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ON THE LINEARIZATION TECHNIQUE FOR QUASIMONOTONIC OPTIMIZATION PROBLEMS

by

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1. Introduction

We consider the maximization problem of a quasimonotonic function $f: D \to \mathbf{R}$ on a closed set $D \subseteq \mathbf{R}^n$. For solving this problem a linearization technique consisting in the successively (exactly or approximatively) solution of certain optimization problems with the same feasible set D and with linear objective functions is used.

This linearization technique was employed by several authors for solving certain optimization problems with linear constraints and a quasimonotonic objective function such as: KUCHER B.M. [9], ŢIGAN S. [13], [14], BHATT S. K. [3] (in the case of linear constraints), BECTOR C.R., JOLLY P. L. [1], ŢIGAN S. [14] (for integer linear constraints), BECTOR C. R., BHATT S.K. [2] (for interval linear constraints).

In this paper sufficient conditions for the convergence (finite or infinite) of this linearization method for quasimonotonic optimization problems are given. We will show that this method can be applied to solve certain quasimonotonic optimization problems on the graphs, when, for instance, the set D consists of all the spanning trees or of all the elementary paths between two fixed vertices. In particular, when the objective function is a fractional one the linearization technique is equivalent to some known algorithms for fractional optimization in graphs (see [4], [5], [7], [11]).

2. Definition and preliminary results

Let $X \subseteq \mathbf{R}^n$ be a non-void open convex set and let $f: X \to \mathbf{R}$ be a differentiable function on X. We denote by

$$\nabla f(x') = \left(\frac{\partial f(x')}{\partial x_1}, \ldots, \frac{\partial f(x')}{\partial x_n}\right)$$

the gradient of f in the point x'.

On the set X the function f is said to be:

(i) pseudoconcave if $\nabla f(x') \cdot (x'' - x') \leqslant 0$ implies $f(x'') \leqslant f(x')$, for every x', x'' in X;

(ii) pseudoconvex if $-f$ is pseudoconcave;

(iii) quasiconcave if $f(x') \leqslant f(x'')$ implies $f(x') \leqslant f(tx' + (1-t)x'')$ (or, equivalently, $f(x') \leqslant f(x'')$ implies $\nabla f(x') \cdot (x'' - x') \geqslant 0$) for every x', x'' in X and $t \in]0, 1[$;

(iv) quasiconvex if $-f$ is quasiconcave;

(v) quasimonotonic if f is both quasiconcave and quasiconvex;

(vi) pseudomonotonic if f is both pseudoconcave and pseudoconvex.

Let $D \subseteq X$ be a non-void closed set. We will denote by co(D) the convex hull of the set D, that is:

$$\text{co}(D) = \left\{y \in \mathbf{R}^n : \exists x^i \in D,\ \exists t_i \geqslant 0\ (i = 1, 2, \ldots, k), \text{ such that } y = \sum_{i=1}^{k} t_i x^i \text{ and } \sum_{i=1}^{k} t_i = 1\right\}.$$

It is known (see [12], Theorem 17.2, p.158) that if D is a closed bounded set then co(D) is a closed bounded set too.

We consider the following optimization problem:

P. Find

$$s = \max\{f(x) : x \in D\},$$

where f is a differentiable function on the convex set X, and D is a closed bounded non-void subset of X.

We associate to problem P the following optimization problem with convex feasible set:

$P1$. Find

$$s_1 = \max\{f(x) : x \in \text{co}(D)\}.$$

The following theorem states sufficient conditions that the problems P and $P1$ have the same optimal solutions in D.

THEOREM 1. *If the function f is quasiconvex and differentiable on the convex set X and D is a closed bounded non-void subset of X, then $s = s_1$. Also, $x' \in D$ is an optimal solution of problem P if and only if it is an optimal solution of problem $P1$.*

Proof. Obviously, since $D \subseteq \text{co}(D)$, we have:

$$s \leqslant s_1. \tag{2.1}$$

On the other hand, since f is continuous and the set co(D) is compact, there exists $x' \in$ co (D) such that $f(x') = s_1$. If $x' \in D$ then the theorem is evidently true. Suppose that $x' \notin D$. Then there exist the points $x^i \in D$, $i \in \{1, 2, \ldots, k\} = K$, and the real numbers $t_i \geqslant 0$, $i \in K$, such that:

$$\sum_{i=1}^{k} t_i = 1 \text{ and } x' = \sum_{i=1}^{k} t_i x^i.$$

Since f is quasiconvex, it follows that:

$$s_1 = f(x') \leqslant \max\{f(x^i) : i \in K\} = f(x^{i'}), \tag{2.2}$$

for some i' in K. But $x^{i'} \in D$, so we have:

$$f(x^{i'}) \leqslant s. \tag{2.3}$$

But, by (2.2) and (2.3), we get: $s_1 \leqslant s$.

This last inequality and (2.1) yield the equality $s = s_1$, between the optimal values of the problems P and $P1$. The last part of the theorem follows from this equality and the inclusion $D \subseteq \text{co}(D)$.

Further we need the following result due to KORTANEK and EVANS [8].

THEOREM 2. *Let f be a continuously differentiable function defined on the open convex set $X \subseteq \mathbf{R}^n$. Consider the two following problems: (I). $\max\{f(x) : x \in C\}$ and (II'). $\max\{\nabla f(x') \cdot x : x \in C\}$, where C is a closed set contained in X and $x' \in C$. Then x' is an optimal solution for (I) if and only if x' is an optimal solution for (I') provided either one of the following conditions holds:*

(a) f is pseudoconcave on X;

(b) f is quasiconcave on X and $\nabla f(x') \neq 0$.

Theorem 2 gives a caracterization of the optimal solutions for the problem. P, when the feasible set D is convex. Now using the theorems 1 and 2, we will derive, with the supplimentary hypotheses that f is quasiconvex, a version of Theorem 2, supposing only that the feasible set is a closed bounded (possible non-convex) set.

THEOREM 3. *Let f be a continuously differentiable quasiconvex function on the convex set X and let D be a closed bounded non-void subset of X. Let suppose in addition for the function f that either one of the conditions (a) or (b) of Theorems 2 holds. Then $x' \in D$ is an optimal solution of the problem P if and only if x' is an optimal solution for the following linearized problem:*

$$P(x').\ \max\{\nabla f(x') \cdot x : x \in D\}.$$

Proof. By Theorem 1, $x' \in D$ is an optimal solution for P iff x' is an optimal solution for $P1$. By Theorem 2, x' is an optimal solution for $P1$ iff x' is an optimal solution for the problem:

$$P1(x'). \ \max \{\nabla f(x') \cdot x : x \in \mathrm{co}(D)\}.$$

And again, by virtue of Theorem 1, $x' \in D$ is an optimal solution for the problem $P1(x')$ if and only if x' is an optimal solution for the problem $P(x')$. By this sequence of equivalences the proof is complete.

The Theorem 4 below follows directly from quasiconvexity definition (see, for instance, [6], P.27 (ix), pp. 29–30).

THEOREM 4. *Let f be a differentiable quasiconvex function on the convex set X and let x', x'' be in X. If we have:*

$$(2.4) \qquad \nabla f(x') \cdot x' < \nabla f(x') \cdot x'',$$

then $f(x') < f(x'')$.

We note that some versions of this theorem was used in [10], [9], [13] to derive a simplex criterion to change a basis for quasimonotonic programming with linear constraints.

3. Algorithms

The theorems 3 and 4 suggest that maximizing a quasimonotonic function on a closed bounded set D is equivalent to maximizing certain linear functions on D. The algorithms below envisage to find a sequence of points in D converging (finitely or infinitely) to a point x in D for which Theorem 3 holds. This is done by solving a certain number of linearized problems.

A l g o r i t m 1

Step 1. Choose $x_0 \in D$ and take $i = 0$.

Step 2. Solve the linearized problem:

$$P(x^i). \ \text{Find}$$

$$(3.1) \qquad s_i = \max \{\nabla f(x^i) \cdot x : x \in D\}.$$

Let x^{i+1} be an optimal solution of the problem $P(x^i)$.

Step 3. (i) If $\nabla f(x^i) x^i < s_i$, then go to Step 2 with i replaced by $i + 1$.

(ii) If $\nabla f(x^i) \cdot x^i = s_i$, stop. By Theorem 3, x^i is an optimal solution for the problem P.

We will give an approximative version of the algorithm 1. For this, we consider the sequence of real numbers (t_i) such that:

$$(3.2) \qquad t_i \geqslant 0, \ \lim_{i \to \infty} t_i = 0.$$

A l g o r i t h m 2

Step 1. Choose $x_0 \in D$ and take $i = 0$.

Step 2. (i) If there exists $x \in D$ such that:

$$(3.3) \qquad \nabla f(x^i) \cdot x > \nabla f(x^i) \cdot x^i,$$

then go to Step 3.

(ii) If

$$(3.4) \qquad \nabla f(x^i) \cdot x \leqslant \nabla f(x^i) \cdot x^i, \ \forall \ x \in D,$$

stop.

Step 3. Find $x^{i+1} \in D$ such that:

$$(3.5) \qquad \nabla f(x^i) \cdot x^{i+1} > \max \{s_i - t_i, \ \nabla f(x^i) \cdot x^i\},$$

where s_i is the optimal value of the problem $P(x^i)$ (see, (3.1)). Replace i by $i + 1$ and go to Step 2.

R e m a r k 3.1. We note that the algorithm 1 can be derived from the algorithm 2 by taking $t_i = 0$, for every natural i. Also, if D is a finite set, in the algorithm 2 we can replace the condition (3.5), by

$$(3.6) \qquad \nabla f(x^i) \cdot x^{i+1} > \nabla f(x^i) \cdot x^i.$$

4. Convergence results

We will state a general convergence result (Theorem 6 below) for the algorithm 2 and, by Remark 3.1, also for the algorithm 1. After that, we will give sufficient conditions for the finite convergence of these algorithms.

THEOREM 5. *Let f be a quasimonotonic differentiable function. Then, whenever condition* (3.5) *from algorithm* 2 *holds, we have*:

$$f(x^{i+1}) > f(x^i).$$

Proof. From the condition (3.5), one gets:

$$\nabla f(x^i) \cdot x^{i+1} > \nabla f(x^i) \cdot x^i,$$

whence, by Theorem 4, it follows $f(x^{i+1}) > f(x^i)$.

THEOREM 6. *Let f be a quasimonotonic continuously differentiable function veriffying at least one of the conditions (a) or (b) from Theorem 2, and let D be a closed bounded set. Then one of the following situations holds*:

(i) *If the condition* (3.4) *is fulfiled for some i, then the algorithm 2 is inished after a finite number of iterations and x^i is an optimal solution or the problem P.*

(ii) *If the condition* (3.4) *is not realized for any i, then every limit point x' of the sequence (x^i) is an optimal solution of the problem P and*

$$(4.1) \qquad f(x') = \lim_{i \to \infty} f(x^i) = \max \{f(x) : x \in D\}.$$

Proof. (i) The condition (3.4) implies that x^i is an optimal solution for the problem $P(x^i)$, whence by Theorem 3, one gets that x^i is an optimal solution for the problem P.

(ii) Let $x' = \lim_{k\to\infty} x^{i_k}$, where (x^{i_k}) is a convergent subsequence of the sequence (x^i). The set D being closed, it follows that $x' \in D$. Also, by (3.5), for every natural k, we have:

$$\nabla f(x^{i_k}) \cdot x^{i_k+1} > s_{i_k} - t_{i_k} = \max\{\nabla f(x^{i_k}) \cdot x : x \in D\} - t_{i_k},$$

that is:

$$\nabla f(x^{i_k}) \cdot x^{i_k+1} > \nabla f(x^{i_k}) \cdot x - t_{i_k},\ \forall\, x \in D. \tag{4.2}$$

By continuity of the gradient of f and by (3.2), taking $k \to \infty$ in (4.2), we get:

$$\nabla f(x') \cdot x' \geqslant \nabla f(x') \cdot x,\ \forall\, x \in D.$$

Therefore x' is an optimal solution for the problem $P(x')$, whence according to the theorem 3, it results that x' is an optimal solution for the problem P.

To prove (4.1), we remark that, by Theorem 5, the sequence $(f(x^i))$ is strictly increasing. Also, by the first part of the theorem, this sequence is upper bounded by $f(x')$. Therefore it is a convergent sequence. But, since it possesses a subsequence $(f(x^{i_k}))$ which converges to $f(x')$, it follows that (4.1) holds.

THEOREM 7. *Suppose that the assumptions on the function f in Theorem 6 hold. Assume also there exists a finite set $D' \subseteq D$, such that:*

$$\max\{f(x) : x \in D\} = \max\{f(x) : x \in D'\}.$$

If $x^i \in D'$, whenever condition (3.5) holds, then Algoritm 2 is finished after a finite number of iterations.

Proof. Since, by Theorem 5, the sequence $(f(x^i))$ is strictly increasing, it follows that in the sequence (x^i) do not exist two identical elements. Hence, the set D' being finite, one arrives after a finite number of iterations that condition (3.4) is fulfiled. Thus by Theorem 6, the algorithm is finished after a finite number of iterations.

Remark 4.1. The assumption of the theorem 7 is evidently verified when the set D is finite. It happens so, for instance, when the feasible set D is defined by a system of linear constraints with integer variables [1], [4], or in some optimization problems in graphs (see section 5 below) when feasible set D is a finite set of subgraphs. When D is defined by a system of linear constraints (with continuously variables [2], [3], [9], [13]), it has, in general, an infinite number of elements, but there exists a finite subset D' containing all extremal points of D, which verifies the assumption of Theorem 7.

Remark 4.2. We note that in the hypotheses of Theorem 7, a version of Algorithm 2 obtained by replacing the condition (3.5) by (3.6) in Step 3, converges finitely too. It happens so, in some simplex algorithms for the quasimonotonic programming [9], [13].

5. Applications to optimization problems in graphs

Let $G = (V, U)$ be a connected graph with $|V| = n$ vertices and $|U| = m$ arcs, $U = \{u_1, u_2, \ldots, u_m\} \subseteq V \times V$.

We denote by E a certain set of subgraphs of the graph G. Such sets of subgraphs can be taken, for instance, the set of spanning trees of the graph G, or the set of elementary paths between two fixed vertices.

Given a subgraph A in E we denote by $U(A)$ the set of its arcs and we associate to A the vector $X(A) \in \mathbb{R}^m$, having the components:

$$x_i(A) = \begin{cases} 0, & \text{if } u_i \in U - U(A), \\ 1, & \text{if } u_i \in U(A), \end{cases} \quad i = 1, 2, \ldots, m.$$

Also, we define the set:

$$C(E) = \{X(A) : A \in E\}.$$

DEFINITION 5.1. *A function $f: E \to \mathbb{R}$ is said to be pseudoconcave (respectively quasiconvex, quasiconcave, pseudoconvex, quasimonotonic or linear) on E if there exists a pseudoconcave (respectively quasiconvex, quasiconcave, pseudoconvex, quasimonotonic or linear) function $\tilde{f}$: co$(C(E)) \to \mathbb{R}$, such that:*

$$f(A) = \tilde{f}(X(A)),\ \forall\, A \in E.$$

We call the function $\tilde{f}$ an extension of the function f.

We note that the fractional objective functions considered in some fractional optimization problems (on the paths set [7], [11], on the spanning trees set [4], [7] and on the cycles set [5]) are both pseudoconcave and quasiconvex functions in the sense of the definition 5.1.

Suppose now that for each arc $u_i \in U$ are given two nonnegative weight a_i and b_i. Then the function $f: E \to \mathbb{R}$, defined by

$$f(A) = \sum_{u_i \in U(A)} a_i + \sqrt{\Big(\sum_{u_i \in U(A)} a_i\Big)^2 + \sum_{u_i \in U(A)} b_i + c},\ \forall\, A \in E,\ c > 0,$$

is quasimonotonic because the extention $\tilde{f}$: co$(C(E)) \to \mathbb{R}$ of f where:

$$\tilde{f}(x_1, x_2, \ldots, x_m) = \sum_{i=1}^m a_i x_i + \sqrt{\Big(\sum_{i=1}^m a_i x_i\Big)^2 + \sum_{i=1}^m b_i x_i + c},$$

is quasimonotonic on the set co$(C(E))$.

Let E be a set of subgraphs of G and let f be a function which is both pseudoconcave and quasiconvex on E. We consider the following quasimonotonic optimization problem on the graph G:

PG. Find $A' \in E$, such that:

$$f(A') = \max\{f(A) : A \in E\}.$$

If $\tilde{f}$ is an extension of f, then the problem PG can be restated in the following manner:

PG'. Find $A' \in E$, such that:

$$\tilde{f}(X(A')) = \max\{\tilde{f}(X) : X \in C(E)\}.$$

The problem PG can be solved by applying the linearization algorithm 1 to the problem PG'. Thus the solving of the nonlinear optimization problem PG can be reduced to the solving of a finite number of „linear" optimization problems on the set of subgraphs E. For these linear problems, in some particular cases, there exist efficient algorithms (see, for instance, [7], when E is the spanning trees set, the paths set, the cycles set, etc.).

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