordered topological linear space, and $f: M \to Y$ a rationally s-convex mapping which is precontinuous at a point $x_0 \in M$. Then f is continuous at every interior point of M.*

Proof. Apply theorem 2.2.2 and theorem 3.2.1. ■

Obviously, any mapping from M into an ordered topological linear space is precontinuous at each point $x_0 \in \text{int } M$ at which it is continuous. Thus corollary 3.2.2 provides the following result.

Corollary 3.2.3. *Let M be a convex set, Y a locally full ordered topological linear space, and $f: M \to Y$ a rationally s-convex mapping which is continuous at a point $x_0 \in \text{int } M$. Then f is continuous at every interior point of M.*

Theorem 3.2.1 provides also the following two corollaries, the first of them being obtained in view of the remark preceding corollary 3.2.3.

Corollary 3.2.4. *Let M be a convex set, x_0 an interior point of M, Y a locally full ordered topological linear space, and $f: M \to Y$ a rationally s-convex mapping. Then, f is continuous at x_0 if and only if it is precontinuous at this point.*

Corollary 3.2.5. *Let M be a convex set, x_0 an interior point of M, Y a locally full ordered locally bounded topological linear space, and $f: M \to Y$ a rationally s-convex mapping. Then, f is continuous at x_0 if and only if there exists a neighbourhood U of x_0 contained in M and such that $\{f(x) : x \in U\}$ is a bounded set.*

Proof. Suppose f is continuous at x_0. Since Y is a locally bounded topological linear space, its origin possesses a bounded neighbourhood V. Then there exists a neighbourhood U of x_0 contained in M and such that $f(x) - f(x_0) \in V$ for every $x \in U$. Consequently, the set $\{f(x) : x \in U\}$ is bounded.

Conversely, if there exists a neighbourhood U of x_0 contained in M and such that the set $\{f(x) : x \in U\}$ is bounded, then, by proposition 2.2.1, f is precontinuous at x_0. Applying theorem 3.2.1, it results that f is continuous at x_0. ■

Before deriving another corollary of theorem 3.2.1, we recall that a topological space is said to be *closure-sequential* if for any subset A and any $x \in \text{cl } A \setminus A$ there exists a sequence in A converging to x.

The following result is due to Averbukh V. I. and Smolyanov O. G. For its proof we refer the reader to YAMAMURO S. [11, p. 140].

PROPOSITION 3.2.6. *Let X be closure-sequential, and let x_{mn} $(m, n \in N)$ be elements of X. If the following conditions are satisfied:*

(i) $(x_{mn})_{m \in N}$ *converges for each* $n \in N$ *to an element* $x_n \in X$;

(ii) $(x_n)_{n \in N}$ *converges to an element* $x_0 \in X$;

then there are in N strictly increasing sequences $(m_k)_{k\in N}$ *and* $(n_k)_{k\in N}$ *such that* $(x_{m_k n_k})_{k\in N}$ *converges to* x_0.

Using this proposition, we state the following result.

Corollary 3.2.7. *Let X be a closure-sequential topological linear space, M a convex set,* x_0 *an interior point of M, Y a locally full ordered topological linear space, and* $f: M \to Y$ *a rationally s-convex mapping. Then, f is continuous at* x_0 *if and only if for each sequence* $(x_n)_{n\in N}$ *in M converging to* x_0 *the sequence* $(f(x_n))_{n\in N}$ *is bounded.*

Proof. Suppose f is continuous at x_0. If $(x_n)_{n\in N}$ is a sequence in M converging to x_0, then $(f(x_n))_{n\in N}$ converges to $f(x_0)$, and is therefore bounded.

Conversely, assume that for each sequence $(x_n)_{n\in N}$ in M converging to x_0 the sequence $(f(x_n))_{n\in N}$ is bounded. We show first that in this case f is precontinuous at x_0. If f is not precontinuous at x_0, there exists a neighbourhood V of the origin of Y such that

$$\{f(x) - f(x_0) : x \in U\} \not\subseteq nV \text{ for all } n \in N$$

whenever $U \in \mathcal{U}$, where $\mathcal{U}$ denotes the family of all neighbourhoods of x_0 contained in M. For each $n \in N$ we can choose $x_n(U) \in U$, where $U \in \mathcal{U}$, such that $f(x_n(U)) - f(x_0) \notin nV$. Introduce in $\mathcal{U}$ an ordering as follows: $U_1 \leqslant U_2$ if $U_2 \subseteq U_1$. For each $n \in N$ the net $(x_n(U))_{U\in\mathcal{U}}$ converges then to x_0. Since X is closure-sequential, there is a sequence $(x_{mn})_{m\in N}$ in $\{x_n(U) : U \in \mathcal{U}\}$ which also converges to x_0. By proposition 3.2.6, there are in N strictly increasing sequences $(m_k)_{k\in N}$ and $(n_k)_{k\in N}$ such that $(x_{m_k n_k})_{k\in N}$ converges to x_0. Since

$$f(x_{m_k n_k}) - f(x_0) \notin n_k V \text{ for all } k \in N,$$

we conclude that

$$\{f(x_{m_k n_k}) - f(x_0) : k \in N\}$$

is not bounded. Thus the sequence $(f(x_{m_k n_k}))_{k\in N}$ is not bounded too. But this contradicts our hypothesis. Hence f is precontinuous at x_0. By theorem 3.2.1, it follows then that f is continuous at x_0. ∎

3.3. Continuity of Upper Semi-Continuous Rationally s-Convex Mappings. In this section we discuss the connection between the upper semi-continuity and the continuity of rationally s-convex mappings.

THEOREM 3.3.1. *Let M be a convex set,* x_0 *a point of M, Y an ordered topological linear space with the boundedness property and with* $\operatorname{int} Y_+ \neq \emptyset$, *and* $f: M \to Y$ *a rationally s-convex mapping which is upper semi-continuous at* x_0. *Then f is continuous at* x_0.

Proof. It should be noted that f is locally majorized at x_0. Therefore the result follows immediately from theorem 3.1.1. ∎

Remark. In theorem 3.3.1 the hypothesis that Y has the boundedness property cannot be dropped. Indeed, the mapping f considered in the remark after theorem 3.1.1 is, as it has been shown there, not continuous at $x_0 = 0$. But it is upper semi-continuous at x_0. To prove this assertion, let α be any interior point of Y_+. Then we have $\alpha(t) > 0$ for all $t \in [1/2, 1]$, since if there were a point $t_0 \in [1/2, 1]$ with $\alpha(t_0) = 0$, the sequence $(\alpha_n)_{n\in N}$ with

$$\alpha_n(t) = \alpha(t) - \frac{1}{n} \text{ for each } t \in [1/2, 1]$$

would be in $Y \setminus Y_+$, although it converges to α. Take as ε the smallest of the numbers min $\{\sqrt[4]{\alpha(t)} : t \in [1/2, 1]\}$ and 1. The set $[-\varepsilon, \varepsilon]$ is a neighbourhood of x_0 contained in M and for every $x \in [-\varepsilon, \varepsilon]$ one has

$$(f(x))\,(t) \leqslant x^4 \leqslant \varepsilon^4 \leqslant \alpha(t) \text{ for all } t \in [1/2, 1],$$

i.e. $f(x) \leqslant \alpha$. Hence f is upper semi-continuous at x_0.

Corollary 3.3.2. *Let M be a convex set, Y an ordered topological linear space with the boundedness property and with* $\operatorname{int} Y_+ \neq \emptyset$, *and* $f: M \to Y$ *a rationally s-convex mapping which is upper semi-continuous at a point* $x_0 \in M$. *Then f is continuous at every interior point of M.*

Proof. Apply theorem 2.3.2 and theorem 3.3.1. ∎

THEOREM 3.3.3. *Let* x_0 *be an interior point of M, Y an ordered topological linear space with* $\operatorname{int} Y_+ \neq \emptyset$, *and* $f: M \to Y$ *a mapping which is continuous at* x_0. *Then f is upper semi-continuous at* x_0.

Proof. If α is any interior point of Y_+, then $[-\alpha, \alpha]$ is a neighbourhood of the origin of Y. Therefore there exists a neighbourhood U of x_0 contained in M and such that

$$f(x) \in f(x_0) + [-\alpha, \alpha] \text{ for all } x \in U.$$

Hence we have (2.3.2). Thus f is upper semi-continuous at x_0. ∎

Remark. Our concept of an upper semi-continuous mapping, introduced by definition 2.3.1, is a natural generalization of the concept of upper semi-continuity known for real-valued functions. Of course, it is possible to generalize the latter concept also in another way for mappings taking values in an ordered topological linear space Y. For instance, we can call a mapping $f: M \to Y$ upper semi-continuous at $x_0 \in M$ if for each $\alpha \in Y_+ \setminus \{o\}$ there exists a neighbourhood U of x_0 contained in M and such that (2.3.2) holds. But in this case a continuous mapping is not necessarily upper semi-continuous. To show this, let Y be the Euclidean space R^2 equipped with the coordinatewise ordering. The mapping $f: Y \to Y$ defined by $f(x) = x$ for all $x \in Y$ is then continuous at the origin of Y, but for $\alpha = (1, 0)$ there exists no neighbourhood U of the origin of Y such that $f(x) \leqslant \alpha$ for every $x \in U$.

Corollary 3.3.4. *Let M be a convex set,* x_0 *an interior point of M, Y an ordered topological linear space with the boundedness property and with* int $Y_+ \neq \emptyset$, *and* $f: M \to Y$ *a rationally s-convex mapping. Then, f is continuous at* x_0 *if and only if it is upper semi-continuous at this point.*

Proof. Apply theorem 3.3.3 and theorem 3.3.1. ■

3.4. Final Conclusions. By using the results presented in sections 3.1—3.3, we are able to state the following two theorems. The first of them gives local characterizations of the continuity of a rationally s-convex mapping at an interior point, while the second gives global characterizations of the continuity at the whole interior.

THEOREM 3.4.1. *Let M be a convex set,* x_0 *an interior point of M, Y an ordered topological linear space with the boundedness property and with* int $Y_+ \neq \emptyset$, *and* $f: M \to Y$ *a rationally s-convex mapping. Then the following statements are equivalent*:

(A_1) *f is locally majorized at* x_0.

(A_2) *There exist a neighbourhood W of the origin of X and an element* $\alpha \in Y$, *such that* $x_0 + W \subseteq M$ *and* (2.1.1) *holds for every rational number* $a \in [0, 1]$ *and every* $x \in W$.

(A_3) *f is locally order-bounded at* x_0.

(A_4) *There exists a neighbourhood U of* x_0 *contained in M and such that the set* $\{f(x) : x \in U\}$ *is bounded.*

(A_5) *f is precontinuous at* x_0.

(A_6) *f is continuous at* x_0.

(A_7) *f is upper semi-continuous at* x_0.

(A_8) *For each* $\alpha \in$ int Y_+ *there exists a neighbourhood U of* x_0 *contained in M and such that* (2.3.3) *holds.*

(A_9) $(x_0, \alpha_0) \in$ int $E(f)$ *for each* $\alpha_0 \in f(x_0) +$ int Y_+.

(A_{10}) *There exists an* $\alpha_0 \in f(x_0) +$ int Y_+ *such that* $(x_0, \alpha_0) \in$ int $E(f)$.

THEOREM 3.4.2. *Let M be a convex set, Y an ordered topological linear space with the boundedness property and with* int $Y_+ \neq \emptyset$, *and* $f: M \to Y$ *a rationally s-convex mapping for which there exists a point* $x_0 \in$ int M *such that one (and hence all) of the statements* (A_1), ..., (A_{10}) *in theorem* 3.4.1 *is true. Then the following (equivalent) statements hold*:

(B_1) *f is locally majorized at every interior point of M.*

(B_2) *f is locally order-bounded at every interior point of M.*

(B_3) *f is precontinuous at every interior point of M.*

(B_4) *f is continuous at every interior point of M.*

(B_5) *f is upper semi-continuous at every interior point of M.*

(B_6) int epi $f = E(f)$.

(B_7) $E(f)$ *is open.*

We notice that the set $E(f)$ which occurs in the statements (A_9), (A_{10}), (B_6) and (B_7) is defined by (2.3.4), while epi f which occurs in (B_6) is defined as follows

$$\text{epi } f = \{(x, \alpha) \in X \times Y : x \in M, f(x) \leq \alpha\}.$$

Combining theorem 3.4.2 and theorem 1.2.1, we obtain the following result.

Corollary 3.4.3. *Let M be an open convex set, let Y be an ordered topological linear space with the boundedness property, with* int $Y_+ \neq \emptyset$ *and with* Y_+ *sequentially closed, and let* $f: M \to Y$ *be a rationally s-convex mapping for which there exists a point* $x_0 \in M$ *such that one (and hence all) of the statements* (A_1), ..., (A_{10}) *in theorem* 3.4.1 *is true. Then f is s-convex and continuous on M.*

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