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ON A METHOD OF THIRD ORDER IN FRÉCHET SPACES

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Let X be a real Fréchet space with a quasinorm induced by an invariant distance d i.e. $||x|| = d(x, 0)$ (see e.g [4, p. 14]) and $P : X \to X$ a continuous mapping. We shall note by $[x', x''; P]$ and $[x', x'', x'''; P]$ the symmetrical divided difference of the first, respectively the second order of the mapping P in the points $x', x'', x''' \in X$. Let's consider the equation

$$(1) \qquad P(x) = x - \Phi(x) = 0.$$

If we have the sequence (x_n) in X, then the sequence (u_n) is defined by $u_n = \Phi(x_n)$, and $\Gamma_n = [x_n, u_n; P]^{-1}$ the inverse of the linear mapping $[x_n, u_n; P]$, if it exists. There exist the results concerning the existence of the solution of the equation (1) in Banach spaces [1], as the limit of the sequence given by

$$(2) \qquad x_{n+1} = x_n - \Gamma_n[P(x_n) + P(x_n - \Gamma_n P(x_n))], \quad n = 0, 1, 2, \ldots$$

The purpose of this paper is to generalize these results for the case of the Fréchet spaces.

Remark. In this paper we use the quasinorm of the linear mappings L having Liepschitz-property i.e. there exists $M > 0$ so that for every

$$x \in X, \ ||L(x)|| \leqslant M||x||$$

and

$$||L|| = \inf \{M > 0 : ||L(x)|| \leqslant M||x|| \text{ for every } x \in X\}.$$

THEOREM. *We suppose that there is a point $x_0 \in X$ and the numbers B_0, d_0, K and M so that:*

(*i*) *there exists $\Gamma_0 = [x_0, u_0; P]^{-1}$ and $||\Gamma_0|| \leqslant B_0$;*

(*ii*) $||\Gamma_0|| \cdot ||P(x_0)|| = ||\Gamma_0|| \cdot ||x_0 - \Phi(x_0)|| \leqslant B_0||x_0 - u_0|| \leqslant d_0$;

(iii) $\sup\{\, ||[x', x''; \Phi]|| : x', x'' \in \bar{S}\} \leqslant M$ *and*

$\sup\{\, ||[x', x'', x'''; P]|| : x', x'', x''' \in \bar{S}\} \leqslant K$;

(iv) $B_0(||P(x_0)|| + ||P(y_0)||) \leqslant d_0(1 + B_0KMd_0) = \eta_0$,

where $y_0 = x_0 - \Gamma_0 P(x_0)$;

(v) $h_0 = B_0K(1+M)\eta_0 < \frac{4}{9}$,

where $\bar{S} = \{x \in X : ||x - x_0|| \leqslant r\}$, $r = \eta_0\left(2 + \frac{5}{B_0}\right)$.

In these conditions there exists the sequence (x_n) *defined by* (2) *having the following properties:*

(j) $x^* = \lim\limits_{n\to\infty} x_n$, $x^* \in \bar{S}$ *exists and* x^* *is a solution of equation* (1);

(jj) the rapidity of the convergence of the sequence (x_n) *to the solution* x^* *of the* (1) *is given by*

$$||x_n - x^*|| \leqslant \frac{9}{4}\left(\frac{5}{9}\right)^n\left(\frac{51}{25}h_0\right)^{3^n - 1}\eta_0, \qquad n = 1, 2, \ldots$$

Remark. The radius of $\bar{S}$ and the numerical estimation in inequality (jj) are not optimal but they are chosen for the convenience of calculation.

Proof. We consider the mapping

$$F : X \to X,\ F(x) = x - \Gamma_0 P(x).$$

We have

$$F(x_0) = F(u_0) = y_0,\ [x', x''; F] = I - \Gamma_0[x', x''; P]$$

and

$$[x', x'', x'''; F] = -\Gamma_0[x', x'', x'''; P]$$

therefore

$$[x_0, u_0; F] = I - \Gamma_0[x_0, u_0; P] = 0$$

and

$$[x_0, u_0, y_0; F] = -\Gamma_0[x_0, u_0, y_0; P],$$

where I is the identity mapping of X. According to the definition of the divided differences [2] we obtain

$$[x_0, u_0, y_0; F](y_0 - x_0)(y_0 - u_0) = -\Gamma_0 P(y_0).$$

We have thus $P(y_0) = [x_0, u_0, y_0; P](y_0 - x_0)(y_0 - u_0)$ wich by

$$||y_0 - x_0|| = ||\Gamma_0 P(x_0)|| \leqslant d_0 \leqslant \eta_0 < r,\ ||u_0 - x_0|| = ||P(x_0)|| \leqslant \frac{d_0}{B_0} \leqslant \frac{\eta_0}{B_0} < r,$$

that means y_0, $u_0 \in \bar{S}$ and

(3) $$||P(y_0)|| \leqslant K||y_0 - x_0|| \cdot ||y_0 - u_0||.$$

Further we have

$$y_0 - u_0 = \Gamma_0 P(u_0) = \Gamma_0[x_0, u_0; \Phi](x_0 - u_0),$$

from which using $||x_0 - u_0|| \leqslant \frac{d_0}{B_0}$ we obtain

(4) $$||y_0 - u_0|| \leqslant Md_0.$$

From (3) we get

(5) $$||P(y_0)|| \leqslant d_0^2 KM.$$

By (i) and (2) we can construct the point x_1, thus $u_1 = \Phi(x_1)$ too. We check the conditions (i)—(v) for the point x_1 with analogous numbers B_1, d_1, K and M. Using (2) by (i), (ii) and (5) it rezults:

(6) $$||x_1 - x_0|| \leqslant ||\Gamma_0 P(x_0)|| + ||\Gamma_0 P(y_0)|| \leqslant \eta_0 < r,$$

which means that $x_1 \in \bar{S}$.

Considering that $x_1 \in \bar{S}$ we obtain

(7) $$||u_1 - u_0|| = ||\Phi(x_1) - \Phi(x_0)|| = ||[x_1, x_0; \Phi](x_1 - x_0)|| \leqslant \leqslant M||x_1 - x_0|| \leqslant M\eta_0.$$

According to the definition of the divided differences we have

$$P(x_1) = [x_0, y_0, u_0; P](y_0 - u_0)\Gamma_0 P(y_0) + [x_1, x_0, y_0; P](x_1 - x_0)(x_1 - y_0).$$

Using (4), (5), (6) and the equality $x_1 = y_0 - \Gamma_0 P(y_0)$ we obtain

(8) $$||P(x_1)|| \leqslant \frac{B_0^2 K^2 \eta_0^2}{B_0}(1+M)^2\eta_0 = \frac{h_0^2}{B}\eta_0 < \frac{\eta_0}{B_0}.$$

From (8) it rezults

$$||x_0 - u_1|| = ||x_0 - \Phi(x_1)|| = ||x_0 - x_1 + P(x_1)|| \leqslant \eta_0\left(1 + \frac{1}{B_0}\right),$$

therefore $u_1 \in \bar{S}$.

Using the condition (iii), (6) and (7) we may write:

(9) $$||\Gamma_0([x_0, u_0; P] - [x_1, u_1; P])|| \leqslant \Gamma_0([x_0, u_0; P] - - [x_1, u_0; P])|| + ||\Gamma_0([x_1, u_0; P] - [x_1, u_1; P])|| \leqslant \leqslant B_0K(\eta_0 + \eta_0 M) = B_0K(M+1)\eta_0 = h_0 < 1.$$

Because

$$\{\Gamma_0[x_1, u_1; P]\}^{-1} = \{I - \Gamma_0([x_0, u_0; P] - [x_1, u_1; P])\}^{-1}$$

from (9) it rezults the existence of the mapping

$$\{\Gamma_0[x_1, u_1; P]\}^{-1},$$

and the inequality

$$\{\Gamma_0[x_1, u_1; P]\}^{-1} \leqslant \frac{1}{1-h_0}.$$

Using further the equality

$$\{\Gamma_0[x_1, u_1; P]\}^{-1}\Gamma_0 = \Gamma_1$$

we obtain

$$\text{(10)} \qquad ||\Gamma_1|| \leqslant \frac{B_0}{1-h_0} = B_1,$$

which means that (i) is satisfied for the points x_1 and u_1 with the number B_1. By (8) and (10) we have

$$||\Gamma_1|| \cdot ||P(x_1)|| \leqslant \frac{h_0^2}{1-h_0}\eta_0 = d_1.$$

Thus the hypothesis (ii) for x_1 is verified with the number d_1. From the existence of the mapping $\Gamma_1 = [x_1, u_1; P]^{-1}$ it rezults that we can construct x_2. If $y_1 = x_1 - \Gamma_1 P(x_1)$, then $y_1 \in \bar{S}$, because

$$||y_1 - x_0|| \leqslant ||y_1 - x_1|| + ||x_1 - x_0|| \leqslant ||\Gamma_1 P(x_1)|| + \eta_0 \leqslant$$

$$\leqslant \left(\frac{h_0^2}{1-h_0} + 1\right)\eta_0 \leqslant \frac{61}{45}\eta_0 < r.$$

Making the same reasoning we used for obtaining $P(y_0)$ we get

$$||\Gamma_1|| \cdot ||P(y_1)|| \leqslant B_1 d_1^2 KM,$$

whence using (8) it results

$$||x_2 - x_1|| \leqslant B_1(||P(x_1)|| + ||P(y_1)||) \leqslant d_1(1 + B_1 d_1 KM) = \eta_1,$$

which means that (iv) is satisfied for the points x_1 with the numbers B_1, d_1, K, M and the following relations are true:

$$\eta_1 = \frac{h_0^2}{1-h_0}\left(1 + \frac{h_0^2}{(1-h_0)^2}\right)\eta_0 \leqslant \frac{1}{5}\left(\frac{17}{5}\right)^2 h_0^2\,\eta_0 \leqslant \eta_0.$$

We have

$$||x_2 - x_0|| \leqslant ||x_2 - x_1|| + ||x_1 - x_0|| \leqslant \eta_1 + \eta_0 < r,$$

and

$$h_1 = B_1 K(M+1)\eta_1 = \frac{B_0}{1-h_0}K(M+1)\,d_1(1 + B_1 d_1 KM) \leqslant$$

$$\leqslant \left(\frac{68}{75}\right)^2 h_0 < h_0 < \frac{4}{9}.$$

So the numbers B_1, d_1, K and M verify the conditions $(i)-(v)$. By induction we can prove the following relations:

$$B_n = \frac{B_{n-1}}{1-h_{n-1}};$$

$$d_n = \frac{h_{n-1}^2}{1-h_{n-1}}\eta_{n-1},$$

$$\eta_n = d_n(1 + B_n d_n KM);$$

$$h_n = B_{n-1}K(M+1)\eta_{n-1}$$

and

$$x_n, u_n, y_n \in \bar{S}$$

for all pozitive integers. From the former inequalities we obtain

$$\text{(11)} \qquad h_n \leqslant \frac{25}{51}\left(\frac{51}{25}h_0\right)^{3^n}$$

and

$$\text{(12)} \qquad \eta_n \leqslant \left(\frac{5}{9}\right)^n \left(\frac{51}{25}h_0\right)^{3^n-1}\eta_0.$$

Using the inequality $||x_{n+1} - x_n|| \leqslant \eta_n$, and the relation (12) we obtain

$$\text{(13)} \qquad ||x_{n+p} - x_n|| \leqslant \eta_n + \eta_{n+1} + \ldots + \eta_{n+p-1} < \left(\frac{5}{9}\right)^n\left(\frac{51}{25}h_0\right)^{3^n-1}\eta_0.$$

The space X being complete it rezults the existence of the limit of the sequence (x_n), and $\lim\limits_{n\to\infty} x_n = x^* \in \bar{S}$. For $p \to \infty$, the inequality (13) gives the rapidity of the convergence of the sequence (x_n).

We shal prove that the point x^* is the solution of the equation (1). By the inequality

$$||P(x_n)|| \leqslant \frac{d_n}{B_n},$$

using the relation $B_n \geqslant B_0$ for all $n = 1, 2, 3, \ldots$ and the formula of d_n, it rezults

$$(14) \qquad ||P(x_n)|| \leqslant \frac{9}{4B_0} h_{n-1}^2 \eta_{n-1}.$$

The inequality (14), using (11) and (12) gives

$$\lim_{n\to\infty} ||P(x_n)|| = ||P(x^*)|| = 0 \text{ thus } P(x^*) = 0.$$

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