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ON THE ACCELERATION WAVES IN SIMPLE FLUIDS

by

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1. Introduction

In the theories of the dynamical behavior of continua, there are several ways of describing the dissipative effects. The oldest way emploied a viscous stress, as is done in the theory of Navier-Stokes fluids. It is undoubted the fact that, this theory is appropriate for the mechanical behavior of many real fluids.

However, the experiments (see for example [1] §116 or [2] §1) have shown that it is not sufficient to predict all the phenomena which may be observed. In order to remove this inconvenience, many researchers have proposed and studied various concepts of a fluid, each more general than its predecesors. One of the most general of them, is that of a simple fluid [1] §32, [2], §7, [3] II §4.14 and so on.

The aim of this work, is to present a general study of the acceleration waves, propagating through a such fluid. As a simple fluid is a special case of a simple material, some of our results could be obtained by specializing of those from [4], but we have found that for fluids it is easier and more instructiv to start again from first principles. In section 2 we give the basic equations and the smooth assumptions upon the constitutive functionals. In the sequel it was presented different aspects with respect to singular surfaces, compatibility conditions and acceleration waves. The last two sections are devoted to the study of acceleration waves propagation in simple fluids.

2. Basic equatios and smooth assumptions

The constitutive equation of a simple fluid, as it was deduced in [1] §32 is

$$T = -P(\rho)\mathbf{1} + \underset{s=0}{\overset{\infty}{\mathcal{H}}}(G(s)\,;\,\rho) \tag{2.1}$$

where T is the stress tensor, $G(\cdot)=F^t(\cdot)^T F^t(\cdot) - \mathbf{1}$ with $F^t(s)=F(t-s)$[1] the history of the relative deformation gradient, ρ is the density, p is a hydrostatic pressure, $\mathbf{1}$ is the unit tensor and $\mathcal{H}$ is a response functional which must satisfy the isotropy relation

$$Q\underset{s=0}{\overset{\infty}{\mathcal{H}}}(G(s)\,;\,\rho)Q^T = \underset{s=0}{\overset{\infty}{\mathcal{H}}}(QG(s)Q^T\,;\,\rho) \tag{2.2}$$

identically in G, ρ and the orthogonal tensor Q.

Some special classes of simple fluids are those of integral type[2] [1] §37. The constitutive equation of the fluids of the integral type of the first order is (for its finding it was used (2.2) too)

$$T = \left[-P(\rho) + \int_0^\infty \lambda(\rho, s)\ \mathrm{tr}\ G(s)ds\right]\mathbf{1} + \int_0^\infty \mu(\rho, s)G(s)ds \tag{2.3}$$

where λ and μ are material scalar functions.

If the fluid is incompressible[3], the equations (2.1) and (2.3) must be replied by

$$T = -P\mathbf{1} + \underset{s=0}{\overset{\infty}{\mathcal{H}}}(G(s)) \tag{2.4}$$

and

$$T = -P\mathbf{1} + \int_0^\infty \mu(s)\,G(s)ds \tag{2.5}$$

In the next, we shall need of some smoothness hypothesis on the constitutive functional $\mathcal{H}$ and on the function p. Such we shall assume that $\mathcal{H}$ has for its domain of definition an open subset $\mathcal{D}$ of a normed

[1] It will be understood that ρ and T are functions of the particle x and time t and $F^t(s)$ depend also of x.

[2] Their constitutive equations have been obtained as asymptotic approximations of those of simple fluids.

[3] In this case, the density ρ at a particle cannot depend on time, and hence it can be omited in the constitutive equation.

space H and it is continuously Fréchet-differentiable throughout this domain. As functions of ρ, $\mathcal{H}$ and p are also assumed to be continuous differentiable.

Finally, for later use, we give below the forms of the balance laws (conservation of mass and balance of momentum)

$$\dot{\rho} + \rho\ \mathrm{div}\ \dot{x} = \dot{\rho} + \rho\ \mathrm{tr}\ D = 0 \tag{2.6}$$

and

$$\mathrm{div}\, T + \rho b = \rho\ddot{x},\quad T = T^T \tag{2.7}$$

where $D = \frac{1}{2}(L + L^T)$ is called stretching tensor [1] §24, $L = \mathrm{grad}\ \dot{x}$ and b is the body force.

3. Singular surfaces and compatibility conditions

Let Σ be a moving regular surface whose equation in the Cartesian coordinate sistem x_k, is

$$\Sigma(x,\ t) = 0 \tag{3.1}$$

Its normal velocity U_n is defined by [5] the relation (177.6) or [6] Ch. XII the relation 3, where the term of displacement velocity is used)

$$U_n = -\frac{\partial\Sigma}{\partial t}\Big/|\mathrm{grad}\ \Sigma|\,n \tag{3.2}$$

where

$$n = \mathrm{grad}\ \Sigma/|\mathrm{grad}\ \Sigma| \tag{3.3}$$

is the unit normal. The magnitude of the normal velocity

$$U_n = U_n \cdot n = -\frac{\partial\Sigma}{\partial t}\Big/|\mathrm{grad}\ \Sigma| \tag{3.4}$$

is just the rate of advance of the surface, as seen by an observer at rest. The quantity (see [5] the relation (183.5) or [7] the relation (4.11))

$$U = U_n - \dot{x}\cdot n \tag{3.5}$$

which is a measure of the normal speed of the surface Σ with respect to the material particles that are instantaneously situated upon it, will be called local speed of propagation.

Let $\psi(x,\ t)$ be now a continuous and differentiable function of x and t on each side of the moving surface Σ. Its jump $[\psi]$ across Σ is defined by

$$[\psi] = \psi^+ - \psi^- \tag{3.6}$$

where ψ^+ and ψ^- are the limiting values of ψ on the two sides of this surface.

DEFINITION 3.1. *A singular surface with respect to the function ψ is defined as a surface across which $[\psi] \neq 0$.*

If the function ψ is continuous across Σ, but its partial derivatives $\psi_{,k}$ are discontinuous, then the following geometrical conditions of compatibility[4] [8] II §2)

$$[\psi_{,k}] = [n^k \psi_{,k}] n_k \tag{3.7}$$

are true.

The corresponding kinematical condition has also the special form [8] II §3)

$$\left[\frac{\partial \psi}{\partial t}\right] = - U_n [n^k \psi_{,k}] \tag{3.8}$$

DEFINITION 3.2. *A singular surface Σ for which $[\psi] = 0$ and at least one of its partial derivatives $\psi_{,k}$ or $\frac{\partial \psi}{\partial t}$ is discontinuous, will be said to be singular of order one, relative to the function ψ.*

In the following we shall retain our attention upon singular surfaces of second order.

4. Acceleration waves

Let us now consider a motion, whose velocity field is

$$\dot{x} = \dot{x}(x, \; t) \tag{4.1}$$

DEFINITION 4.1. *A surface Σ is called an acceleration wave or a second order wave with respect to our motion, if:*

a) *$\dot{x}$ is a continuous function of x and t jointly for all x and t.*

b) *$\ddot{x}$ and* grad *$\dot{x}$ have jump discontinuities across Σ but are continuous in x and t jointly everywhere else.*

c) *the function $t \to G(x, \cdot)$ has values in H and is continuously differentiable with respect to its norm*[5]).

For an acceleration wave the following geometrical and kinematical conditions of compatibility ([5] §190 or [8] §5,6)

$$[x^k, {}_{ij}] = a^k n_i n_j, \quad a^k = [n^i n^j x^k, {}_{ij}] \tag{4.2}$$

and

$$[\ddot{x}^k] = U^2 a^k \tag{4.3}$$

can be easy obtained from (3.7) and (3.8).

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[4] In [5] § 175 it is given as Maxwell's theorem.

[5] More precisely, $G(x,.)$ is a smooth function of t with respect to the H-norm [9] § 5. A such condition limits the wildness of the past history of a point.

DEFINITION 4.2. *The vector a, whose components are given by (4.2) will be called the amplitude of the wave.*

DEFINITION 4.3. *An acceleration wave for which*

$$a \times n = 0 \; or \; a \cdot n = 0 \tag{4.4}$$

is called a longitudinal or transversal one.

In terms of a we can easy write the next two useful kinematical conditions of compatibility

$$[L] = - U a \otimes n \tag{4.5}$$

and

$$[D] = - \frac{U}{2} (a \otimes n + n \otimes a), \quad [\mathrm{tr}\ D] = - U a \cdot n \tag{4.6}$$

Other restrictions on the possible discontinuities in the derivatives will result from (2.6) and (2.7). Such conditions will be called dynamical conditions of compatibility. In the assumption that the body force is a continuous function of x and t, they are

$$[\dot{\rho}] + \rho [\mathrm{tr}\ D] = 0 \tag{4.7}$$

and

$$[\mathrm{div}\ T] = \rho [\ddot{x}] \tag{4.8}$$

Now, by combining (4.3), (4.8) and a simple consequence of (3.8)

$$U [\mathrm{div}\ T] = - [\dot{T}] n \tag{4.9}$$

we attain to

$$[\dot{T}] n + \rho \, U^3 a = 0 \tag{4.10}$$

5. The velocity and the amplitude of acceleration waves

In order to characterize the two specific sizes of acceleration waves, let us firstly derive (2.1) with respect to t[6])

$$\dot{T} = - \rho \frac{dp}{d\rho} \mathbf{1} + \partial_\rho \underset{s=0}{\overset{\infty}{\mathscr{H}}} (G(s)\, ; \; \rho) \dot{\rho} + \delta_G H (G(s)\, ; \; \rho / \dot{G}(s)) \tag{5.1}$$

This relation, together with ([10] the relation [1.3.15)$_1$)

$$\dot{G}(s) = - \frac{'d}{ds} G(s) - L^T G(s) - G(s) L - L^T - L$$

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[6] Here, the notation „/" is meant to indicate the linear dependence of the functiona $\rho_G \mathscr{H}$ with respect to its last argument.

(4.5), $(4.6)_2$, (4.7) and the definition (4.1) lead to

$$[\dot{T}] = \rho U \left[-\frac{dp}{d\rho} \mathbf{1} + \partial_\rho \overset{\infty}{\underset{s=0}{T}} (G(s)\,;\ \rho) \right] a \cdot n + \tag{5.2}$$
$$+ 2U \delta_G \overset{\infty}{\underset{s=0}{T}} (G(s)\,;\ \rho/C(s))\ a \otimes n$$

where $C(s) = G(s) + \mathbf{1}(s)$ with $\mathbf{1}(s) = \mathbf{1}$ for each $s \in [0, \infty)$.

Now, introducing (5.2) in (4.10) it obtain

THEOREM 4.1. *The amplitude a and the velocity u of an acceleration wave traveling in the direction n of a simple fluid of the type* (2.1), *must obey the propagation condition*

$$Q(n)a = u^2 a \tag{5.3}$$

wher e

$$Q_{ij}(n) = \frac{dp}{d\rho} n_i n_j - \partial_\rho \overset{\infty}{\underset{s=0}{\mathscr{H}_{ik}}} (G(s)\,;\ \rho)\, n_k n_j -$$
$$- \frac{2}{\rho} \partial_{G_{em}} \overset{\infty}{\underset{s=0}{\mathscr{H}_{ik}}} (G(s)\,;\ \rho/C_{ej}(s))\, n_m n_k$$

are the components of the instantaneous acoustic tensor [4] §2.

Remark 5.1. The relation (5.3) tell us that any real right proper vector a of $Q(n)$ is a possible amplitude vector if its corresponding eigenvalue is real and positive. Generally, $Q(n)$ have not real positive eigenvalues and than real proper vectors.

Following [1] §71 we can conclude that if

$$V \cdot Q(n) v > 0$$

for any unit vector V, than in a given direction n, there is at least one real amplitude with positive velocity.

Furthermore, if a is a such real amplitude, then

$$U^2 = (a \cdot Q(n) a)/a$$

where a is the norm of the vector a.

Remark 5.2. For fluids of integral type of the first order, the acoustic tensor $Q(n)$ takes the simplified form

$$Q(n) = \left\{ \left[\frac{dp}{d\rho} \int_0^\infty \partial_\rho \lambda(\rho, s)\ \operatorname{tr} G(s) ds \right] \mathbf{1} - \right. \tag{5.4}$$
$$\left. - \int_0^\infty \partial_\rho \mu(\rho, s)\ G(s) ds \right\} n \otimes n - \frac{n \cdot n}{\rho} \int_0^\infty \mu(\rho, s)\, C(s) ds -$$
$$- \frac{n \otimes n}{\rho} \int_0^\infty [2\lambda(\rho, s) + \mu(\rho, s)]\ C(s) ds$$

If this tensor will be simmetric, i.e. if

$$n \otimes n\ G(\cdot) = G(\cdot)\ n \otimes n$$

then its characteristic equation

$$|Q(n) - u^2 \mathbf{1}| = 0$$

will have only real roots.

Remark 5.3. If the fluid is perfect then the corresponding acoustic tensor

$$Q(n) = \frac{dp}{d\rho} n \otimes n \tag{5.5}$$

has only one real eigenvalue $u^2 = \frac{dp}{d\rho}$ and

$$u = \sqrt{\frac{dp}{d\rho}} \tag{5.6}$$

is just the acoustic propagation speed [6] XII the relation 32).

Furthermore, from (5.3) and (5.5) we find known result [6] II §8

$$a \times n = 0 \tag{5.7}$$

i.e., an acceleration wave in a perfect fluid is necessary longitudinal.

Remark 5.4. If the fluid is incompressibil ρ is a constant, such that $Q(n)$ will take simplified forms. In the liniar case, for example

$$Q(n) = -\frac{1}{\rho} [n \cdot n \mathbf{1} + n \otimes n] \int_0^\infty \mu(s)\ C(s) ds \tag{5.8}$$

Generally, from $(4.6)_2$ and (4.7) it results

$$a \cdot n = 0 \tag{5.9}$$

i.e., in incompressible simple fluids only transversal acceleration waves are possible.

6. Homothermal acceleration waves

In this section we shall allow the stress to be affected not only by the history of the strain, but also the histories of thermodynamic variables. Such, in view of [10] §1.3. we can write T under the form

$$T = -P(\rho)\mathbf{1} + \overset{\infty}{\underset{s=0}{\mathscr{H}}} ({}_r\Gamma^t(s)\,;\ \theta,\ \rho) \tag{6.1}$$

where ${}_r\Gamma^t = (G, {}_r\theta^t, {}_r\bar{g}^t)$ is the past history of Γ^t, θ is the temperature, $\bar{g} = F^T g$ and $g = \operatorname{grad}\ \theta$.

The liniarized form of (6.1) [10] §1.3.) is

$$T = -P(\rho)\mathbf{1} + \left[\nu(\rho,\ 0)\,\theta + \int_0^\infty \nu'(\rho,\ s)\,\theta^t(s)ds + \right.$$

(6.2)

$$\left. + \int_0^\infty \lambda(\rho,\ s)\ \mathrm{tr}\, G(s)ds\right]\mathbf{1} + \int_0^\infty \mu(\rho,\ s)\ G(s)ds$$

The smooth assumptions upon the functional $\mathcal{H}$ are an easy extension of those of the second section and we shall suppose them to be tacit satisfied.

DEFINITION 6.1. *A singular surface ε with respect to a motion, characterized by the fields $\chi(x,\ t)$ and $\theta(x,\ t)$, will be called an acceleration wave if:*

a) *χ and θ are continuous functions of x and t jointly for all x and t.*

b) *$\ddot{\chi}$, grad $\dot{\chi}$ and $\dot{\theta}$ have jump discontinuities across ε, but are continuous in x and t jointly everywhere else.*

c) *for each x, the function $t \to {}_r\Gamma^t(x,\ \cdot)$ has values in H and is continuously differentiable with respect to its norm.*

DEFINITION 6.2. *An acceleration wave for which*

(6.5) $$[\dot{\theta}] = 0,\ [\mathrm{grad}\ \theta] = 0$$

will be called a homothermal wave[7] [11] §3.

Now, derivating T from (6.1) and taking into account the precedent relation of (5.2) and [10] the relation $(1.3.15)_2$)

(6.4) $$\dot{\bar{g}}(s^t) = -\frac{d}{ds}\,\bar{g}^t(s) - L^T\bar{g}^T(s)$$

we attain to

$$\dot{T} = -\frac{dp}{d\rho}\dot{\rho}\mathbf{1} + \partial_\rho \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho)\dot{\rho} + \partial_\theta \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho)\dot{\theta} -$$

(6.5) $$-\ \delta_G \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho/L^TC(s) + C(s)L) - \delta_g \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho/L^T\bar{g}^t(s)) -$$

$$-\ \delta_\Gamma \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho\left/\frac{d}{ds}\,{}_r\Gamma^t(s)\right.)$$

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[7] It is a well known fact that an acceleration wave in a definite conductor of heat is homothermal [4] § 6.

From (6.5), (4.5), $(4.6)_2$ and (4.7) it result that for a homothermal wave

$$[\dot{T}] = \rho u\left[-\frac{dp}{d\rho} + \partial_\rho \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho)\right]a\cdot n +$$

(6.6) $$+\ 2v\,\delta_G \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho/C(s))a\otimes n +$$

$$+\ u\left[\delta_g \underset{s=0}{\overset{\infty}{T}}({}_r\Gamma^t(s)\,;\ \theta,\ \rho/a\otimes n\bar{g}^t(s))\right]$$

This last relation together with (4.10) lead to

THEOREM 6.1. *The amplitude a and the velocity u of a homothermal acceleration wave, traveling in the direction n of a simple fluid of the type (6.1) must obey the propagation condition*

(6.7) $$Q^*(n)a = u^2a$$

where

$$Q^*_{ij}(n) = \frac{dp}{d\rho}n_in_j - \partial_\rho \underset{s=0}{\overset{\infty}{\mathcal{H}}}{}_{ik}({}_r\Gamma^t(s)\,;\ \theta,\ \rho)n_kn_j -$$

$$-\ \frac{2}{\rho}\,\delta_{G_{lm}} \underset{s=0}{\overset{\infty}{\mathcal{H}}}{}_{ik}({}_r\Gamma^t(s)\,;\ \theta,\ \rho/C_{lj}(s))n_mn_k -$$

$$-\ \frac{1}{\rho}\,\delta_{g_l} \underset{s=0}{\overset{\infty}{\mathcal{H}}}{}_{ik}({}_r\Gamma^t(s)\,;\ \theta,\ \rho/\bar{g}^t_m(s))n_jn_k$$

are the components of instantaneous acoustic tensor corresponding to ρ, θ and ${}_r\Gamma^t(\cdot)$.

Remark 6.1. In the case of the linearized theory when T is given by (6.2), this tensor has the simplified form

$$Q^*(n) = Q(n) - \left[\theta\partial_\rho\nu(\rho,\ 0) + \int_0^\infty \partial_\rho\nu'(\rho,\ s)\,\theta^t(s)ds\right]n\otimes n$$

while, for incompressible fluids

$$Q^*(n) = Q(n)$$

Obviously, in each case $Q^*(n)$ is simmetric if and only if $Q(n)$ do so.

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