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SOME PRACTICAL APPROXIMATION METHODS FOR NONLINEAR EQUATIONS

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1. The goal of this paper is to construct some simple procedure for the approximation of the solutions of a given nonlinear equation, with practical facilities and with a good efficiecy.

Let $f : \Omega \to \mathbf{R}$, with $\Omega \subset \mathbf{R}$ be a given function and one considers the equation

(1) $$f(x) = 0, \; x \in \Omega.$$

Let, also, $F : D \to \mathbf{R}^n$, where $D \subset \Omega^n$, be an approximation method, a F-method, of the solutions of the equation (1) and

(2) $$x_0, x_1, \ldots, x_{n-1}, x_n, \ldots$$

with $x_i = F(x_{i-n}, \ldots, x_{i-1})$, the sequence generated by F, for a given $(x_0, \ldots, x_{n-1}) \in D$.

If $x^* \in \Omega$ is a solution of (1), the number $p = p(F)$ with the property that

$$\lim_{i \to \infty} \frac{x^* - F(x_{i-n+1}, \ldots, x_i)}{(x^* - x_i)^p} = C \neq 0,$$

where C is a constant, is named the convergence order of F and C is the asymptotic error.

A F-method depends of some informations about the function f. Usually, these information are values of the function f and certain of its derivatives at some previous approximations. Next, will be considered only this case.

Let $v_i = v_i(F)$ be the evaluation number of $f^{(i)}$ and $C_i = C_i(f)$ the number of arithmetic operations for one evaluation of $f^{(i)}$, if $f^{(i)}$ is a rational function or the number of arithmetic operations for the approximation of the function $f^{(i)}$ with a given error, if it is not rational.

The value [5]

$$CP(F;f)=\sum_i v_i C_i + C(F) \tag{3}$$

where $C(F)$ is the combinatorial cost, is named the complexity of the method F.

The value

$$E(F;f)=\frac{\log_2 p(F)}{CP(F;f)} \tag{4}$$

where $p(F)$ is the convergence order of F is named the efficiency of F.

It is also important to merition that for large classes of functions the methods with a good efficiency are the simple methods, as the Newton like methods, the secant method etc.

Next, will be studied such simple methods, generated by the inverse interpolation of Birkhoff type procedure.

2. Let $x^*\in\Omega$ be a solution of the equation (1) and $V(x^*)$ a neighbourhood of x^*. One supposes that the function f has an inverse $g=f^{-1}$ on $V(x^*)$.

Now, if $y_k=f(x_k)$, where $x_k\in V(x^*)$, $k=0,1,\ldots,m$, are approximations of x^*, $r_k\in \mathrm{N}$ and $I_k=\{0,1,\ldots,r_k\}$, $k=0,1\ldots,m$, then if there exist $g^{(j)}(y_k)$, $j\in I_k$, $k=0,1,\ldots,m$, one considers the Birkhoff type interpolation problem: to determine the polynomial P of minimum degree which satisfies the conditions:

$$P^{(j)}(y_k)=g^{(j)}(y_k),\quad j\in I_k,\ k=0,1,\ldots,m.$$

It is known [3] that, if the functionals $g^{(j)}(y_k)$, $j\in I_k$, $k=0,1,\ldots,m$ possesses the interpolation property, then the interpolating polynomial P exists and it is unique. Let B_n be the corresponding interpolation operator, with n the degree of the interpolating polynomial.

We have

$$(B_n g)(y)=\sum_{k=0}^{m}\sum_{j\in I_k} b_{kj}(y)g^{(j)}(y_k)$$

where b_{kj} are the fundamental interpolating polynomials, i.e.

$$b_{kj}^{(p)}(y_\nu)=0,\ k\neq\nu,\ p\in I_\nu$$

$$b_{kj}^{(p)}(y_k)=\delta_{pj},\ p\in I_k,$$

for $k,\nu=0,1,\ldots,m$, $j\in I_k$.

We also have the corresponding interpolation formula

$$g=B_n g+R_n g \tag{5}$$

where $R_n g$ is the remainder term.

Using the property that $B_n h=h$, for any $h\in \mathrm{P}_n$ — the set of all polynomials of the degree at most n, it follows that for $g\in C^{n+1}[V(0)]$, where $V(0)=f(V(x^*))$,

$$(R_n g)(y)=\int_{V(0)}\varphi(t;y)f^{(n+1)}(t)\,dt$$

where

$$\varphi(t;y)=\frac{1}{n!}\left\{(y-t)_+^n-\sum_{k=0}^{m}\sum_{j\in I_k} b_{kj}(y)\left[(y-t)_+^n\right]_{y=y_k}^{(j)}\right\}.$$

Taking into account that $x^*=g(0)=(B_n g)(0)+(R_n g)(0)$, $(B_n g)(0)$ defines a new approximation to x^*. Hence

$$F(x_0,\ldots,x_m)=(B_n g)(0) \tag{6}$$

is an approximation method for x^*.

Next, we consider some particular cases.

Case 1. $m=1,\ r_0=1,\ r_1=0,\ I_0=\{1\},\ I_1=\{0\}$.

The interpolation formula (5) becomes

$$g(y)=(B_1g)(y)+(R_1g)(y),$$

where

$$(B_1g)(y)=(y-y_1)\,g'(y_0)+g(y_1)$$

and for $y_0<y_1$

$$(R_1g)(y)=\frac{1}{2}(y-y_1)(y+y_1-2y_0)g''(\eta) \tag{6'}$$

with $\eta\in[y_0,y_1]$, for any $y\in[y_0,y_1]$. In this way, we obtain the method F_1 defined by

$$F_1(x_0,x_1)=x_1-\frac{f(x_1)}{f'(x_0)}.$$

THEOREM 1. *Let*

i) $f(x_0)<0<f(x_1)$

ii) $f'(x)$ *exists, it is finite and* $f'(x)>0$ *for* $x\in[x_0,x_1[$.

iii) $f''(x)$ *exists, it is finite and* $f''(x)\leqslant 0$ *for* $x_0<x<x_1$.

Then

1) *the equation* (1) *has a unique solution* x^*, *with* $x_0<x^*<x_1$.

2) *the sequence* (x_n) *defined by the method* F_1, i.e.

$$x_{n+1}=F_1(x_0,x_n),\ n=1,2,\ldots \tag{7}$$

converges to x^*. *Furthermore, we have*

$$x^* \leqslant x_{n+1} \leqslant x_n, \; n = 1,2, \ldots . \tag{8}$$

P r o o f. The existence and the uniqueness follow immediately from the assumptions i) and ii).

By induction, first we prove the inequalities (8). Indeed, we have $x^* < x_1$. Suppose that $x^* \leqslant x_n$. Then, using the remainder expression (6), one obtains

$$x^* - x_{n+1} = (R_1 g)(0) = \frac{1}{2} f(x_n)[f(x_n) - 2f(x_0)] \frac{f''(\xi_n)}{[f'(\xi_n)]^3}$$

and, from the inequalities $f(x_n) \geqslant 0$ $(x_n \geqslant x^*)$, $f(x_0) < 0, f'(x) > 0, f''(x) \leqslant \leqslant 0$ for $x \in]\, x_0, x_1[$, it follows that $x^* - x_{n+1} \leqslant 0$, i.e. $x^* \leqslant x_{n+1}$. From (7) it also follows that

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_0)} \leqslant 0, \text{ i.e. } x_{n+1} \leqslant x_n \text{ and (8) is proved.}$$

From (8), we have that the sequence (x_n) is decreasing and bounded, so it is convergent. Let $\tilde{x}^*$ be its limit. Then (7) implies

$$\lim_{n\to\infty} \frac{f(x_n)}{f'(x_0)} = \lim_{n\to\infty} (x_{n+1} - x_n) = 0$$

hence $\lim\limits_{n\to\infty} f(x_n) = f(\tilde{x}^*) = 0$ and $x_0 < \tilde{x}^* < x_1$. From the uniqueness of x^* it follows that $\tilde{x}^* = x^*$ and the theorem is completely proved.

The utility of the method F_1 grows up if it is combined with Newton's method.

COROLLARY 1. *In the assumptions of theorem 1, the sequence*

$$x_0, x_1, \ldots, x_n, \ldots$$

generated by the iteration

$$x_{2n} = x_{2n-2} - \frac{f(x_{2n-2})}{f'(x_{2n-2})}, \; n = 1,2,\ldots \tag{9}$$

$$x_{2n+1} = x_{2n-1} - \frac{f(x_{2n-1})}{f'(x_{2n-2})}, \; n = 1,2,\ldots \tag{10}$$

converges to x^*. *Furthermore,* $x^* \in [x_{n-1}, x_n]$ *for any* $n \in \mathrm{N}$.

Indeed, the iteration (9) is just Newton's method, which in the given conditions satisfy the inequalites $x_n \leqslant x_{n+1} \leqslant x^*$, $n = 0,1,\ldots$, while by theorem 1, we have $x^* \leqslant x_{n+1} \leqslant x_n$, $n = 1,2,\ldots$.

R e m a r k 1. The combined method (9)–(10) has the advantage that both components need the evaluation of the first derivative f' at the same point.

R e m a r k 2. A more simple combined method is obtained from (9)–(10) for x_0 fix, i.e.

$$x_{2n} = x_{2n-2} - \frac{f(x_{2n-2})}{f'(x_0)}, \; n = 1,2,\ldots$$

$$x_{2+1} = x_{2n-1} - \frac{f(x_{2n-1})}{f'(x_0)}, \; n = 1,2\ldots$$

which uses the evaluation of the first derivative only at x_0.

This last method is very useful for the problems with a large evaluation cost of f'.

Case 2. $m = 2$, $r_0 = 1$, $r_1 = 1$, $I_0 = \{1\}$, $I_1 = \{0, 1\}$. The interpolation formula is

$$g(y) = (B_2 g)(y) + (R_2 g)(y)$$

with

$$(B_2 g)(y) = \frac{(y - y_1)^2}{2(y_0 - y_1)} g'(y_0) + g(y_1) + \frac{(y - y_1)(-y + 2y_0 - y_1)}{2(y_0 - y_1)} g'(y_1) \tag{11}$$

and

$$(R_2 g)(y) = \int_{y_0}^{y_1} \varphi(y; t)\, g'''(t)\, dt$$

where

$$\varphi(y; t) = \frac{(y - t)_+^2}{2} - \frac{(y - y_1)(2y_0 - y - y_1)}{2(y_0 - y_1)} (y_1 - t) - \frac{(y_1 - t)^2}{2}.$$

As φ does not change the sign on $[y_0, y_1]$ we obtain, for $g \in C^3 [y_0, y_1]$

$$(R_2 g)(y) = \frac{1}{12} (y - y_1)^2 (2y + y_1 - 3y_0) g'''(\eta) \tag{12}$$

with $y_0 \leqslant \eta \leqslant y_1$.

Thus, we have the method F_2 defined by

$$F_2(u,v) = v - \frac{f_1(v)}{2[f(v) - f(u)]} \left[\frac{f(v) - 2f(u)}{f'(v)} + \frac{f(v)}{f'(u)} \right] \tag{13}$$

and

$$x^* - F_2(u,v) = \frac{1}{12} f^2(v)[f(v) - 3f(u)] g'''(\eta') \tag{14}$$

where η' is in the interval defined by u and v.

THEOREM 2. *If f satisfies the assumptions of theorem* 1, $g'''(y) \leqslant 0$ *for* $y \in [y_0, y_1]$ *and* (x_n) *is the sequence generated by* F_2 *with the initial values* x_0, x_1, i.e.

$$x_{n+1} = F_2(x_0, x_n), \; n = 1,2,\ldots$$

hen:

1°. $x^* \leqslant x_{n+1} \leqslant x_n$, $n = 1, 2, \ldots$

2°. if $x_0 := x_1$; $x_1 := x_0$, *then* $x_{n+1} \leqslant x_n \leqslant x^*$, $n = 1, 2, \ldots$.

3°. $\lim_{n\to\infty} x_n = x^*$.

P r o o f. It follows easy by induction. Let us consider in detail the case 1°. We have $x^* < x_1$. Suppose that $x^* \leqslant x_n$. Then, from (14), one obtains

$$x^* - x_{n+1} = \frac{1}{12} f^2(x_n)[f(x_n) - 3f(x_0)]g'''(\eta_n) \leqslant 0, \text{ i.e. } x^* \leqslant x_{n+1}.$$

From (13), we also have

$$(15) \qquad x_{n+1} - x_n = -\frac{f(x_n)}{2[f(x_n) - f(x_0)]}\left[\frac{f(x_n) - 2f(x_0)}{f'(x_n)} + \frac{f(x_n)}{f'(x_0)}\right] \leqslant 0$$

for any $n = 1, 2, \ldots$. Thus 1° is proved. On the same way can be proved 2°. To prove 3° we also consider first case 1°. Then, the sequence (x_n) is decreasing and bounded, hence convergent. Let $\tilde{x}^*$ be its limit. From (15) we have

$$\lim_{n\to\infty} \frac{f(x_n)}{2[f(x_n) - f(x_0)]}\left[\frac{f(x_n) - 2f(x_0)}{f'(x_n)} + \frac{f(x_n)}{f'(x_0)}\right] = \lim_{n\to\infty}(x_n - x_{n+1}) = 0$$

i.e. $\lim_{n\to\infty} \delta_n f(x_n) = 0$ with $\delta_n > 0$ for any $n = 1, 2, \ldots$. Hence $\lim_{n\to\infty} f(x_n) = 0 = f(\tilde{x}^*)$. As $x^* \in]x_0, x_1[$, it follows that $\tilde{x}^* = x^*$, x^* being the unique solution of the equation (1) in this interval.

In the case 2°, the proof of 3° is analogous.

R e m a r k 3. The cases 1° and 2° can be combined to approximate x^* from both parts.

Case 3. $m = 2$, $r_0 = 0$, $r_1 = 1$, $I_0 = \{0\}$, $I_1 = \{0,1\}$.

In this case we have a Hermite interpolation problem. The interpolation formula is

$$(16) \qquad g(y) = (H_2 g)(y) + (R_2 g)(y)$$

where

$$(H_2 g)(g) = \frac{(y - y_1)^2}{(y_0 - y_1)^2} g(y_0) + \frac{(y - y_0)(2y_1 - y - y_0)}{(y_0 - y_1)^2} g(y_1) + \frac{(y - y_0)(y - y_1)}{y_0 - y_1} g'(y_1)$$

and

$$(R_2 g)(y) = \frac{(y - y_0)(y - y_1)^2}{6} g'''(\eta), \; y_0 \leqslant \eta \leqslant y_1.$$

Let F_H be the iteration method generated by (14), i.e.

$$(17) \qquad F_H(x_0, x_1) = x_1 - \left[\frac{f(x_1)}{f(x_0) - f(x_1)}\right]^2 (x_1 - x_0) - \frac{f(x_1)}{f(x_0) - f(x_1)} \frac{f(x_0)}{f'(x_1)}$$

and

$$(18) \qquad x^* - F_H(x_0, x_1) = -\frac{f(x_0) f^2(x_1)}{6} g'''(\eta_0).$$

THEOREM 3. *If f satisfies the conditions of theorem* 1, $g'''(y) \geqslant 0$ *for* $y \in [y_0, y_1]$ *and* (x_n) *is the sequence generated by*

$$x_{n+1} = F_H(x_{n-1}, x_n), \; n = 1, 2, \ldots$$

then x^* belongs to the interval defined by two consecutive approximations, i.e. $(x^* - x_n)(x^* - x_{n+1}) \leqslant 0$ *for any* $n = 1, 2, \ldots$, *and* $\lim_{n\to\infty} x_n = x^*$.

P r o o f. By (17)—(18) and the hypothesis of theorem, it follows that $x_2 < x_1$ and $x_2 \leqslant x^*$. As $f(x_2) < 0$ ($f(x_2) = 0$ means that $x^* = x_2$) and $f(x_1) > 0$, (17)—(18) imply that $x_3 > x_2$ and $x^* \leqslant x_3$. Let now suppose that $x_{n-1} \leqslant x^* \leqslant x_n$. Then $f(x_{n-1}) \leqslant 0$, $f(x_n) \geqslant 0$ and it follows that $x_{n+1} \leqslant x_n$ and $x_{n+1} \leqslant x^*$, hence the first conclusion is proved. To prove that $\lim_{n\to\infty} x_n = x^*$, we observe by the construction of F_H that $x_n \in]x_0, x_1[$ for any $n = 2, \ldots$, i.e. the sequence (x_n) is bounded, hence it contains a subsequence (x_{n_k}) with $x_{n_k} \to \tilde{x}^*$ as $k \to \infty$. Thus, $x_{n_{k+1}} - x_{n_k} \to 0$ as $k \to \infty$ and from (17), we have $\lim_{k\to\infty} f(x_{n_k}) = 0 = f(\tilde{x}^*)$, with $\tilde{x}^* \in]x_0, x_1[$. Also, the uniqueness of x^* implies $\tilde{x}^* = x^*$.

COROLLARY 2. *If in theorem* 3 *the condition* $g''' \geqslant 0$ *is changes by* $g''' \leqslant 0$ *then*:

1°. *if* (x_n) *is defined by the formula*

$$(19) \qquad x_{n+1} = F_H(x_0, x_n), \; n = 1, 2, \ldots$$

then

$$x^* \leqslant x_{n+1} \leqslant x_n, \quad n = 1, 2, \ldots$$

2°. *if* (x_n) *is also generated by* (19) *but with* $x_0 := x_1$ *and* $x_1 := x_0$, then

$$x_n \leqslant x_{n+1} \leqslant x^*, \; n = 1, 2, \ldots$$

3°. *in each cases*, $\lim_{n\to\infty} x_n = x^*$.

The proof is a simple verification based on the relations (17)—(18

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