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A PROPERTY OF THE HIRSCHMANN–WIDDER AND LEVIATAN OPERATORS

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In this paper, we show that the Hirschman-Widder and Leviatan Operators preserve the non-concavity with respect to the Tchebycheff System $(1, x^{d_1})$, $d_1 \geqq 1$ on the closed interval $[0, 1]$.

This property completes the results obtained by D. Leviatan in [2], [3] and by Luciana Lupaş in her Dissertation [4].

Let us consider a function $w \in C[0, 1]$ such that $w(x) > 0$ for all $x \in (0, 1)$ and let be $u(x) = \int_0^x w(s)\, ds$.

The functions 1, u form a Tchebycheff System on $[0, 1]$.

Assume that g is twice continuously differentiable on the open interval $(0, 1)$.

If, moreover, w has a continuous derivative on the open interval $(0, 1)$, we may define the following operators:

$$g(x) \to (D_1 g)(x) = \frac{d}{dx} g(x) \qquad x \in (0, 1)$$

$$g(x) \to (D_2 g)(x) = \frac{d}{dx} \frac{g(x)}{w(x)} \qquad x \in (0, 1)$$

$$g(x) \to (E_2 g)(x) = (D_2 D_1 g)(x) \quad x \in (0, 1)$$

LEMMA 1. *If* $(E_2 g)(x) \geqq 0$ *for any* $x \in (0, 1)$, *then* g *is non-concave with respect to the Tchebycheff System* $(1, u)$ *on* $(0, 1)$. This lemma is a particular case of Theorem 2.1, chapter XI, § 2 in the S. Karlin and W. I. Studden's book [1].

Let us consider a sequence $(d_m)_{m=1}^{\infty}$ of real numbers with the following properties:

$$0 = d_0 < d_1 < \ldots < d_m < \ldots \tag{1}$$

$$\lim_{m\to\infty} d_m = +\infty \tag{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{d_k} = +\infty \tag{3}$$

Introduce the functions $x \to p_{mk}(x)$ by:

$$p_{mk}(k) = \begin{cases} (-1)^{m-k} d_{k+1} \ldots d_m [d_k, \ldots, d_m;\ x^t] \text{ if } m = 1, 2, \ldots \text{ and } k = 0, 1, \ldots, m-1 \\ x^{d_m} \quad \text{if} \quad k = m \quad m = 1, 2, \ldots \end{cases} \tag{4}$$

where $[a_1, a_2, \ldots, a_j;\ x^t]$ is the divided difference of the function $t \to g(t) = x^t$ on the points $a_1, a_2, \ldots, a_j$. Define the nodes t_{mk} by:

$$t_{mk} = \begin{cases} \left[\left(1 - \frac{d_1}{d_{k+1}}\right)\left(1 - \frac{d_1}{d_{k+2}}\right) \ldots \left(1 - \frac{d_1}{d_m}\right)\right]^{\frac{1}{d_1}} \text{ if } m = 1, 2, \ldots \text{ and } k = 0, 1, \ldots, m-1 \\ 1 \quad \text{if} \quad k = m \quad m = 0, 1, 2, \ldots \end{cases} \tag{5}$$

The Hirschman-Widder Operators $H_m : C[0, 1] \to C[0, 1]$ are defined as:

$$H_m(f)(x) = \sum_{k=0}^{m} f(t_{mk}) p_{mk}(x) \tag{6}$$

It is well-known that $p_{mk}(x) \geq 0$ for $0 \leq k \leq m$, $m = 1, 2, \ldots$, and $0 \leq x \leq 1$. So $H_m : C[0, 1] \to C[0, 1]$ is a positive linear operator.

Denote by

$$q_{mk}(x) = \frac{d_k}{d_m} p_{mk}(x); \qquad q_{mk}(x) \geq 0 \quad \text{on} \quad [0, 1]$$

Let

$$z_{mk} = t_{m-1}$$

In the paper [2], D. Leviatan has introduced the following positive linear operators:

$$x \in (0, 1] \qquad M_k(f)(x) = \sum_{m=k}^{\infty} f(z_{mk}) q_{mk}(x) \qquad k = 1, 2, \ldots$$

Since $\lim_{x\to 0_+} M_k(f)(x) = f(0)$, we may define

$$M_k(f)(0) = f(0)$$

By this definition $M_k : C[0, 1] \to C[0, 1]$

LEMMA 2. *If* p_{mk} $k = 0, 1, \ldots, m$ *are the generalized polynomials defined by the formula* (4) *then for every* $x \in (0, 1]$ *we have:*

(i) [3] $p'_{mk}(x) = \frac{1}{x}[d_k p_{mk}(x) - d_{k+1} p_{m,k+1}(x)]$ $k = 0, 1, \ldots, m-1$;

$$m = 1, 2, \ldots$$

$$p'_{mm}(x) = \frac{1}{x} p_{mm}(x)$$

(ii)
$$p''_{mk}(x) = \frac{d_k(d_k - 1)p_{mk}(x) + d_{k+1}(1 - d_k - d_m)p_{m,k+1}(x)}{x^2} + \frac{d_{k+1} d_{k+2} p_{m,k+2}(x)}{x^2}$$

$$k = 0, 1, \ldots, m-2 \qquad m = 2, 3, \ldots$$

$$p''_{m,m-1}(x) = \frac{d_{m-1}(d_{m-1} - 1)p_{m,m-1}(x) + d_m(1 - d_{m-1} - d_{k+1})p_{m,m}(x)}{x^2}$$

$$p''_{mm}(x) = \frac{1}{x^2} d_m(d_m - 1) p_{mm}(x)$$

LEMMA 3. *If* $f \in C[0, 1]$ *is non-concave with respect to the Tchebycheff System* $(1, x^{d_1})$ $d_1 \geq 1$ *on* $[0, 1]$ *then for every system of points*

$$0 \leq t_{mk} < t_{m,k+1} < t_{m,k+2} \leq 1$$

defined by the relations (5), *the following inequality holds:*

$$\left(1 - \frac{d_1}{d_{k+2}}\right) f(t_{m,k+2}) - \left(1 + \frac{d_{k+1}}{d_{k+2}} - \frac{d_1}{d_{k+2}}\right) f(t_{m,k+1}) + \frac{d_{k+1}}{d_{k+2}} f(t_{mk}) \geq 0 \tag{7}$$

PROPOSITION 1. *If* $f \in C[0, 1]$ *is non-concave with respect to the Tchebycheff System* $(1, x^{d_1})$, $d_1 \geq 1$, *on* $[0, 1]$, *then for* $m = 1, 2, \ldots$ $H_m(f)$ *is also non-concave on* $[0, 1]$ *with respect to this system.*

Proof. Since $H_m(f) \in C[0, 1]$, we need only show that $H_m(f)$ is non-concave on $(0, 1)$ with respect to $(1, x^{d_1})$.

Let $w(x) = d_1 x^{d_1 - 1}$, then $w \in C[0, 1] \cap C^1(0, 1)$, $w(x) > 0$ for any $x \in (0, 1)$. Let $x^{d_1} = \int_0^x w(s)\, ds$.

Observe that $H(f) \in C^2(0, 1)$.

According to Lemma 1, we must show that $(E_2 H_m(f))(x) \geq 0$ for any $x \in (0, 1)$.

$$(E_2 H_m(f))(x) = \frac{H''_m(f)(x) w(x) - H'_m(f)(x) w'(x)}{w^2(x)} \tag{8}$$

Therefore

$$\operatorname{sgn}(E_2H_m(f))(x) = \operatorname{sgn}[H''_m(f)(x)w(x) - H'_m(f)(x)w'(x)].$$

For $x \in (0, 1)$

$$\operatorname{sgn}[H''_m(f)(x)d_1x^{d_1-1} - H'_m(f)(x)d_1(d_1-1)x^{d_1-2}] =$$
$$= \operatorname{sgn}[H''_m(f)x^2 - H'_m(f)(x)x(d_1-1)]$$

$$H''_m(f)(x)x^2 - H'_m(f)(x)x(d_1-1) = \sum_{k=0}^{m-2}[d_k(d_k-1)p_{mk}(x) +$$
$$+ d_{k+1}(1-d_k-d_{k+1})p_{m,k+1}(x) + d_{k+1}d_{k+2}p_{m,k+2}(x)]f(t_{mk}) +$$
$$+ [d_{m-1}(d_{m-1}-1)p_{m,m-1}(x) + d_m(1-d_{m-1}-d_m)p_{mm}(x)]f(t_{m,m-1}) +$$
$$+ d_m(d_m-1)p_{mm}(x)f(t_{mm}) - \sum_{k=0}^{m-1}(d_1-1)d_kp_{mk}(x)-(d_1-1)d_{k+1}p_{m,k+1}(x)]f(t_{mk}) -$$
$$- (d_1-1)d_mp_{mm}(x)f(t_{mm}) = \sum_{k=0}^{m} c_{mk}p_{mk}(x).$$

Compute the coefficient $C_{m,k+2}$ of $p_{m,k+2}(x)$, $k = 0, 1, \ldots, m-2$

$$c_{m,k+2} = d_{k+2}(d_{k+2}-1)f(t_{m,k+2}) + d_{k+2}(1-d_{k+1}-d_{k+2})f(t_{m,k+1}) +$$
$$+ d_{k+1}d_{k+2}f(t_{mk}) - (d_1-1)d_{k+2}f(t_{m,k+2}) + (d_1-1)d_{k+2}f(t_{m,k+1}) =$$
$$= d_{k+2}^2\left[\left(1-\frac{d_1}{d_{k+2}}\right)f(t_{m,k+2}) - \left(1+\frac{d_{k+1}}{d_{k+2}}-\frac{d_1}{d_{k+2}}\right)f(t_{m,k+1}) + \right.$$
$$\left. + \frac{d_{k+1}}{d_{k+2}}f(t_{mk})\right].$$

Therefore

(9) $$\frac{C_{m,k+2}}{d_{k+2}^2} = \left(1-\frac{d_1}{d_{k+2}}\right)f(t_{m,k+2}) - \left(1+\frac{d_{k+1}}{d_{k+2}}-\frac{d_1}{d_{k+2}}\right)f(t_{m,k+1}) + \frac{d_{k+1}}{d_{k+2}}f(t_{mk}).$$

Comparing the relations (7) with (9) we can conclude that:

$$C_{m,k+2} \geq 0 \quad k = 0, 1, \ldots, m-2$$

On the other hand, because $d_0 = 0$, we have $C_{m0} = 0$.

$$C_{m1} = d_1(d_1-1)f(t_{m1}) + d_1(1-d_1)f(t_{m0}) - (d_1-1)d_1(t_{m1}) +$$
$$+ (d_1-1)d_1f(t_{m0}) = 0.$$

Therefore

(10) $$C_{mk} \geq 0 \quad k = 0, 1, \ldots, m \text{ and } \forall x \in (0, 1), \ (E_2H_m(f))(x) \geq 0$$

Proposition 1 is proved.

In the papers [2] and [3] it has been shown that the following lemma holds.

LEMMA 4. *(i) For* $0 < x \leq 1 \quad M_k(1)(x) \equiv 1 \quad k \geq 0$

(ii) If i is continuous on $[0, 1]$, *then* $\lim_{k\to\infty} M_k(f)(x) = f(x)$ *uniformly on* $[0, 1]$.

(iii) For $0 < x \leq 1$ *and* $0 \leq k \leq m = 1, 2, \ldots,$ *the following relations hold:*

$$\frac{d}{dx}q_{mk}(x) = x^{-1}[d_mq_{mk}(x) - d_{m-1}q_{m-1,k}(x)]$$

$$\frac{d}{dx}q_{mk}(x) = x^{-1}d_m[q_{mk}(x) - q_{m,k+1}(x)]$$

$$q_{mk}(x) = 0 \quad for \quad m < k.$$

(iv) If f is bounded on $[0, 1]$, *then for* $0 < x \leq 1$ *and for any* $k \geq 0$ *the following equality is satisfied:*

$$\frac{d}{dx}M_k(f)(x) = \sum_{m=k}^{\infty}\frac{d}{dx}q_{mk}(x)f(z_{mk}).$$

In the same way, the following lemma may be established.

LEMMA 5. *For* $0 < x \leq 1$ *and* $0 \leq k \leq m = 1, 2, \ldots$ *the following relations are true:*

(i) $$q''_{mk}(t) = x^{-2}[d_k(d_k-1)q_{mk}(x) + d_k(1-d_k-d_{k+1})q_{m,k+1}(x) +$$
$$+ d_kd_{k+1}q_{m,k+2}(x)]$$

(ii) $$q''_{kk}(t) = \frac{d_k(d_k-1)q_{kk}(x)}{x^2}$$

$$q''_{k,k-1}(x) = x^{-2}[d_k(d_k-1)q_{k,k-1}(x_{k-1}) + d(1-d_{k-1}-d_k)q_{k-1,k-1}(x)]$$
$$k = 1, 2, \ldots$$

$$q''_{mk}(x) = x^{-2}[d_m(d_m-1)q_{mk}(x) + d_{m-1}(1-d_{m-1}-d_m)q_{m-1,k}(x) +$$
$$+ d_{m-1}d_{m-2}q_{m-2,k}(x)] \quad (q_{mk}(x) = 0 \quad k > m).$$

(iii) If f is bounded on $[0, 1]$, *then for* $0 < x \leq 1$ *and for any* $k \geq 0$ *the following equality holds:*

$$\frac{d^2}{dx^2}M_k(f)(x) = \sum_{m=k}^{\infty}\frac{d^2}{dx^2}q_{mk}(x)f(z_{mk})$$

LEMMA 6. *If* $f \in C[0,1]$ *is a non-concave function on* $[0,1]$ *with respect to* $(1, x^{d_1})$, $d_1 \geq 1$, *then for every system of three distinct points* $0 \leq \leq z_{m+2,k} < z_{m+1,k} < z_{mk} \leq 1$ *we have*:

$$f(z_{mk})\left(1-\frac{d_1}{d_m}\right)-f(z_{m+1,k})\left(1+\frac{d_{m+1}}{d_m}-\frac{d_1}{d_m}\right)+f(z_{m+2,k})\frac{d_{m+1}}{d_m} \geq 0$$

PROPOSITION 2. *If* $f \in C[0,1]$ *is non-concave on* $[0,1]$ *with respect to the Tchebycheff System* $(1, x^{d_1})$, $d_1 \geq 1$, then for $k = 1, 2, \ldots$ $M_k(f)$ *is also non-concave on* $[0,1]$ *with respect to this system.*

Proof. Since $M_k f \in C[0,1]$, we need only show that $M_k(f)$ is non-concave on $(0,1)$ with respect to $(1, x^{d_1})$, $d_1 \geq 1$.

According to Lemma 1, we must establish that $(E_2 M(f))(x) \geq 0$ for any $x \in (0,1)$.

$$(E_2 M_k(f))(x) = \frac{M_k''(f)(x)w(x) - M_k'(f)(x)w'(x)}{w^2(x)}$$

Therefore

$$\operatorname{sgn}(E_2M_k(f))(x) = \operatorname{sgn}(M_k''(f)(x)w(x) - M_k'(f)(x)w'(x)) =$$
$$= \operatorname{sgn}(M_k''(f)(x)x^2 - (d_1-1)M_k'(f)(x)x)$$

$$x^2M_k''(f)(x)\ \ (d_1-1)xM_k'(f)(x) = d_k(d_k-1)q_{kk}(x)f(z_{kk}) +$$
$$+ d_{k+1}(d_{k+1}-1)q_{k+1,k}(x)f(z_{k+1,k}) + d_k(1-d_k-d_{k+1})f(z_{k+1,k}) -$$
$$-(d_1-1)d_kq_{kk}(x)f(z_{kk}) + \sum_{m=k}^{\infty} x^2q''_{m+2,k}(x)f(z_{m+2,k}) -$$
$$-(d_1-1)\sum_{m=k}^{\infty} xq'_{m+1,k}(x)f(z_{m+1,k}).$$

The expression $\sum_{m=k}^{\infty} x^2q''_{m+2,k}(x)f(z_{m+2,k}) - (d_1-1)\sum_{m=k}^{\infty} xq'_{m+1,k}(x)f(z_{m+1,k}) =$

$$=\sum_{m=k}^{\infty}\{[d_{m+2}(d_{m+2}-1)q_{m+2,k}(x) + d_{m+1}(1-d_{m+1}-d_{m+2})q_{m+1,k}(x) +$$
$$+ d_md_{m+1}q_{mk}(x)]f(z_{m+2,k}) - (d_1-1)(d_{m+1}q_{m+1,k})(x) - d_mq_{mk}(x))f(z_{m+1,k})\}$$

Therefore

$$x^2M_k''(f)(x) - (d_1-1)xM_k'(f)(x) =$$
$$= [d_k(d_k-1)f(z_{kk}) - (d_1-1)d_kf(z_{kk}) + d_k(1-d_k-d_{k+1})f(z_{k+1,k})]$$
$$q_{kk}(x) + d_{k+1}(d_{k+1}-1)f(z_{k+1,k})q_{k+1,k}(x) +$$

$$+\sum_{m=k}^{\infty}\{[d_{m+2}(d_{m+2}-1)q_{m+2,k}(x) + d_{m+1}(1-d_{m+1}-d_{m+2})q_{m+1,k}(x) +$$
$$+d_md_{m+1}q_{mk}(x)]f(z_{m+2,k}) - (d_1-1)(d_{m+1}q_{m+1,k}(x) - d_mq_{mk}(x))f(z_{m+1,k})\} =$$
$$= \sum_{m=k}^{\infty} C_{mk}q_{mk}(x)$$

Compute the coefficients C_{mk} $m = k, k+1, \ldots$

$$C_{kk} = d_k^2\left[\left(1-\frac{d_1}{d_k}\right)f(z_{kk}) - \left(1-\frac{d_1}{d_k}+\frac{d_{k+1}}{d_k}\right)f(z_{k+1,k}) + \frac{d_{k+1}}{d_k}f(z_{k+2,k})\right]$$

$$C_{k+1,k} = d_{k+1}^2\left[\left(1-\frac{d_1}{d_{k+1}}\right)f(z_{k+1,k}) - \left(1-\frac{d_1}{d_{k+1}}+\frac{d_{k+2}}{d_{k+1}}\right)f(z_{k+2,k}) + \right.$$
$$\left. + \frac{d_{k+2}}{d_{k+1}}f(z_{k+3,k})\right]$$

$$C_{m+2,k} = d_{m+2}^2\left[\left(1-\frac{d_1}{d_{m+2}}\right)f(z_{m+2,k}) - \left(1-\frac{d_1}{d_{m+2}}+\frac{d_{m+3}}{d_{m+2}}\right)f(z_{m+3,k}) + \right.$$
$$\left. + \frac{d_{m+3}}{d_{m+2}}f(z_{m+4,k})\right] \qquad m = k, k+1, \ldots$$

From Lemma 6, it follows that $C_{mk} \geq 0$ $m = k, k+1, \ldots$ Since $q_{mk}(x) \geq \geq 0$ for any $x \in [0,1]$, we have for every $x \in (0,1)$, $(E_2M_k(f))(x) \geq 0$. Therefore $M_k(f)$ is non-concave on $[0,1]$ with respect to the Tchebycheff System $(1, x^{d_1})$, $d_1 \geq 1$.

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