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EXTENSIONS OF SOME FINITE DIFFERENCE EQUATIONS FOR THE CASE OF DISTRIBUTIONS

by

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0. Let $\mathcal{D}_x = \mathcal{D}_x(\mathbf{R})$, $x \in \mathbf{R}$, be the space of testing functions of a real variable and $\mathcal{D}_{x,y} = \mathcal{D}_{x,y}(\mathbf{R}^{p+1})$, $x \in \mathbf{R}$, $y \in \mathbf{R}^p$ the space of testing functions of $p+1$ variables. $\mathcal{D}'_x = \mathcal{D}'_x(\mathbf{R})$ and $\mathcal{D}'_{x,y} = \mathcal{D}'_{x,y}(\mathbf{R}^{p+1})$ denote the corresponding dual spaces. Further, we use the notations: $D^k = \frac{d^k}{dx^k}$ for the derivatives of the functions belonging to $\mathcal{D}_x$, and $D^k_x = \frac{\partial^k}{\partial x^k}$, $D^k_{y_i} = \frac{\partial^k}{\partial y_i^k}$, where y_i, $i = 1, 2, \ldots, p$ are the coordinates of the vector y, for the derivatives of the functions belonging to $\mathcal{D}_{x,y}$.

1. Let $a \in \mathbf{R}^p$. We consider the following operators on the space $\mathcal{D}_{x,y}$

$$J(a) : \mathcal{D}_{x,y} \to \mathcal{D}_{x,y}$$

$$J(a)[\varphi](x, y) = \varphi\left(x - \sum_{i=1}^{p} a_i y_i, \; y\right) \tag{1}$$

for any $\varphi \in \mathcal{D}_{x,y}$, where a_i, $i = 1, 2, \ldots, p$ are the coordinates of the vector a, and

$$I(a) : \mathcal{D}_{x,y} \to \mathcal{D}$$

$$I(a)[\varphi](x) = \int_{\mathbf{R}^p} J(a)[\varphi](x, y)dy \tag{2}$$

for any $\varphi \in \mathcal{D}_{x,y}$.

We remark that

$$I(a)[\varphi] = < 1_y,\ J(a)[\varphi] >$$

for any $\varphi \in \mathfrak{D}_{x,y}$, where 1_y is the regular distribution defined by the function $1_y = 1$, for any $y \in \mathbf{R}^p$.

We observe also the evident properties

$$J(a)\ \ J(b) = J(b)\ \ J(a) = J(a+b)\ ;\ a \in \mathbf{R}^p,\ b \in \mathbf{R}^p$$
$$I(a)\ \ J(b) = I(b)\ \ J(a) = I(a+b)\ ;\ a \in \mathbf{R}^p,\ b \in \mathbf{R}^p$$

the succession of the operators being from right to left in the case of the product IJ.

Finally, we observe that

$$I(a)[\varphi](x) = \frac{1}{|a_i|} \int\limits_{\mathbf{R}^p} P_{a_i}[\varphi](x, y)\,dy$$

for any $\varphi \in \mathfrak{D}_{x,y}$, where

$$P_{a_i}[\varphi](x,y) = \varphi(y_i, y_1, \ldots, y_{i-1}, \eta, y_{i+1}, \ldots, y_p)$$

for any $\varphi \in \mathfrak{D}_{x,y}$, where

$$\eta = \frac{1}{a_i}\left(x - \sum_{k \neq i} a_x x_k - y_i\right)$$

2. We consider now the following operators on the space $\mathfrak{D}'_{x,y}$, respectively $\mathfrak{D}'_x$

$$\tilde{J}(a) : \mathfrak{D}'_{x,y} \to \mathfrak{D}'_{x,y}$$

(3) $$\langle \tilde{J}(a)[T], \varphi\rangle = \langle T,\ J(a)[\varphi]\rangle$$

for any $T \in \mathfrak{D}'_{x,y}$ and $\varphi \in \mathfrak{D}'_{x,y}$; and

$$\tilde{I}(a) : \mathfrak{D}'_x \to \mathfrak{D}'_{x,y}$$

(4) $$\langle \tilde{I}(a)[T],\ \ \varphi\rangle = \langle T,\ I(a)[\varphi]\rangle$$

for any $T \in \mathfrak{D}'_x$ and $\varphi \in \mathfrak{D}_{x,y}$.

The following properties are immediate:

$$\tilde{J}(a)\ \ \tilde{J}(b) = \tilde{J}(b)\ \tilde{J}(a) = \tilde{J}(a+b)\ ;\ a \in \mathbf{R}^p,\ b \in \mathbf{R}^p$$
$$\tilde{J}(a)\ \ \tilde{I}(b) = \tilde{J}(b)\tilde{I}(a) = \tilde{I}(a+b)\ ;\ a \in \mathbf{R}^p,\ b \in \mathbf{R}^p$$

PROPOSITION 1. *If $f \in \mathfrak{D}'_x$ is a regular distribution, then $\tilde{I}(a)[f]$ is the regular distribution defined by the function g for which we have*

$$g(x, y) = f\left(x + \sum_{i=1}^{p} a_i y_i\right)$$

where $(x, y) \in \mathbf{R} \times \mathbf{R}^p$.

P r o o f :

For any $\varphi \in \mathfrak{D}_{x,y}$, we have

$$\langle \tilde{I}(a)[f],\ \varphi\rangle = \langle f,\ I(a)[\varphi]\rangle = \int\limits_{\mathbf{R}} f(x)dx \int\limits_{\mathbf{R}^p} \varphi\left(x - \sum_{i=1}^{p} a_i y_i,\ y\right) dy =$$

$$= \int\limits_{\mathbf{R}^{p+1}} f(x)\varphi\left(x - \sum_{i=1}^{p} a_i y_i,\ \ y\right)dxdy = \int\limits_{\mathbf{R}^{p+1}} f\left(x + \sum_{i=1}^{p} a_i y_i\right)\varphi(x,\ y)dxdy$$

C o r o l l a r y 1. *If $\alpha \in \mathfrak{D}'_x$ is the regular distribution defined by the function $\alpha \in C^\infty_{\mathbf{R}}$, then $\tilde{I}(a)[\alpha]$ is the regular distribution defined by the function $\beta \in C^\infty_{\mathbf{R}\times\mathbf{R}^p}$ for which*

$$\beta(x,\ y) = \alpha\left(x + \sum_{i=1}^{p} a_i y_i\right)$$

where $(x,\ y) \in \mathbf{R} \times \mathbf{R}^p$.

PROPOSITION 2. *If $\alpha \in C^\infty_{\mathbf{R}}$, then for any $T \in \mathfrak{D}'_x$, we have*

$$\tilde{I}(a)[\alpha T] = \tilde{I}(a)[\alpha]\ \ \tilde{I}(a)[T]$$

On the right side we have the product of the distribution $\tilde{I}(a)[T] \in \mathfrak{D}'_{x,y}$ and the function $\tilde{I}(a)[\alpha] \in C^\infty_{\mathbf{R}\times\mathbf{R}^p}$.

P r o o f :

For any $\varphi \in \mathfrak{D}_{x,y}$, we have

$$\langle \tilde{I}(a)[\alpha]\ \ \tilde{I}(a)[T],\ \varphi\rangle = \langle \tilde{I}(a)[T],\ \varphi\tilde{I}(a)[\alpha]\rangle =$$
$$= \langle T,\ I(a)[\varphi\tilde{I}(a)[\alpha]]\rangle = \langle T,\ \alpha I(a)[\varphi]\rangle =$$
$$= \langle \alpha T, I(a)[\varphi]\rangle = \langle \tilde{I}(a)[\alpha T],\ \varphi\rangle$$

PROPOSITION 3. *For any $T \in \mathfrak{D}'_x$ we have the following rules*

$$D_x\tilde{I}(a)[T] = \tilde{I}(a)[DT]$$
$$D_{y_i}\tilde{I}(a)[T] = a_i\tilde{I}(a)[DT]$$

for every $i = 1, 2, \ldots, p$.

P r o o f :

For any $\varphi \in \mathfrak{D}_{x,y}$, we have

$$\langle \tilde{I}(a)[DT],\ \varphi\rangle = \langle DT,\ I(a)[\varphi]\rangle =$$
$$= -\langle T,\ D(I(a)[\varphi])\rangle = -\langle T.\ I(a)[D_x\varphi]\rangle =$$
$$= -\langle \tilde{I}(a)[T],\ D_x\varphi\rangle = \langle D_x\tilde{I}(a)[T],\ \varphi\rangle$$

The first rule is proved. We shall prove the second rule for an arbitrary i. We have, for any $\varphi \in \mathfrak{D}_{x,y}$,

$$\langle D_{y_i}\tilde{I}(a)[T],\ \varphi\rangle = \langle \tilde{I}(a)[T],\ -D_{y_i}\varphi\rangle =$$

$$= \langle T,\ -I(a)[D_{y_i}\varphi]\rangle = \left\langle T,\ -\frac{1}{|a_i|}\int\limits_{\mathbf{R}^p} P_{a_i}[D_{y_i}\varphi]\mathrm{d}y\right\rangle =$$

$$= \left\langle T,\ -\frac{1}{|a_i|}\int\limits_{\mathbf{R}^p}\frac{\partial}{\partial\eta}P_{a_i}[\varphi]\mathrm{d}y\right\rangle =$$

$$= \left\langle T,\ -\frac{a_i}{|a_i|}\int\limits_{\mathbf{R}^p}\frac{\partial}{\partial x}P_{a_i}[\varphi]\mathrm{d}y\right\rangle =$$

$$= \left\langle a_i T,\ -\frac{1}{|a_i|}\frac{d}{dx}\int\limits_{\mathbf{R}^p}P_{a_i}[\varphi]\mathrm{d}y\right\rangle =$$

$$= \langle a_i DT,\ I(a)[\varphi]\rangle = \langle a_i\tilde{I}(a)[DT],\ \varphi\rangle$$

(For the signification of $P_{a_i}[\varphi]$ and of η, see the end the section 1).

PROPOSITION 4. *If for $T \in D'_x$ we have $y_i\tilde{I}(a)[T] = 0$ for an $i \in \in \{1, 2, \ldots, p\}$ and for an $a \in \mathbf{R}^p$, then $T = 0$.*

Proof:

Let $\chi \in \mathfrak{D}_y$ such that $\int\limits_{\mathbf{R}^p} y_i\chi(y)\mathrm{d}y = 1$. Let $\psi(x, y) = J(-a)[\varphi\chi](x, y)$ where $\varphi \in \mathfrak{D}_x$ is arbitrary. Then we have

$$\langle y_i\tilde{I}(a)[T],\ \psi\rangle = \langle \tilde{I}(a)[T],\ y_i J(-a)[\varphi\chi]\rangle =$$

$$= \langle T,\ I(a)[y_i J(-a)[\varphi\chi]]\rangle = \langle T,\ 1_y,\ y_i J(a)[J(-a)[\varphi\chi]]\rangle\rangle =$$

$$= \langle T,\ \langle 1_y,\ y_i\varphi\chi\rangle\rangle = \left\langle T,\ \varphi\int\limits_{\mathbf{R}^p} y_i\chi\ \mathrm{d}y\right\rangle = \langle T,\ \varphi\rangle$$

From $\langle T, \varphi\rangle = 0$ for any $\varphi \in \mathfrak{D}_x$ results $T = 0$.

3. We define for the testing functions of $p+1$ variables the following „finite difference operator of order p", which operator maps these functions in the space single variable testing functions:

$$\Delta_p : \mathfrak{D}_{x,y}(\mathbf{R}^{p+1}) \to \mathfrak{D}_x(\mathbf{R})$$

$$(5)\quad \Delta_p = I(1, 1, \ldots, 1) - [I(0, 1, \ldots, 1) + \ldots + I(1, 1, \ldots, 0)] + [I(0, 0, 1, \ldots, 1) + \ldots + I(1, 1, 1, \ldots, 0, 0)] + \ldots + (-1)^p I(0, 0, \ldots, 0)$$

For the distributions belonging to $\mathfrak{D}'_x(\mathbf{R})$ we define „the finite difference operator of order p" in the following manner:

$$\tilde{\Delta}_p : \mathfrak{D}'_x(\mathbf{R}) \to \mathfrak{D}'_{x,y}(\mathbf{R}^{p+1})$$

$$(6)\quad \langle\tilde{\Delta}_p[T],\ \varphi\rangle = \langle T,\ \Delta_p[\varphi]\rangle$$

for any $T \in \mathfrak{D}'_x(\mathbf{R})$ and $\varphi \in \mathfrak{D}_{x,y}(\mathbf{R}^{p+1})$.

We observe that if f is a regular distribution defined by the locally integrable function f, then $\tilde{\Delta}_p[f]$ is the regular distribution defined by the function

$$g(x, y) = f(x + y_1 + \ldots + y_p) - f(x + y_2 + \ldots + y_p) - \ldots - f(x + y_1 + \ldots + y_{p-1}) + f(x + y_3 + y_4 + \ldots + y_p) + \ldots + f(x + y_1 + y_2 + \ldots + y_{p-2}) + \ldots + (-1)^p f(x)$$

PROPOSITION 5. *For any $T \in \mathfrak{D}'_x(\mathbf{R})$ we have*

$$\tilde{\Delta}_p[T] = y_1 y_2 \ldots y_p\tilde{I}(\theta_1, \theta_2, \ldots, \theta_p)[D^p T]$$

where $\theta_i \in (0, 1)$, $i = 1, 2, \ldots, p$.

Proof:

Let $p = 1$. In this case we have

$$\Delta_1 : \mathfrak{D}_{x,y_1}(\mathbf{R}^2) \to \mathfrak{D}_x(\mathbf{R})\ ;\ \Delta_1 = I(1) - I(0)\ ;\ \tilde{\Delta}_1 : \mathfrak{D}'_x(\mathbf{R}) \to \mathfrak{D}_{x,y_1}(\mathbf{R}^2)$$

For any $T \in \mathfrak{D}'_x(\mathbf{R})$ and $\varphi \in \mathfrak{D}_{x,y_1}(\mathbf{R}^2)$, we have

$$\langle\tilde{\Delta}_1[T],\ \varphi\rangle = \langle T,\ \Delta_1[\varphi]\rangle = \langle T_x,\ \langle 1_{y_1},\ (J(1) - J(0))[\varphi](x, y_1)\rangle\rangle =$$

$$= \langle T_x,\ \langle 1_{y_1},\ -y_1 D_x J(\theta_1)[\varphi](x, y_1)\rangle = \langle T_x,\ \langle 1_{y_1},\ J(\theta_1)[-D_x y_1\varphi](x, y_1)\rangle\rangle =$$

$$= \langle T,\ I(\theta_1)[-D_x y_1\varphi)]\rangle = \langle T,\ -DI(\theta_1)[y_1\varphi]\rangle = \langle y_1\tilde{I}(\theta_1)[DT],\ \varphi\rangle$$

We obtain

$$\tilde{\Delta}_1[T] = y_1\tilde{I}(\theta_1)[DT]$$

where $\theta_1 \in (0, 1)$.

Now let $p = 2$. In this case we have

$$\Delta_2 : \mathfrak{D}_{x,y}(\mathbf{R}^3] \to \mathfrak{D}_x(\mathbf{R}),\ \Delta_2 = I(1, 1) - I(1, 0) - I(0, 1) + I(0, 0),$$

$$\tilde{\Delta}_2 : \mathfrak{D}'_x(\mathbf{R}) \to D'_{x,y}(\mathbf{R}^3)$$

We observe that for any $\varphi \in \mathfrak{D}_{x,y}(\mathbf{R}^3)$

$$(I(1, 1) - I(0, 1))[\varphi] = -DI(\theta_1, 1)[y_1\varphi]$$

and

$$(I(1, 0) - I(0, 0))[\varphi] = -DI(\theta_1, 0)[y_1\varphi]$$

occur, where $\theta_1 \in (0, 1)$. Hence

$$\Delta_2[\varphi] = -D(I(\theta_1,1) - I(\theta_1, 0)[y_1\varphi]$$

Since

$$(I(\theta_1, 1) - I(\theta_1, 0))[y_1\varphi] = -DI(\theta_1, \theta_2)[y_1y_2\varphi]$$

where $\theta_2 \in (0, 1)$, we obtain

$$\Delta_2[\varphi] = D^2I(\theta_1, \theta_2)[y_1y_2\varphi]$$

where $\theta_i \in (0, 1)$; $i = 1, 2$.

Hence

$$\langle\widetilde{\Delta}_2[T], \varphi\rangle = \langle T, \Delta_2[\varphi]\rangle =$$

$$= \langle T, D^2I(\theta_1, \theta_2)[y_1y_2\varphi]\rangle = \langle y_1y_2\, \widetilde{I}(\theta_1, \theta_2)[D^2T], \varphi\rangle$$

for any $T \in \mathfrak{D}'_x(\mathbf{R})$ and $\varphi \in \mathfrak{D}_{x,y}(\mathbf{R}^3)$.

In a similar way we can prove the relation from the proposition 5 for an arbitrary p.

4. Now let $p = 1$. For the testing — functions from $D_{x,h}$, where in this case h is a scalar variable, we define the following operator which maps these functions in the space $\mathfrak{D}_x$:

$$\Delta_1^n : \mathfrak{D}_{x,h} \to \mathfrak{D}_x$$

$$\Delta_1^n = \Delta_1^{n-1}(J(1) - J(0)),\ n \geqslant 2,\ \Delta_1^1 = \Delta_1$$

On the space $\mathfrak{D}'_x$, we define now the operator

$$\widetilde{\Delta}_1^n : \mathfrak{D}'_x \to \mathfrak{D}'_{x,h}$$

in the following manner

$$\langle\widetilde{\Delta}_1^n[T], \varphi\rangle = \langle T, \Delta_1^n[\varphi]\rangle$$

for any $T \in \mathfrak{D}'_x$, $\varphi \in \mathfrak{D}_{x,h}$ and $n \geqslant 1$.

If T is defined by a locally integrable function f, then

$$\langle\widetilde{\Delta}_1^1[T], \varphi\rangle = \langle T, \Delta_1^1[\varphi]\rangle = \int_{\mathbf{R}} f(x) \cdot \Delta_1^1[\varphi](x, h)\mathrm{d}y =$$

$$= \int_{\mathbf{R}^2} f(x)[\varphi(x-h, h) - \varphi(x, h)]\mathrm{d}x\mathrm{d}h = \int_{\mathbf{R}^2} [f(x+h) - f(x)]\ \varphi(x, h)\mathrm{d}x\mathrm{d}h$$

We conclude that in this case $\widetilde{\Delta}_1^1[T]$ is the distribution defined by the function $\alpha(x, h) = f(x+h) - f(x)$. Generally, if T is defined by the function f, then $\widetilde{\Delta}_1^n[T]$ is the distribution defined by the function

$$\beta(x, h) = f(x+nh) - C_n^1 f(x+(n-1)h) + C_n^2 f(x+(n-2)h) + \ldots + (-1)^n f(x)$$

PROPOSITION 6. *For any $T \in \mathfrak{D}'_x$, we have*

$$\widetilde{\Delta}_1^n[T] = h^n\widetilde{I}(n\theta)[D^nT]$$

occurs, where $\theta \in (0,1)$, $n \geqslant 1$.

Proof:

Let $n = 1$. For any $\varphi \in \mathfrak{D}_{x,h}$, we have

$$\langle\widetilde{\Delta}_1^1[T], \varphi\rangle = \langle T, \Delta_1^1[\varphi]\rangle = \langle T, \Delta_1[\varphi]\rangle =$$

$$= \left\langle T_x, \int_{\mathbf{R}} [\varphi(x-h, h) - \varphi(x, h)]\mathrm{d}h \right\rangle =$$

$$= \left\langle T_x, \int_{\mathbf{R}} -h\varphi'_x(x-\theta h, h)\mathrm{d}h \right\rangle = \langle T_x, \langle 1_h, -h\varphi'_x(x-\theta h, h)\rangle\rangle =$$

$$= \langle T_x, \langle 1_h, J(\theta)[-D_xh\varphi]\rangle\rangle = \langle T, I(\theta)[-D_xh\varphi]\rangle =$$

$$= \langle T, -D_xI(\theta)[h\varphi]\rangle = \langle h\widetilde{I}(\theta)[DT], \varphi\rangle$$

hence

$$\widetilde{\Delta}_1^1[T] = h\widetilde{I}(\theta)[DT]$$

where $\theta \in (0, 1)$.

In order to apply the proof by induction, we observe that

$$\langle\widetilde{\Delta}_1^n[T], \varphi\rangle = \langle T, \Delta_1^n[\varphi]\rangle = \langle T, \Delta_1^{n-1}(J(1) - J(0))[\varphi]\rangle =$$

$$= \langle\widetilde{\Delta}_1^{n-1}[T], (J(1) - J(0))[\varphi]\rangle$$

for any $T \in \mathfrak{D}'_x$ and $\varphi \in \mathfrak{D}'_{x,h}$.

Now, by induction hypothesis we have

$$\langle\widetilde{\Delta}_1^n[T], \varphi\rangle = \langle h^{n-1}I((n-1)\theta)[D^{n-1}T], (J(1) - J(0))[\varphi]\rangle =$$

$$= \left\langle h^{n-1}D^{n-1}T, \int_{\mathbf{R}} [\varphi(x-h-(n-1)\theta h, h) - \varphi(x-(n-1)\theta h, h)]\mathrm{d}h \right\rangle =$$

$$= \left\langle h^{n-1}D^{n-1}T, \int_{\mathbf{R}} -h\varphi'_x(x-(n-1)\theta h - \theta h, h)\mathrm{d}h \right\rangle =$$

$$= \langle h^n\widetilde{I}(n\theta)[D^nT], \varphi\rangle$$

for any $T \in \mathfrak{D}'_x$, $\varphi \in \mathfrak{D}_{x,h}$ and the proposition is proved.

5. Let be the functional equation

$$\widetilde{\Delta}_p[T] = 0$$

where $T \in \mathfrak{D}'_x$.

Using the proposition 5 we can write

$$y_1 y_2 \ldots y_p \widetilde{I}(\theta_1, \theta_2, \ldots, \theta_p)[D^p T] = 0$$

Hence $D^p T = 0$, which relation leads to the

PROPOSITION 7. *The general solution of the functional equation*

$$\widetilde{\Delta}_p[T] = 0$$

in distributions is a regular distribution defined by an arbitrary polynomial of degree at most $p - 1$.

We consider also the functional equation

$$\widetilde{\Delta}_1^n[T] = 0$$

where $T \in \mathcal{D}_x$.

Using the proposition 6 we can write

$$h^p \widetilde{I}(n\theta)[D^n T] = 0$$

We deduce $D^n T = 0$ and we have the

PROPOSITION 8. *The general solution of the functional equation*

$$\widetilde{\Delta}_1^n[T] = 0$$

in distributions is a regular distribution defined by an arbitrary polynomial of degree at most $n - 1$.

Corollary 2. *The general solution of the Fréchet functional equation*

$$f(x + y_1 + y_2 + \ldots + y_p) - f(x + y_2 + y_3 + \ldots + y_p) - \ldots - f(x + $$
$$+ y_1 + y_2 + \ldots + y_{p-1}) + \ldots + (-1)^p f(x) = 0$$

in the class of the locally integrable functions is an arbitrary polynomial of degree at most $p - 1$.

Corollary 3. *The general solution of the functional equation*

$$f(x + nh) - C_n^1 f(x + (n-1)h) + \ldots + (-1)^n f(x) = 0$$

in the class of locally integrable functions is an arbitrary polynominal of degree at most $n - 1$.

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