MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 12, Nº 1, 1983, pp. 19—23

EQUIVALENCE RELATION IN PROBABILISTIC SYNTOPOGENOUS SPACES

by

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In [4], K. MENGER introduced three types of distinguishability for pairs of points of a probabilistic metric space, depending upon the behaviour of the distance distribution function near zero. If (X, F) is a PM space, $x, y \in X$ and $t_{xy} = \inf \{\alpha : F_{xy}(\alpha) > 0\}$, then we say that x and y are:
(A) certainly-distinguishable if $t_{xy} > 0$;
(B) barely-distinguishable if $t_{xy} = 0$ and $F_{xy}(0^+) = 0$;
(C) perhaps-indistinguishable if $F_{xy}(0^+) > 0$.

The above mentioned types of distinguishability were reconsidered by B. SCHWEIZER ([5]) who defined two relations on X as follows:
(1) $x\rho y$ iff x and y are perhaps-indistinguishable, i.e. iff (C) holds;
(2) $x\delta y$ iff x and y are not certainly-distinguishable, i.e. iff either (B) or (C) holds.

In the following we refer only to the relation ρ. In [5, Th.1] B. SCHWEIZERR. proof that, if (X, F, T) is a Menger space, where T is a t-norm such that:
(3) $T(a, b) > 0$ whenever $a > 0$ and $b > 0$,
then ρ is an equivalence relation on X and R. J. EGBERT shows that the quotient space $\dot{X} = X/\rho$ can be endowed with an adequate probabilistic metric; in some conditions $\dot{x}$ and $\dot{y}$ are perhaps-indistinguishable in $\dot{X}$ iff $\dot{x} = \dot{y}$ (see [2, Th. 31]).

In [3] we introduce a probabilistic variant of the Császár's syntopogenous structures (see [1]) — the probabilistic syntopogenous structure. Using these structures, we define, in a natural way, the probabilistic uniformities and the probabilistic proximities; the relations between these structures and the probabilistic metric structures are analogous with the relations between the uniform structures, the proximity structures and the metric structures on a set.

Let I be the closed unit interval and let $T : I \times I \to I$ be a t-function ($T(a_1, b_1) \leqslant T(a_2, b_2)$ if $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$); for every $a \in I$ let $\mathcal{S}_a$ be a non-empty family of topogenous orders on X, such that for every $<_1, <_2 \in \mathcal{S}_a$ there is $< \in \mathcal{S}_a$ with $<_1, <_2 \subseteq <$. A probabilistic syntopogenous structure (pss) on X is a pair $(\mathcal{S}, T)$, where $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$, satisfying the following conditions:

(PSO) $\mathcal{S}_0 = \{<_0\}$, where $A <_0 B$ iff $A = \Phi$ or $B = X$,

(PS1) for every $a, b \in I$ with $a < b$ and for every $< \in \mathcal{S}_a$ there exists $<' \in \mathcal{S}_b$ such that $< \subseteq <'$.

(PS2) for every $< \in \mathcal{S}_{T(a,b)}$ there exist $<' \in \mathcal{S}_a$ and $<'' \in \mathcal{S}_b$ such that $A < B$ implies that $A <' C <'' B$ for some $C \subseteq X$.

A probabilistic syntopogenous space is a triple $(X, \mathcal{S}, T)$, where $(\mathcal{S}, T)$ is a pass on X.

We say that a pss $(\mathcal{S}, T)$ on X, $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$, is symmetrical, perfect, or simple if each $\mathcal{S}_a$, $a \in I$ is symmetrical, perfect, or consist of a single topogenous order. If (X, F, T) is a Menger space, where T is a left-continuous t-norm, then for every $\alpha > 0$, $a \in I$, we define $<_{\alpha, a} \subseteq \mathfrak{P}(X) \times \mathfrak{P}(X)$, letting $A <_{\alpha, a} B$ if $A \times (X - B) \subseteq \{(x, y) : F_{xy}(\alpha) \leqslant a\}$. For every $a \in (0, 1]$, let $\mathcal{S}_a = \{<_{\alpha, b} : \alpha > 0,\ b < a\}$ and $\mathcal{S}_0 = \{<_0\}$; then the pair $(\mathcal{S}, T)$, where $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$, is a perfect and symmetrical pss on X — the pss induced by the probabilistic metric of the space (see [3, Th. 6.1.]).

The purpose of this paper is to give an extension of Schweizer's and Egbert's results as well as a probabilistic variant of a Császár's theorem concerning the simple, symmetrical and perfect syntopogenous structures.

We omit the proof of the following easily established

LEMMA. *Let (X, F, T) be a Menger space under a left-continuous t-norm T, let ρ be the relation* (1) *and let $(\mathcal{S}, T)$, $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$ be the pss induced by the probabilistic metric of the space. Then $x \rho y$ iff there exists $a > 0$ such that for every $< \in \mathcal{S}_a$ we have $x \not< X - y$.*

DEFINITION 1. *Let $(\mathcal{S}, T)$ be a pss on X, where $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$; we define a relation $\rho \subseteq X \times X$ letting $x \rho y$ iff there exists $a \in (0, 1]$ such that, for every $< \in \mathcal{S}_a$, we have $x \not< X - y$; taking into account the result of lemma, is natural to say that x and y are perhaps-indistinguishable if $x \rho y$.*

Now, we give a generalization of Schweizer's theorem (see [5, Th. 1])

THEOREM 1. *Let $(\mathcal{S}, T)$ be a symmetrical pss on X under a t-function T which satisfies the condition* (3). *Then, the relation ρ is an equivalence relation on X.*

P r o o f. For every topogenous order $<$ on X and for every $x \in X$ we have $x \not< X - x$; hence ρ is a reflexive relation. Because $(\mathcal{S}, T)$ is symmetrical it follows that ρ is a symmetrical relation. Finally, if $x \rho y$ and $y \rho z$ then there exist $a, b \in (0, 1]$ such that for every $<' \in \mathcal{S}_a$ and $<'' \in \mathcal{S}_b$ we have $x \not<' X - y$, $y \not<'' X - z$, where $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$. From

(3), $c = T(a, b) > 0$; suppose that there exists $< \in \mathcal{S}_c$ such that $x < X - z$; then, from (PS2), there exists $<' \in \mathcal{S}_a$, $<'' \in \mathcal{S}_b$ and $C \subseteq X$ such that $x <' C <'' X - z$. Because $y \not<'' X - z$, it follows that $y \in C$; hence $C \subseteq X - y$ and therefore $x <' X - y$. But this is a contradiction; thus ρ is a transitive relation.

Because every probabilistic proximity and every probabilistic uniformity are induced by an unique simple and symmetrical pss, resp. by a symmetrical and perfect pss (see [3, Th. 4.1, and 5.1.]), this result is fulfiled in every probabilistic proximity space as well as in every probabilistic uniformity space.

Let $(\mathcal{S}, T)$ be a symmetrical and perfect pss on X under a t-function T which satisfies (3), $\mathcal{S} = \{\mathcal{S} : a \in I\}$, let ρ be the equivalence relation associated to $(\mathcal{S}, T)$ (see Theorem 1) and let $\dot{X} = X/\rho$ be the quotient space; for every $a \in I$ and $< \in \mathcal{S}_a$ we define $\dot{<} \subseteq \mathfrak{P}(\dot{X}) \times \mathfrak{P}(\dot{X})$ letting $\dot{A} \dot{<} \dot{B}$ iff for every $\dot{x} \in \dot{A}$, $\dot{y} \in \dot{X} - \dot{B}$ there is $u \in \dot{x}$ such that $u < X - \dot{y}$ or there is $v \in \dot{y}$ such that $v < X - \dot{x}$. Let $\dot{\mathcal{S}}_a = \{\dot{<} : < \in \mathcal{S}_a\}$ and $\dot{\mathcal{S}}_a = \{\dot{\mathcal{S}}_a : a \in I\}$. The following theorem is an extension of the above mentioned Egbert's result.

THEOREM 2. *$(\dot{\mathcal{S}}, T)$ is a symmetrical and perfect pss on $\dot{X}$; in addition, if $(\dot{\mathcal{S}}, T)$ fulfils the condition:*

(4) *for every $\dot{x} \in \dot{X}$ there exists $a \in (0, 1]$ such that for every $u, v \in \dot{x}$ and $< \in \mathcal{S}_a$ we have $u \not< X - v$,*

then $\dot{x}$ and $\dot{y}$ are perhaps-indistinguishable in $(\dot{X}, \dot{\mathcal{S}}, T)$ if and only if $\dot{x} = \dot{y}$.

P r o o f. It follows directly from the definition of the family $\dot{\mathcal{S}}_a$ that it is a non-empty family of symmetrical and perfect topogenous orders on $\dot{X}$, directed by $\subseteq$. (PSO) We suppose that $\dot{A} \dot{<}_0 \dot{B}$ and $\dot{A} \neq \Phi$ and $\dot{B} \neq \dot{X}$; then, there exist $\dot{x} \in \dot{A}$, $\dot{y} \in \dot{X} - \dot{B}$ such that $u <_0 X - \dot{y}$ for some $u \in \dot{x}$, or $v <_0 X - \dot{x}$ for some $v \in \dot{y}$. It follows that $X - \dot{y} = X$ or $X - \dot{x} = X$ and this is a contradiction, because $y \in \dot{y}$ and $x \in \dot{x}$. (PS1) is obvious. We shall show that the condition (*PS*2) also is satisfed. Let $\dot{<} \in \dot{\mathcal{S}}_{T(a,b)}$; by the definition of $\dot{\mathcal{S}}_{T(a,b)}$, $< \in \mathcal{S}_{T(a,b)}$ and, hence, there exist $<' \in \mathcal{S}_a$, $<'' \in \mathcal{S}_b$ such that $A < B$ implies $A <' C <'' B$ for some $C \subseteq X$. Let $\dot{A} \dot{<} \dot{B}$; for every $\dot{x} \in \dot{A}$ and $\dot{y} \in \dot{X} - \dot{B}$ we have:

a). there exists $u \in \dot{x}$ such that $u < X - \dot{y}$, or

b). there exists $v \in \dot{y}$ such that $v < X - \dot{x}$.

a). $u < X - \dot{y}$ implies that there exists $C \subseteq X$ such that $u <' C <'' X - \dot{y}$. Let $\dot{C} = \{\dot{z} : z \in C\}$; for every $\dot{w} \in \dot{X} - \dot{C}$ we have $C \subseteq X - \dot{w}$ (for every $z \in C$ we have $\dot{z} \in \dot{C}$, hence $\dot{z} \neq \dot{w}$, therefore $z \notin \dot{w}$). Thus $u <' X - \dot{w}$ for every $\dot{w} \in \dot{X} - \dot{C}$, hence $\dot{x} \dot{<}' \dot{C}$. On the other hand, for every $\dot{x} \in \dot{C}$ there exists $x \in \dot{x}$ such that $x \in C <'' X - \dot{y}$, hence $x <'' X - \dot{y}$ herefore $\dot{C} \dot{<}'' \dot{X} - \dot{y}$.

b). Similary, we proof that there exists $\dot{C} \subseteq \dot{X}$ such that $\dot{x} \dot{<} '\dot{C} \dot{<} ''\dot{X}-\dot{y}$. Therefore, for every $\dot{x} \in \dot{A}$ and $\dot{y} \in \dot{X} - \dot{B}$ there exists $\dot{C}_{xy} \subseteq \dot{X}$ such that $\dot{x} \dot{<} '\dot{C}_{xy} \dot{<} ''\dot{X} - \dot{y}$. Let $\dot{C} = \bigcup_{\dot{x} \in \dot{A}} \bigcap_{\dot{y} \in \dot{X}-\dot{B}} \dot{C}_{xy}$; because $\dot{<}'$ and $\dot{<}''$ are biperfect, it follows that $\dot{A} \dot{<} '\dot{C} \dot{<} ''\dot{B}$. Hence $(\dot{\mathcal{S}}, T)$ is a symmetrical and perfect pss on $\dot{X}$.

Now, we suppose that $\dot{x}$ and $\dot{y}$ are perhaps-indistinguishable in the space $(\dot{X}, \dot{\mathcal{S}}, T)$ with the condition (4) and $\dot{x} \neq \dot{y}$. From the definition 1 and from the definition of $\dot{\mathcal{S}}$, there exists $a \in (0, 1]$ such that for every $\dot{<} \in \dot{\mathcal{S}}_a$ we have $\dot{x} \not\dot{<} \dot{X} - \dot{y}$ and $\dot{y} \not\dot{<} \dot{X} - \dot{x}$. Because $<$ is a symmetrical and perfect topogenous order on X, it follows that there exist $x_< \in \dot{x}$, $y \in \dot{y}$ such that $x \not< X - y_<$ and $y \not< X - x_<$. From (4) there exits $b \in (0, 1]$ such that for every v, $w \in \dot{y}$ and $< \in \mathcal{S}_b$ we have $v \not< X - w$. Let $c = T(a, b) > 0$ (from (3)); $\dot{x} \neq \dot{y}$ implies that there exists $< \in \mathcal{S}_c$ such that $x < X - y$. From (PS2) there exist $<' \in \mathcal{S}_a$, $<'' \in \mathcal{S}_b$ and $C \subseteq X$ such that $x <' C <'' X - y$. It follows that there exists $y_{<'} \in \dot{y}$ such that $x < 'X - y_{<'}$. Hence $C \not\subseteq X - y_{<'}$, so $y_{<'} \in C$. But this is a contradiction because $y_{<'}$, $y \in \dot{y}$ implies $y_{<'} \not< ''X - y$.

We say that $(\dot{\mathcal{S}}, T)$ is the quotient pss on $\dot{X}$ and we note $\dot{\mathcal{S}} = \mathcal{S}/\rho$

In [1, 14.1], A. Császár shows that if $\mathcal{T} = \{<\}$ is a simple, symmetrical and perfect syntopogenous structure on X and $x\rho y$ iff $x \not< X - y$ then ρ is an equivalence relation and the quotient order $</\rho$ on X/ρ is the inclusion $\subseteq$. The following theorem is a probabilistic variant of this result.

THEOREM 3. *Let* $(\ , T)$ *be a simple, symmetrical and perfect pss on* X *under a t-function* T *which satisfies the condition* (3) *and let* $(\dot{\mathcal{S}}, T)$ *be the quotient pss on* $\dot{X} = X/\rho$, *where* ρ *is the perhaps-indistinguishability relation on* X; *then* $\dot{\mathcal{S}} = \{\dot{\mathcal{S}}_a : a \in I\}$ *where* $\dot{\mathcal{S}}_a = \{\subseteq\}$ for every $a \in (0, 1]$.

Proof. Let $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$, where $\mathcal{S}_a = \{<_a\}$, and $<_a$ is a symmetrical and perfect topogenous order on X for every $a \in I$. Then $\dot{\mathcal{S}}_a = \{\dot{<}_a\}$ for every $a \in I$; the condition $\dot{<}_a \subseteq \subseteq$ is obvious. Now, if $\dot{A} \subseteq \dot{B}$ then, for every $u \in \dot{x} \in \dot{A}$ and $v \in \dot{y} \in \dot{X} - \dot{B}$, we have that u and v are not perhaps-indistinguishable $(\dot{x} \cap \dot{y}) = \Phi)$; hence, for every $a \in (0, 1]$ $u <_a X - v$. Because $<_a$ is symmetrical and perfect we have $\dot{x} \dot{<}_a \dot{X} - \dot{y}$, hence $\dot{A} \dot{<}_a \dot{B}$. Therefore $\subseteq \subseteq \dot{<}_a$, so that $\dot{<}_a = \subseteq$ for every $a \in (0, 1]$.

Remark 1. If $(\mathcal{S}, T)$ is a simple, symmetrical and perfect pss on X then $\dot{x}$ and $\dot{y}$ are perhaps-indistinguishable in $\dot{X}$ iff $\dot{x} = \dot{y}$. Indeed, $\dot{x}$ and $\dot{y}$ are perhaps-indistinguishable iff there exists $a \in (0, 1]$ such that $\dot{x} \dot{<}_a \dot{X} - \dot{y}$ where $\dot{\mathcal{S}}_a = \{\dot{<}_a\}$ and $\dot{\mathcal{S}} = \{\dot{\mathcal{S}}_a : a \in I\}$ is the quotient pss; because $\dot{<}_a = \subseteq$ we have $\dot{x} = \dot{y}$.

Remark 2. Let $(\mathcal{S}', T)$, $(\mathcal{S}'', T)$ be two simple, symmetrical and perfect pss on X, where $\mathcal{S}' = \{\mathcal{S}'_a : a \in I\}$, $\mathcal{S}'' = \{\mathcal{S}''_a : a \in I\}$ and $\mathcal{S}'_a = \{<'_a\}$, $\mathcal{S}''_a = \{<''_a\}$ for every $a \in I$, and let ρ_1 and ρ_2 be the perhaps-indistinguishability relations associated to $(\mathcal{S}', T)$ and $(\mathcal{S}'', T)$; then $\mathcal{S}' = \mathcal{S}''$ iff $\rho_1 = \rho_2$.

Now, if ρ is an equivalence relation on X, we define a relation $< \subseteq \mathfrak{P}(X) \times \mathfrak{P}(X)$ letting $A < B$ iff for every $x \in A$, $y \in X - B$ $(x, y) \notin \rho$. Then $\mathcal{S}_\rho = \{<\}$ is a simple, symmetrical and perfect syntopogenous structure on X; if T is a t-function which satisfies the condition:

(3′) $T(a, b) = 0$ iff $a = 0$ or $b = 0$,

then $(\mathcal{S}, T)$, $\mathcal{S} = \{\mathcal{S}_a : a \in I\}$ where $\mathcal{S}_a = \mathcal{S}_\rho$ for every $a \in (0, 1]$, is the unique simple, symmetrical and perfect pss on X for which ρ is the perhaps-indistinguishability relation.

REFERENCES

[1] Császár Á., *Fondements de la topologie générale*, Akadémiai Kiadó, Budapest, 1960.
[2] Egbert R. J., *Products and quotients of probabilistic metric spaces*, Pacific J. Math. **24** 437–455 (1968).
[3] Florescu, L., *Structures syntopogènes probabilistes*, Publicationes Mathematicae Debrecen **28** 15–24 (1981).
[4] Menger, K., *Probabilistic geometry*, Proc. Nat. Acad. Sci. U.S.A., **37** 226–229 (1951).
[5] Schweizer, B., *Equivalence relations in probabilistic metric spaces*, Bul. Inst. Politehnic Iaşi **10** (14) 67–70 (1964).

Received 8.I.1982.

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