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A GENERALIZATION OF SET-VALUED METRIC PROJECTIONS

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1. Introduction

Let X be a normed linear space, and M a linear subspace of X. The set-valued mapping $x \to P_M(x)$, where

$$P_M(x) = \{m_0 \in M \mid \|x - m_0\| = \text{dist}\ (x, M)\}$$

is called the metric projection of X onto M, and each $p_M(x) \in P_M(x)$ is called a best approximation of x out of M. For some $x \in X$ it is possible that $P_M(x) = \emptyset$, but when X is reflexive and M closed (it suffices M reflexive), then this will never happen. Among the general properties of P_M we notice the following two: for $x \in X$ with $P_M(x) \neq \emptyset$ we have:

(1.1) $\|x - p_M(x)\| \leqslant \|x\| \quad (p_M(x) \in P_M(x))$

(1.2) $\|x - p_M(x)\| = \|x\|$ for some (all) $p_M(x) \in P_M(x)$ iff $0 \in P_M(x)$

When X is reflexive and strictly convex, and M closed then P_M is a well-defined operator (in general non-linear) which assigns to each $x \in X$ its unique best approximation $P_M x$. So, one possible way to generalize the (single-valued) metric projection P_M is to consider a map $P: X \to X$ satisfying (1.1) and (1.2) where we replace $P_M(x)$ $(p_M(x))$ by Px. Such a map is called [12] a B-operator. If the range of P is included in M, for some closed linear subspace M of X, then the B-operator P is said to be on M ([1], [12]) if for $x \in X \setminus M$ we have $Px = 0$ if and only if $P_M x = 0$. Such operatos have been used to construct P_M (note that for this problem only the points $x \notin M$ are interesting, since otherwise $P_M x = x$). Clearly, when X is reflexive and strictly convex, and M closed, then each P_M is a B-operator on M.

B-operators (on M) were first introduced by B. ATLESTAM and F. SULLIVAN in [1] (in connection with methods of calculating best approximations on finite dimensional subspaces of L^p), but an extensive study of B-operators (B-operators on M) and their applications were done by F. SULLIVAN in [12].

In this paper we enlarge the class of B-operators (on some closed subspaces) when X is an arbitrary normed linear space, in such a way, that for each linear subspace $M \subset X$ (not necessarily closed), P_M belongs to this class. Then we must consider not only single-valued mappings defined on the whole X, but set-valued mappings, their domains being subsets of X, and the set-valued mappings satisfy conditions similar with (1.1) and (1.2). Such set-valued mappings will be called B-set-valued mappings (see Definition 2.1 below), and in an appropriate way we define B-set-valued mappings on M, M a linear subspace (see Definition 2.2 below). The results of this paper may be regarded as generalizations for B-set-valued mappings of the results of [12] for B-operators. The difficulties which appear here are more or less comparable with the ones which appear when the results on P_M when M is a Chebyshev subspace of X (i.e., when $P_M(x)$ is a singleton for each $x \in X$), are generalized for the case when M is an arbitrary subspace of X (see e.g., [11], [4]). We notice that even when P is a B-set-valued mapping on M, its domain being X and M a Chebyshev subspace of the reflexive and strictly convex space X, we can not expect to apply the results of [12] for B-operators to the selections $p(x) \in P(x)$, $x \in X$, which are clearly B-operators, since these B-operators are not in general on M (though they be on other subspaces). Of course, we shall use some ideas and techniques of [12], and we let to the reader to compare the results and the proofs given here and the corresponding ones of [12].

Finally, we mention that another generalization of the set-valued metric projection, quite different of the above one was considered in [5].

2. B-SET-Valued mappings and their associated B-SET-Valued mappings

Let X be a normed linear space over the real field R, and X^* its dual space. Throughout this paper the word „subspace" stands for „linear subspace". Let us denote by 2^X the collection of all subsets of X, including the empty set $\emptyset$. Let $P: X \to 2^X$ be a set-valued mapping. We denote by $\mathrm{Dom}(P) = \{x \in X \mid P(x) \neq \emptyset\}$, $P^{-1}(0) = \{x \in X \mid 0 \in P(x)\}$, and for $x \in \mathrm{Dom}\,(P)$ we generally denote the elements of $P(x)$ by $p(x)$.

2.1. DEFINITION. *A set-valued mapping $P: X \to 2^X$ is called a B-set-valued mapping if for each $x \in \mathrm{Dom}(P)$ there exists $c_x \in R$ with $0 \leq c_x \leq ||x||$ such that:*

1) $||x - p(x)|| = c_x$ *for all* $p(x) \in P(x)$.
2) $c_x = ||x||$ *if and only if* $0 \in P(x)$.

It is obvious that for a B-set-valued mapping P, we have $P(0) = \{0\}$ if $0 \in \mathrm{Dom}(P)$, and by $0 \leq c_x \leq ||x||$ and condition 1 we have:

$$||p(x)|| \leq 2||x|| \qquad (x \in \mathrm{Dom}(P),\ p(x) \in P(x)) \tag{2.1}$$

Let us denote by $P(X) = \bigcup\{P(x) \mid x \in \mathrm{Dom}(P)\}$.

2.2. DEFINITION. *If $P(X) \subset M$ for some subspace M of X then the B-set-valued mapping P is said to be on M if $P_M^{-1}(0) \setminus \{0\} = P^{-1}(0) \setminus M$.*

Equivalently, P is on M if $P(X) \subset M$ and for $x \notin M$ we have $0 \in P(x)$ if and only if $0 \in P_M(x)$.

Since $P_{\bar{M}}^{-1}(0) = P_M^{-1}(0)$ where $\bar{M}$ is the closure of M in the norm topology, if P is on M then P is also on $\bar{M}$, so the assumption on P to be on a closed subspace is not more restrictive than to be on an arbitrary subspace.

Clearly, if M is a subspace of X, then the set-valued mapping $P: X \to 2^X$ defined by the conditions $P(x) \subset P_M(x)$, $x \in X$, and if $x \in P_M^{-1}(0)$ then $0 \in P(x)$, is a B-set-valued mapping on M. The next four results give conditions on a B-set-valued mapping P on M for which $P(x) \subset P_M(x)$ for $x \in \mathrm{Dom}(P)$.

We recall (see e.g., [3]) that a normed linear space X is called *strictly convex* if for all $x, y \in X$, $x \neq y$, $||x|| = ||y|| = 1$ we have $||x + y|| < 2$. This is equivalent [10], with the fact that each subspace M of X is a semichebyshev subspace of X, i.e., for each $x \in X$, $P_M(x)$ is either empty or a singleton. Another equivalent condition which will be used in Section 4, is that each $f \in X^* \setminus \{0\}$ attains its norm at most at one point $x \in X$, $||x|| = 1$.

2.3. Remark. If P is a B-set-valued mapping on M, and $x \in \mathrm{Dom}(P)$ such that $P(x) \cap P_M(x) \neq \emptyset$, then $P(x) \subset P_M(x)$. Consequently, if X is strictly convex, then for each B-set-valued mapping P on M, and $x \notin M \setminus \{0\}$, we have $x \in P^{-1}(0)$ if and only if $P(x) = \{0\}$. Indeed, let $m \in P(x) \cap P_M(x)$ and $p(x) \in P(x)$. We have by condition 1 of Definition 2.1 that $||x - p(x)|| = ||x - m||$. Since $m \in P_M(x)$ and $p(x) \in M$ it follows $p(x) \in P_M(x)$. The other assertion follows since P is on M, which is now a semichebyshev subspace of X.

2.4. Remark. Let P be a B-set-valued mapping on M, $x \in \mathrm{Dom}(P) \setminus M$ and $m \in P(x)$. The following assertions are equivalent:

i) $m \in P_M(x)$
ii) $x - m \in \mathrm{Dom}(P)$ and $c_x = c_{x-m}$.
iii) $x - m \in \mathrm{Dom}(P)$ and $c_x \leq c_{x-m}$

Indeed, suppose we have i). Then $0 \in P_M(x - m)$, and P being on M it follows $0 \in P(x - m)$. So, $x - m \in \mathrm{Dom}(P)$ and since $0 \in P(x - m)$ and $m \in P(x)$ we have $c_{x-m} = ||x - m|| = c_x$, i.e., we have ii). Since ii) $\Rightarrow$ iii) is obvious, suppose we have iii). By $m \in P(x)$ and iii) we have $c_x = ||x - m|| \leq c_{x-m} \leq ||x - m||$, and so $c_{x-m} = ||x - m||$. By condition 2 of Definition 2.1 it follows $0 \in P(x - m)$, and P being on M, $0 \in P_M(x - m)$ whence $m \in P_M(x)$.

2.5. Remark. An immediate consequence of Remarks 2.3 and 2.4 is the following: if P is a B-set-valued mapping on M and for some $x \in \in \mathrm{Dom}(P) \setminus M$ we have $P(x) \cap P_M(x) \neq \emptyset$, then we have $c_{x-p}(x) = = c_x$ for each $p(x) \in P(x)$ with $x - p(x) \in \mathrm{Dom}(P)$.

2.6. THEOREM. *Let P be a B-set-valued mapping on M and $x \in \in \mathrm{Dom}(P) \setminus M$. The following assertions are equivalent:*

i) $P(x) \subset P_M(x)$

ii) $0 \in P(x - p(x))$ *for all* $p(x) \in P(x)$.

iii) *For some* $p(x) \in P(x)$ *there exists* $m \in P(x - p(x))$ *such that* $-m \in P(x - p(x) - m)$.

Proof. i) $\Rightarrow$ ii). Let $p(x) \in P(x)$. By i) we have $0 \in P_M(x - p(x))$ and since P is on M, it follows $0 \in P(x - p(x))$, that is ii).

ii) $\Rightarrow$ iii) is obvious for $m = 0$.

iii) $\Rightarrow$ i). The hypothesis iii) and $x \notin M$ imply that $x - p(x)$, $x-p(x) - m \in \mathrm{Dom}(P) \setminus M$. Since $-m \in P(x - p(x) - m)$ we have:

$$c_{x-p(x)-m} = ||(x - p(x) - m) - (-m)|| = ||x - p(x)|| \geq c_{x-p(x)} \tag{2.2}$$

whence by Remark 2.4, $m \in P_M(x - p(x))$. Hence $0 \in P_M(x - p(x) - m)$ and since P is on M, it follows $0 \in P(x - p(x) - m)$. Hence, using (2.2) we have $c_{x-p(x)-m} = ||x - p(x) - m|| = ||x - p(x)||$ and since $p(x) + + m \in P_M(x)$ and $p(x) \in M$, we get $p(x) \in P_M(x)$. So, $p(x) \in P(x) \cap \cap P_M(x)$, whence by Remark, 2.3 we have i), which completes the proof.

We recall (see e.g., [11]) that a set-valued mapping $P: X \to 2^X$ is called *upper (K) semi-continuous (u.(K)s.c.) at* $x \in \mathrm{Dom}(P)$, respectively *lower (K) semi-continuous (l.(K)s.c.) at* $x \in \mathrm{Dom}(P)$, if the relations $x_n \in \mathrm{Dom}(P)$, $\lim_{n\to\infty} x_n = x$, $p(x_n) \in P(x_n)$, $\lim_{n\to\infty} p(x_n) = y \in X$ imply $y \in P(x)$, respectively if the relations $x_n \in \mathrm{Dom}(P)$, $\lim_{n\to\infty} x_n = x$, $p(x) \in P(x)$ imply the existence of $p(x_n) \in \in P(x_n)$ with $\lim_{n\to\infty} p(x_n) = p(x)$. If everywhere above we replace lim by w-lim (i.e., for z_n, $z \in X$ we have $w\text{-}\lim_{n\to\infty} z_n = z$ if for each $f \in X^*$, $\lim_{n\to\infty} f(z_n) = f(z)$), then P is called *sequentially weakly upper (K) semi-continuous* (ω*-w.u.(K)s.c.*) respectively *sequentially weakly lower (K) semi-continuous* (ω*-w.l.(K)s.c.*) at $x \in \mathrm{Dom}(P)$. P is called *(norm-weak)u.(K)s.c.* at $x \in \mathrm{Dom}(P)$ if the relations $x_n \in \mathrm{Dom}(P)$, $\lim_{n\to\infty} x_n = x$, $p(x_n) \in \in P(x_n)$, $w\text{-}\lim_{n\to\infty} p(x_n) = y$ imply $y \in P(x)$. P is called *upper semi-continuous (u.s.c.)* at $x \in \mathrm{Dom}(P)$, respectively *lower semi-continuous (l.s.c.) at* $x \in \mathrm{Dom}(P)$, if for each closed subset $C \subset X$ the relations $x_n \in \mathrm{Dom}(P)$, $\lim_{n\to\infty} x_n = x$, $P(x_n) \cap C \neq \emptyset$ imply $P(x) \cap C \neq \emptyset$, respectively if for each open subset $D \subset X$, the relations $x_n \leq \mathrm{Dom}(P)$, $\lim_{n\to\infty} x_n = x$, $P(x) \cap \cap D \neq \emptyset$ imply $P(x_n) \cap D \neq \emptyset$ for $n \geq n_0$. Clearly, if P is u.s.c. at $x \in \mathrm{Dom}(P)$ then P is u. (K)s.c. at x while [10] P is l.s.c. at $x \in \mathrm{Dom}(P)$ if and only if P is l.(K)s.c. at x.

We recall (see e.g., [3]) that a normed linear space X has *property* (H) if the relations $x_n, x \in X$, $w\text{-}\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} ||x_n|| = ||x||$ imply $\lim_{n\to\infty} x_n = x$, and it is called *uniformly convex* if for x_n, $y_n \in X$, $||x_n|| = = ||y_n|| = 1$, $n = 1, 2, \ldots,$ the relation $\lim_{n\to\infty} ||x_n + y_n|| = 2$ implies $\lim_{n\to\infty} ||x_n - y_n|| = 0$. It is known that if X is uniformly convex then it has property (H), and a uniformly convex Banach space is always reflexive.

The metric projection P_M is always u.(K)s.c. at each $x \in \mathrm{Dom}(P_M)$, M an arbitrary subspace of X, and in uniformly convex Banach spaces X, P_M is continuous at each $x \in X$ for each closed subspace $M \subset X$ (see e.g., [6]). More generally, if X is a strictly convex normed linear space with property (H), then P_M is continuous at every $x \in X$ for each reflexive subspace $M \subset X$. These results will be easy consequences of the following result, if we use the known fact ([11]) that for the metric projection P_M the continuity (semi-continuity) at all $x \in P_M^{-1}(0)$ implies the continuity (semi-continuity) at all $x \in \mathrm{Dom}(P_M)$.

2.7. PROPOSITION. *If X is a strictly convex normed linear space, then each B-est-valued mapping P on the closed subspaces $M \subset X$ is u.(K)s.c. at each $x \in P^{-1}(0) \setminus (M \setminus \{0\})$. If in addition X has property (H) and M is reflexive, then P is both u.s.c. and l.s.c. at each $x \in P^{-1}(0) \setminus (M \setminus \{0\})$.*

Proof. Suppose X strictly convex, P a B-set-valued mapping on the closed subspace M and let $x \in P^{-1}(0) \setminus (M \setminus \{0\})$. By Remark 2.3 we have $P(x) = P_M(x) = \{0\}$.

Let $x_n \in \mathrm{Dom(P)}$, $\lim_{n\to\infty} x_n = x$ and $p(x_n) \in P(x_n)$, $\lim_{n\to\infty} p(x_n) = m \in M$.

By condition 1 of Definition 2.1 we have for all n

$$||x_n - p(x_n)|| \leq ||x_n|| \tag{2.3}$$

and so $||x - m|| = \lim_{n\to\infty} ||x_n - p(x_n)|| \leq \lim_{n\to\infty} ||x_n|| = ||x||$. Since $P_M(x) = = \{0\}$ and $m \in M$, it follows $m = 0$ and so P is *u.(K)s.c.* at x.

Suppose now that X has in addition property (H) and M is reflexive, and let $x_n \in \mathrm{Dom}(P)$, $\lim_{n\to\infty} x_n = x$, and $p(x_n) \in P(x_n)$. We show that $\lim_{n\to\infty} p(x_n) = 0$. By (2.1), $\{p(x_n)\}_{n=1}^{\infty}$ is a bounded sequence of the reflexive space M and so there exists a subsequence $\{p(x_{n_i})\}_{i=1}^{\infty}$ of $\{(px_n)\}_{n=1}^{\infty}$ such that $w\text{-}\lim_{i\to\infty} p(x_{n_i}) = m \in M$. Then $w\text{-}\lim(x_{n_i} - p(x_{n_i})) = x - m$, and since (2.3) holds for all n, we have

$$||x - m|| \leq \liminf_{i\to\infty} \|x_{n_i} - p(x_{n_i})\| \leq \limsup_{i\to\infty} \|x_{n_i} - p(x_{n_i})\| \leq \lim \|x_{n_i}\| = ||x|| \tag{2.4}$$

We have $P_M(x) = \{0\}$, whence by (2.4) we get $m = 0$. Hence, using again (2.4) it follows $\lim_{i\to\infty} \|x_{n_i} - p(x_{n_i})\| = ||x||$. Since X has property (H)., $\lim_{i\to\infty} (x_{n_i} - p(x_{n_i})) = x$ and so $\lim_{n\to\infty} p(x_{n_i}) = 0$. Therefore each weakly convergent subsequence of $\{p(x_n)\}_{n=1}^{\infty}$ converges in norm to zero, whence $\lim_{n\to\infty} p(x_n) = 0$. It is now obvious that P is u.s.c. at x and 1(K)s.c. at x (hence l.s.c. at x).

For $x \in X$ we denote by $[x]$ the linear space spanned by x. When G is a subset of X we shall denote $\operatorname{sp} G$, resp. $\overline{\operatorname{sp}}\, G$, the linear space spanned by G, resp. the closed linear space spanned by G.

To each set-valued mapping $P : X \to 2^X$ we associate another set-valued mapping $P' : X \to 2^X$ with $\operatorname{Dom}(P') \subset \operatorname{Dom}(P)$ in the following way: if $x \in P^{-1}(0) \cup (X \setminus \operatorname{Dom}(P))$ then $P'(x) = P(x)$; if $x \in \operatorname{Dom}(P) \setminus P^{-1}(0)$, let us put

$$a_x = \inf\{\operatorname{dist}(x, [p(x)]) \mid p(x) \in P(x)\} \tag{2.5}$$

If there exists $p(x) \in P(x)$ such that $a_x = \operatorname{dist}(x, [p(x)])$ then $P'(x) = \bigcup\{P_{[p(x)]}(x) \mid p(x) \in P(x), \operatorname{dist}(x, [p(x)]) = a_x\}$; if not, then $P'(x) = \emptyset$.

Let us observe that it can happen that for some $x \in \operatorname{Dom}(P')$ with $P(x)$ a singleton, to have not $P'(x)$ a singleton. If X is strictly convex, then for $x \in \operatorname{Dom}(P')$ we have $P_{[p(x)]}(x)$a singleton for each $p(x) \in P(x)$ but $P'(x)$ can be not a singleton and in this case surely $P(x)$ is not a singleton.

2.8. DEFINITION. *A set-valued mapping P is called orthogonal if $P' = P$*

2.9. Remark. For each set-valued mapping P, the setvalued mapping P' is orthogonal, and P' will be called its *associated orthogonal set-valued mapping*.

2.10. Remark. If P is a set-valued mapping and $x \in \operatorname{Dom}(P)$ with $P(x)$ compact, then $x \in \operatorname{Dom}(P')$ and $P'(x)$ is also compact. Indeed, by the definition of P' we must show the above statements only for $x \notin P^{-1}(0)$. Let $p_n(x) \in P(x)$ and $\lambda_n \in R$ be such that $||x - \lambda_n p_n(x)|| = \operatorname{dist}(x, [p_n(x)])(\leqslant ||x||)$, $\lim \operatorname{dist}(x, [p_n(x)]) = a_x$, where a_x is defined by (2.5). Since $P(x)$ is compact we can suppose $\lim p_n(x) = p(x) \in P(x)$, $p(x) \neq 0$. Then $\{\lambda_n\}_{n=1}^{\infty}$ is a bounded sequence and we can suppose $\lim \lambda_n = \lambda$. We have

$$a_x \leqslant \operatorname{dist}(x, [p(x)]) \leqslant ||x - \lambda p(x)|| = \lim ||x - \lambda_n p_n(x)|| = a_x$$

and so $a_x = \operatorname{dist}(x, [p(x)]) = ||x - \lambda p(x)||$, i.e., $\lambda p(x) \in P'(x)$ and $x \in \operatorname{Dom}(P')$. The proof that $P'(x)$ is compact is similar and we omit it.

2.11. Remark. If P is a B-set-valued mapping, then $(P')^{-1}(0) = P^{-1}(0)$. Indeed, by the definition of P' we must show only the inclusion $\subset$. Let $x \in P(')^{-1}(0) \setminus P^{-1}(0)$. Since $0 \in P'(x)$ and $0 \notin P(x)$, there exists $p(x) \in P(x)$, $p(x) \neq 0$ with $a_x = \operatorname{dist}(x, [p(x)]) = ||x|| \leqslant ||x - p(x)||$, where a_x is defined by (2.5) Since $p(x) \in P(x)$ and P is a B-set-valued mapping, we have $c_x = ||x - p(x)|| \leqslant ||x||$. So, $c_x = ||x||$ whence by condition 2 of Definition 2.1, it follows $0 \in P(x)$, a contradiction.

If P is a B-set-valued mapping, by Remark 2.11 it follows that for $x \in \operatorname{Dom}(P')$ and each $p'(x) \in P'(x)$ we have $p'(x) = \lambda p(x)$, $p(x) \in P(x)$, $\lambda \in R$ with $||x - p'(x)|| = ||x - \lambda p(x)|| = \operatorname{dist}(x, [p(x)])$.

In the sequel we want to see what properties of P are inhereted by P'. Since there are very few in the general case (see Lemma 2.12 below) most of the results will require some conditions on X as well as some additional assumptions on P.

2.12. Lemma. *If P is a B-set-valued mapping, then P' is also a B-set-valued mapping. Moreover if P is on M, then P' is on M.*

Proof. For $x \in \operatorname{Dom}(P')$ let $c'_x = c_x$ for $x \in P^{-1}(0)$ and $c'_x = a_x$ otherwise, where a_x is defined by (2.5). Then clearly $0 \leqslant c'_x \leqslant ||x||$ and condition 1 of Definition 2.1 is obviously satisfaied by P', while condition 2 is satisfied by Remark 2.11 and the fact that P is a B-set-valued mapping. If P us on M, then $P'(X) \subset M$ and the last statement follows using again Remark 2.11.

The next result, which will be useful in the sequel, is a slight generalization of [12], Section 4, Lemma 1, the implication a) $\Rightarrow$ b).

2.13. Lemma. *Let X be a uniformly convex normed linear space and let $\{M_n\}_{n=1}^{\infty}$ be a sequence of subspaces of X. Let $\{x_n\}_{n=1}^{\infty} \subset X$ and $m_n \in M_n$, $n = 1, 2, \ldots$, such that:*

$$\lim ||x_n|| = \lim ||x_n - m_n|| = \lim \operatorname{dist}(x_n, M_n) \tag{2.6}$$

Then $\lim m_n = 0$.

Proof. If $\lim ||x_n|| = 0$, then by (2.6) we have $\lim m_n = 0$. If $\lim ||x_n|| > 0$, then we can suppose that for all n, $||x_n|| > 0$ and $||x_n - m_n|| > 0$. Let us put for all n

$$\alpha_n = \frac{1}{||x_n||} + \frac{1}{||x_n - m_n||}$$

Then by (2.6) we have $\lim \alpha_n = 2/\alpha$, where $\alpha = \lim ||x_n|| > 0$. We have that for all n

$$2 \geqslant \left\| \frac{1}{||x_n||} + \frac{x_n - m_n}{||x_n - m_n||} \right\| = \alpha_n \left\| x_n - \frac{1}{\alpha_n ||x_n - m_n||} m_n \right\| \geqslant \alpha_n \operatorname{dist}(x_n, M_n) \to 2$$

Hence, since X is uniformly convex

$$\lim \left\| \frac{x_n}{||x_n||} - \frac{x_n - m_n}{||x_n - m_n||} \right\| = 0$$

whence by (2.6), $\lim m_n = 0$.

The next two results give conditions on X and the B-est-valued mapping P on M for which P' has some semi-continuity properties. The assumptions on X and M being not weaker then those given in Proposition 2.7, in view of this proposition and Lemma 2.12, we need not consider $x \in (P')^{-1}(0)$.

2.14. THEOREM. *Let X be a uniformly convex normed linear space, P a B-set-valued mapping on the closed subspace M, and $x \in \mathrm{Dom}(P') \setminus ((P')^{-1}(0) \cup M)$.*

i) If P is both u.(K)s.c. and l.(K)s.c. at x, then P' is u.(K)s.c. at x.

ii) If P is u.(K)s.c. at x and $P(x)$ is a singleton, then P' is u.(K)s.c. at x.

iii) If M is reflexive, $P(x)$ compact, and P is both u.s.c. and l.s.c. at x, then P' is u.s.c. at x.

iv) If M is reflexive, $P(x)$ is a singleton and P is u.s.c. at x, then P' is u.s.c. at x.

Proof. Let $x \in \mathrm{Dom}(P') \setminus ((P')^{-1}(0) \cup M)$, $x_n \in \mathrm{Dom}(P')$ such that $\lim x_n = x$, and $p'(x_n) \in P'(x_n)$. We have $p'(x_n) = \lambda_n p(x_n)$ for some $p(x_n) \in P(x_n)$ and $\lambda_n \in R$. If P is u.(K)s.c. at sx, then for all n

$$\|p(x_n)\| \geqslant \beta > 0 \tag{2.7}$$

Indeed, if $\lim p(x_{n_i}) = 0$, then $0 \in P(x) = P'(x)$, contradicting $x \notin (P')^{-1}(0$

i) Suppose $\lim p'(x_n) = m \in M$. We show that $m \in P'(x)$. By (2.7) and (2.1) we have that $\{\lambda_n\}_{n=1}^{\infty}$ is a bounded sequence, and so we can suppose (passing to a subsequence if necessary), that $\lim \lambda_n = \lambda$. If $\lambda = 0$, then by (2.1) we have $\lim p'(x_n) = 0$, and for all n the following relations hold:

$$\|x_n - p'(x_n)\| = \mathrm{dist}\,(x_n, [p(x_n)]) \leqslant \|x_n - p(x_n)\| \leqslant \|x_n\| \tag{2.8}$$

Hence

$$\lim \|x_n\| = \lim \mathrm{dist}\,(x_n, [p(x_n)]) = \lim \|x_n - p(x_n)\| \tag{2.9}$$

By Lemma 2.13 it follows $\lim p(x_n) = 0$, contradicting (2.7). Therefore $\lambda \neq 0$, and so $\lim p(x_n) = m/\lambda$. Since P is u.(K)s.c. at x, it follows $m/\lambda \in P(x)$, and so $m = \lambda p(x)$ for some $p(x) \in P(x)$. By the definition of P' and the assumption on x, there exist $m_0 \in P(x)$ and $\mu \in R$ with $a_x = \|x - \mu m_0\|$, where a_x is defined by (2.5). Since P is l.(K)s.c. at x, there exist $m_n \in P(x_n)$ with $\lim m_n = m_0$. We have for all n

$$\|x_n - \lambda_n p(x_n)\| \leqslant \|x_n - \mu m_n\|$$

and so

$$a_x \leqslant \|x - \lambda p(x)\| = \lim \|x_n - \lambda_n p(x_n)\| \leqslant \lim \|x_n - \mu m_n\| = \|x - \mu m_0\| = a_x$$

Thus, $a_x = \|x - \lambda p(x)\|$, whence $\lambda p(x) \in P'(x)$, which completes the proof of i).

ii) Using the notation of i), the proof of ii) is similar, since if $P(x) = \{p(x)\}$, then $\lim p(x_n) = p(x)$ and the above argument holds replacing m_0 by $p(x)$ and m_n by $p(x_n)$.

iii) Let C be a closed subset of X such that $P'(x_n) \cap C \neq \emptyset$, and let $p'(x_n) \in P'(x_n) \cap C$, where as above $p'(x_n) = \lambda_n p(x_n)$ for some $p(x_n) \in P(x_n)$ and $\lambda_n \in R$, $n = 1, 2, \ldots$. We claim that $\{p_n(x)\}_{\infty}^{n=1}$ has a convergent subsequence. If not, the set $C_1 = \{p(x_n) \mid n = 1, 2, \ldots\}$ is closed and $P(x_n) \cap C_1 \neq \emptyset$ for all n, so by the assumption on P to be u.s.c. at x, it follows $P(x) \cap C_1 \neq \emptyset$. Thus, there is x_{n_1} with $p(x_{n_1}) \in P(x) \cap C_1$. Let $C_2 = \{p(x_n) \mid n < n_1\}$. Then C_2 is closed and $P(x_n) \cap C_2 \neq \emptyset$ for all $n > n_1$, and so $P(x) \cap C_2 \neq \emptyset$, say, $p(x_{n_2}) \in P(x)$ for some $n_2 > n_1$. By repeating the above argument, we find a subsequence $\{p(x_{n_i})\}_{i=1}^{\infty} \subset \{p(x_n)\}_{n=1}^{\infty}$ with $\{p(x_{n_i})\}_{i=1}^{\infty} \subset P(x)$. This contradicts the compactness of $P(x)$, since $\{p(x_{n_i})\}_{i=1}^{\infty}$ has no convergent subsequence. Therefore there exists a convergent subsequence of $\{p(x_n)\}_{n=1}^{\infty}$, and without loss of generality we can suppose $\lim p(x_n) = m \in M$. By (2.7), we have $m \neq 0$. Then $\{\lambda_n\}_{n=1}^{\infty}$ is a bounded sequence and we may assume $\lim \lambda_n = \lambda$. Hence, $\lim p'(x_n) = \lambda m \in C$. By i) above, P' is u.(K)s.c. at x and so $\lambda m \in P'(x)$. Therefore $P'(x) \cap C \neq \emptyset$, which shows that P' is u.s.c. at x.

iv) The proof is similar with iii), using ii) instead of i). This completes the proof of the theorem.

2.15. THEOREM. *Let X be a strictly convex normed linear space with propery (H), P a B-set-valued mapping on the reflexive subspace M and $x_n \in \mathrm{Dom}(P') \setminus ((P')^{-1}(0) \cup M)$. If P is both l.(K)s.c. and (norm-weak)u.(K)s.c. at x, then P' is u.s.c. at x. If in addition $P'(x)$ is a singleton, then P' is l.(K)s.c. at x.*

Proof. Let $x \in \mathrm{Dom}(P') \setminus ((P')^{-1}(0) \cup M)$, $x_n \in \mathrm{Dom}(P')$, $\lim x_n = x$, and C a closed subset of X such that $P'(x_n) \cap C \neq \emptyset$ for all n. Let $p'(x_n) \in P'(x_n) \cap C$. Then $p'(x_n) = \lambda\, p(x_n)$ where $p(x_n) \in P(x_n)$ and $\lambda_n \in R$. By (2.1), $\{p(x_n)\}_{n=1}^{\infty}$ is a bounded sequence of the reflexive space M, thus we may assume that w-$\lim p(x_n) = p(x)$, where $p(x) \in P(x)$ since P is (norm-weak)u.(K)s.c. at x. We have $p(x) \neq 0$, otherwise $P(x) = P'(x) = \{0\}$ in contradiction with $x \notin (P')^{-1}(0)$. Since $\|p(x)\| \leqslant \lim\inf \|p(x_n)\|$, it follows that (2.7) holds for $n \geqslant n_0$ and so may we assume that $\lim \lambda_n = \lambda$. Because P is l.(K)s.c. at x, there are $m_n \in P(x_n)$ with $\lim m_n = p(x)$. We have $\|x_n - p(x_n)\| = \|x_n - m_n\|$ for all n, and so $\lim \|x_n - p(x_n)\| = \|x - p(x)\|$. Since we have also w-$\lim(x_n - p(x_n)) = x - p(x)$ and X has property (H), it follows $\lim p(x_n) = p(x)$, whence $\lim p'(x_n) = \lambda p(x) \in C$. The proof to show that $\lambda p(x) \in P'(x)$ is the same with the last part of the proof of Theorem 2.14 i) and we omit it.

The above proof shows that if $P'(x)$ is a singleton, say, $P'(x) = \{p'(x)\}$, and $p'(x_n) \in P'(x_n)$, then each weakly convergent subsequence of the bounded sequence $\{p'(x_n)\}_{n=1}^{\infty}$ converges in the norm topology to $p'(x)$, whence $\lim p'(x_n) = p'(x)$, which proves that P' is l.(K)s.c. at x when $P'(x)$ is a singleton. This completes the proof of the theorem.

Generalizing the definition of P dominates P when P, P_M are single-valued mappings (see [12]) we give:

2.16. DEFINITION. *If P is a B-set-valued mapping on M, then P dominates P_M (written $P \geqslant P_M$) if for every bounded sequence $\{x_n\}_{n=1}^{\infty} \subset \mathrm{Dom}(P)$ with* dist $(x_n, M) \geqslant \alpha > 0$ *for all n, the existence of a sequence $\{p(x_n)\}_{n=1}^{\infty}$, $p(x_n) \in P(x_n)$ with $\lim p(x_n) = 0$ implies the existence of an index n_0 such that $\{x_n\} n \geqslant n_0 \subset \mathrm{Dom}(P_M)$ and the existence of some $p_M(x_n) \in P_M(x_n)$ $(n \geqslant n_0)$ with $\lim p_m(x_n) = 0$.*

2.1.7. PROPOSITION. *Let X be a normed linear space and P a B-set-valued mapping on the subspace M, with $\mathrm{Dom}(P) = X$, and such that $P \geqslant P_M$. If P is 1.(K)s.c. at each $x \in P_{(0)}^{-1}M$, then P_M is 1.(K)s.c. at each $x \in \mathrm{Dom}(P_n)$.*

Proof. Since P_M is 1.(K)s.c. at each $x \in M$, let $x \in \mathrm{Dom}(P_M) \setminus M$, $x_n \in \mathrm{Dom}(P_M)$ such that $\lim x_n = x$, and let $p_M(x) \in P_M(x)$. Then $x - p_M(x) \in P_M^{-1}(0) \,/\, \{0\}$ and P being on M we have $x - p_M(x) \in P^{-1}(0) / M$. Since $\lim (x_n - p_M(x)) = x - p_M(x)$ and $0 \in P(x - p_M(x))$, there exist $p(x_n - p_M x)) \in P(x_n - p_M(x))$ such that $\lim p(x_n - p_M(x)) = 0$ (since P is 1.(K)s.c. at $x - p_M(x)$). By $P \geqslant P_M$, there exist n_0 and $p_M(x_n - p_M(x)) \in P_M(x_n - p_M(x))$ for $n \geqslant n_0$ such that $\lim p_M(x_n - p_M(x)) = 0$, and so $\lim p_M(x_n) = p_M(x)$, which proves that P_M is 1.(K)s.c. at x.

2.18. PROPOSITION. *If X is a uniformly convex normed linear space and P a B-set valued mapping on the subspace M such that $P \geqslant P_M$, then $P' \geqslant P_M$.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty} \subset \mathrm{Dom}(P')$ is a bounded sequence with $\mathrm{dist}(x_n, M) \geqslant \alpha > 0$ for all n, and there are $p'(x_n) \in P'(x_n)$ with $\lim p'(x_n) = 0$. Then $p'(x_n) = \lambda_n p(x_n)$ for some $p(x_n) \in P(x_n)$ and $\lambda_n \in R$. We may assume that $\{\|x_n\|\}_{n=1}^{\infty}$ is convergent. Then for all n (2.8) holds and since $\lim p'(x_n) = 0$, we get (2.9). By Lemma 2.13, we have $\lim p(x_n) = 0$ and since $P \geqslant P_M$ the result follows.

Generalizing for set-valued mapping the notion of a demi-compact operator ([2], see also [12]) we give:

2.19. DEFINITION. *A set-valued mapping P is called demi-compact if for every bounded sequence $\{x_n\}_{n=1}^{\infty} \subset \mathrm{Dom}(P)$, the existence of some $p(x_n) \in P(x_n)$ with $\{p(x_n)\}_{n=1}^{\infty}$ convergent, implies the existence of a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$.*

2.20. PROPOSITION. *If X is a uniformly convex normed linear space and P a demi-compact B-set-valued mapping, then P' is also demi-compact.*

Proof. Let $\{x_n\}_{n=1}^{\infty} \subset \mathrm{Dom}(P')$ be a bounded sequence and $p'(x_n) \in P'(x_n)$ with $\lim p'(x_n) = w$. Then $p'(x_n) = \lambda_n p(x_n)$ for some $p(x_n) \in P(x_n)$ and $\lambda_n \in R$. We claim that $\{p(x_n)\}_{n=1}^{\infty}$ has a convergent subsequence, whence since P is demi-compact, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence and so P' is demi-compact. Suppose now that $\{p(x_n)\}_{n=1}^{\infty}$ has no convergent subsequence, and we may assume that $\{\|x_n\|\}_{n=1}^{\infty}$ is convergent. Then by

(2.1), $\{p(x_n)\}_{n=1}^{\infty}$ is bounded and since $\lim \lambda_n p(x_n) = w$ we must have $\lim \lambda_n = 0$, and so $\lim p'(x_n) = 0$. We have (2.8) for all n, whence (2.9) holds. By Lemma 2.13 we obtain $\lim p(x_n) = 0$, a contradiction. This completes the proof.

3. Convergence theorems

Let $P: X \to 2^X$ be a set-valued mapping and for $x \in \mathrm{Dom}(P)$ let us define the following sequence of subsets of X by:

(3.1) $$Y_0 = \{x\},\ Y_{n+1} = \bigcup\{y_n - P(y_n) \mid y_n \in Y_n \cap \mathrm{Dom}(P)\},\quad (n = 0, 1, 2, \ldots$$

Clearly, if $\mathrm{Dom}(P) = X$ then $Y_n \neq \emptyset$ for all n. Otherwise we shall make the assumption:

(3.2) $$Y_n \neq \emptyset \qquad (n = 1, 2, \ldots)$$

Note that if (3.2) holds, then by (3.1) we have $Y_n \cap \mathrm{Dom}(P) \neq \emptyset$ for all $n = 0, 1, 2, \ldots$, and if P is a B-set-valued mapping, then $\|y_n\| \leqslant \|x\|$ for all $y_n \in Y_n$, $Y_n \neq \emptyset$.

When (3.2) is fulfilled, we shall be concerned with sequences $\{s_n\}_{n=0}^{\infty}$, $s_n \in Y_n$ with the following property:

(3.3) $$\|s_{n+1}\| \leqslant \|s_n - p(s_n)\| \quad \text{for all } p(s_n) \in P(s_n),\ n = 0, 1, 2, \ldots$$

When P is a B-set valued mapping, then we have $\|s_n - p(s_n)\| \leqslant \|s_n\|$, hence a sequence $\{s_n\}_{n=0}^{\infty}$ satisfying (3.3), satisfies also the condition:

(3.4) $$\lim \|s_n\| = \lim \|s_n - p(s_n)\| \leqslant \|x\|$$

since $\{\|s\|\}_{n=0}^{\infty}$ is a decreasing sequence, hence convergent.

Let P be a B-set-valued mapping with $\mathrm{Dom}(P) = X$. Then sequence $s_n\}_{n=0}^{\infty}$, $s_n \in Y_n$, satisfying (3.3) always exist. Indeed, for $x \in X$ take

(3.5) $$s_0 = x,\ s_{n+1} = s_n - p(s_n),\ p(s_n) \in P(s_n),\ n = 0, 1, 2, \ldots$$

The same conclusion holds under the weaker assumption that for all $x_n \in \mathrm{Dom}(P)$

(3.6) $$(x - P(x)) \cap \mathrm{Dom}(P) \neq \emptyset.$$

Indeed, we must take in (3.5), $p(s_n) \in P(s_n)$ with $s_n - p(s_n) \in \mathrm{Dom}(P)$. Another way to obtain such sequences which, as we shall see in Example 3.2 below could be different of the ones obtained by (3.5), is given in the next result.

3.1. Remark. Let P be a B-set-valued mapping with $P(X) \subset M$, where M is a reflexive subspace of the normed linear space X, and such that (3.6) holds for all $x \in \mathrm{Dom}(P)$. Let $x \in \mathrm{Dom}(P)$, and consider the sets Y_n defined by (3.1) ($Y_n \neq \emptyset$ by (3.6), for all n).

i) Suppose that $\mathrm{Dom}(P)$ is sequentially weakly closed (i.e., the relations $z_n \in \mathrm{Dom}(P)$, $n = 1, 2, \ldots$, $z \in X$, $w-\lim z_n = z$ imply $z \in \mathrm{Dom}(P)$), and P is ω — w.u.(K)s.c. at every $x \in \mathrm{Dom}(P)$. Then there exists $s_n \in Y_n \cap \mathrm{Dom}(P)$ such that

$$\|s_n\| = \inf\{\|y_n\| \mid y_n \in Y_n \cap \mathrm{Dom}(P)\} \tag{3.7}$$

and $\{s_n\}_{n=0}^{\infty}$ satisfies (3.3).

ii) If $\dim M < \infty$ the same conclusion holds if $\mathrm{Dom}(P)$ is closed and P is u.(K)s.c. at every $x \in \mathrm{Dom}(P)$.

To show i) we first note that $Y_n \cap \mathrm{Dom}(P)$ is sequentially weakly compact for all $n = 0, 1, 2, \ldots$. Indeed, for $n = 0$ this being obvious, suppose that $Y_n \cap \mathrm{Dom}(P)$ is sequentially weakly compact and we show that $Y_{n+1} \cap \mathrm{Dom}(P)$ has the property. Let $\{w_k\}_{k=1}^{\infty} \subset Y_{n+1} \cap \mathrm{Dom}(P)$. Then $w_k = z_k - p(z_k)$ where $z_k \in Y_n \cap \mathrm{Dom}(P)$ and $p(z_k) \in P(z_k)$. Since $Y_n \cap \mathrm{Dom}(P)$ is sequentially weakly compact, we can assume $w-\lim z_k = z \in Y_n \cap \mathrm{Dom}(P)$. Hence $\{\|z_k\|\}_{k=1}^{\infty}$ is bounded and by (2.1), $\{p(z_k)\}_{k=1}^{\infty}$ is a bounded sequence of the reflexive space M. So we can assume $w-\lim p(z_k) = p(z)$, where $p(z) \in P(z)$ since P is ω—w.u.(K)s.c. at z. Therefore $\{w_k\}_{k=1}^{\infty}$ has a convergent subsequence to $z - p(z) \in Y_{n+1} \cap \mathrm{Dom}(P)$ (by hypothesis $\mathrm{Dom}(P)$ is sequentially weakly closed). Now, since all $Y_n \cap \mathrm{Dom}(P)$ are sequentially weakly compact, there exists $s_n \in Y_n \cap \mathrm{Dom}(P)$ such that (3.7) holds. Indeed, for each n, let $y_{nk} \in Y_n \cap \mathrm{Dom}(P)$ with $\lim\limits_{k\to\infty} \|y_{nk}\| = \inf\{\|y_n\| \mid y_n \in Y_n \cap \mathrm{Dom}(P)\}$. Since $Y_n \cap \mathrm{Dom}(P)$ is sequentially weakly compact, we may assume $w-\lim\limits_{k\to\infty} y_{nk} = s_n \in Y_n \cap \mathrm{Dom}(P)$. We show that $\{s_n\}_{n=0}^{\infty}$ satisfies (3.7) and (3.3). We have $\in \|s_n\| \leq \liminf\limits_{k\to\infty} \|y_{nk}\| = \lim\limits_{k\to\infty} \|y_{nk}\| = \inf\{\|y_n\| \mid y_n \in Y_n \cap \mathrm{Dom}(P)\} \leq \|s_n\|$, hence (3.7) is proved. Now, since $s_n - P(s_n) \subset Y_{n+1}$ and by (3.6), $(s_n - P(s_n)) \cap \mathrm{Dom}(P) \neq \emptyset$, there exists $p(s_n) \in P(s_n)$ such that $s_n - p(s_n) \in Y_{n+1} \cap \mathrm{Dom}(P)$. Hence $\|s_{n+1}\| = \inf\{\|y_{n+1}\| \mid y_{n+1} \in Y_{n+1} \cap \mathrm{Dom}(P)\} \leq \|s_n - p(s_n)\|$. Since P is a B-set-valued mapping, we have (3.3), which completes the proof of i). The proof of ii) is similar and simpler than that of i). Note that in this case $Y_n \cap \mathrm{Dom}(P)$ is compact for all n.

3.2. Example. Let $X = R^2$ with the Euclidean norm, $M = \{(\alpha, 0) \mid \alpha \in R\}$ and define the B-set-valued mapping P on M, with $\mathrm{Dom}(P) = X$, in the following way: $P((\alpha_1, \alpha_2)) = \left\{\left(\frac{\alpha_1}{2}, 0\right), \left(\frac{3\alpha_1}{2}, 0\right)\right\}$ if $\alpha_1 \geq 0$, and $P((\alpha_1, \alpha_2)) = \left\{\left(\frac{3\alpha_1}{4}, 0\right), \left(\frac{5\alpha_1}{4}, 0\right)\right\}$ if $\alpha_1 < 0$. Then P is u.(K)s.c. at every $x \in X$. For $x = (1, 1)$ we have $Y_0 = \{(1, 1)\}$, $Y_1 = \left\{\left(\frac{1}{2}, 1\right), \left(-\frac{1}{2}, 1\right)\right\}$, $Y_2 = \left\{\left(\frac{1}{4}, 1\right), \left(-\frac{4}{1}, 1\right), \left(\frac{1}{8}, 1\right), \left(-\frac{1}{8}, 1\right)\right\}$, and so on. Take the following sequence $\{s_n\}_{n=0}^{\infty}$, $s_n \in Y_n$ which satisfies (3.7): $s_0 = (1, 1)$, $s_1 = \left(\frac{1}{2}, 1\right)$, $s_2 = \left(\frac{1}{8}, 1\right)$, antd the other s_n, $n \geq 3$, only to satisfy (3.7). Then $\{s_n\}_{n=0}^{\infty}$ is not of the form (3.5).

3.3. THEOREM. *Let P be a B-set-valued mapping on the finite dimensional subspace M, with $\mathrm{Dom}(P)$ closed and P is u.(K)s.c. at every $x \in \mathrm{Dom}(P)$, and let $x \in \mathrm{Dom}(P) \setminus M$. Suppose the sequence $\{Y_n\}$ defined by (3.1) satisfies (3.2). Then a sequence $\{s_n\}_{n=0}^{\infty}$, $s_n \in Y_n \cap \mathrm{Dom}(P)$ which satisfies (3.3), has convergent subsequences, and for each convergent subsequence $\{s_{n_i}\}_{i=0}^{\infty} \subset \{s_n\}_{n=0}^{\infty}$ we have $\lim s_{n_i} = x - p_M(x)$ for some $p_M(x) \in P_M(x)$ which depends on the subsequence.*

Proof. For $s_n \in Y_n$ we have:

$$s_n = x - m_n, \quad m_n \in M \tag{3.8}$$

But $\|s_n\| \leq \|x\|$ and so $\{m_n\}_{n=0}^{\infty}$ is a bounded sequence of the finite dimensional subspace M, and so it has a convergent subsequence, say, $\lim m_{n_i} = m \in M$. Hence $\lim s_{n_i} = x - m$, whence $\{s_n\}_{n=0}^{\infty}$ has convergent subsequences. Let now $\{s_{n_i}\}_{i=0}^{\infty}$ be a convergent subsequence of $\{s_n\}_{n=0}^{\infty}$. By (3.8) we have $\lim m_{n_i} = m \in M$, whence $\lim s_{n_i} = x - m \in \mathrm{Dom}(P)$. We show that $m \in P_M(x)$. Since P is a B-set-valued mapping, using (3.3) we obtain $\|s_{n_i}\| \leq \|s_{n_{i-1}}\| - p(s_{n_{i-1}})\| \leq \|s_{n_{i-1}}\|$ for all i. Since $\|s_n\| \leq \|x\|$ for all n, by (2.1) the sequence $\{p(s_{n_i})\}_{i=0}^{\infty}$ is bounded, and we may assume $\lim p(s_{n_i}) = p(x - m) \in P(x - m)$ since P is u.(K)s.c. at $x - m$. We have:

$$\|x - m\| = \lim \|s_{n_i}\| = \lim \|s_{n_i} - p(s_{n_i})\| = ((x - m) - p(x - m)\|$$

Then $x - m \in P^{-1}(0) \setminus M$, and so $x - m \in P_M^{-1}(0)$. Hence $m = p_M(x) \in P_M(x)$ and so $\lim s_{n_i} = x - p_M(x)$.

An immediate consequence of Theorem 3.3 is:

3.4. COROLLARY. *Under the same hypotheses as in Theorem 3.3 if $P_M(x) = \{p_M(x)\}$, then for each sequence $\{s_n\}_{n=0}^{\infty}$, $s_n \in Y_n \cap \mathrm{Dom}(P)$ satisfying (3.3), we have $\lim s_n = x - p_M(x)$.*

Let P be a B-set valued mapping on M and suppose that all the hypotheses of Theorem 3.3 are satisfied. Then we can define another B-set-valued mapping $\overline{P}$ on M, $\mathrm{Dom}(\overline{P}) \subset \mathrm{Dom}(P)$ in the following way: if $x \in (M \cap \mathrm{Dom}(P)) \cup (X \setminus \mathrm{Dom}(P))$ then $\overline{P}(x) = P(x)$; if $x \in \mathrm{Dom}(P) \setminus M$ and the sets Y_n defined by (3.1) satisfy (3.2), and there exists $s_n \in Y_n \cap \mathrm{Dom}(P)$ such that $\{s_n\}_{n=0}^{\infty}$ satisfies (3.3) then $\overline{P}(x) = \{x - \lim s_{n_i} \mid s_n \in Y_n, \{s_n\}_{n=0}^{\infty}$ satisfies (3.3) and $\{s_{n_i}\}_{i=0}^{\infty}$ is a convergent sub sequence of $\{s_n\}_{n=0}^{\infty}\}$; otherwise $\overline{P}(x) = \emptyset$. By Theorem 3.3 we have for each $x \notin M$ that $\overline{P}(x) \subset P_M(x)$. Then clearly $\overline{P}$ is a B-set-valued mapping, and to show that it is on M we must only check that if $0 \in P_M(x)$ then $0 \in \overline{P}(x)$ for $x \notin M$. If $0 \in P_M(x)$, $x \notin M$, since P is on M we have $0 \in P(x)$, whence $x \in Y_n \cap \mathrm{Dom}(P)$ for all $n = 0, 1, 2, \ldots$, and (3.3) is satisfied for $s_n = x$ and $p(s_n) = 0$. Hence $0 \in \overline{P}(x)$.

3.5. THEOREM. *Let X be a uniformly convex normed linear space and P an orthogonal B-set-valued mapping on the reflexive subspace M. Let $x \in \mathrm{Dom}(P) \setminus M$ and suppose that the sequence $\{Y_n\}_{n=0}^{\infty}$ defined by (3.1) satisfies (3.2). Let $s_n \in Y_n \cap \mathrm{Dom}(P)$ such that $\{s_n\}_{n=0}^{\infty}$ satisfies (3.4). We have:*

i) *If $P \geqslant P_M$ then $\lim s_n = x - p_M(x)$ where $P_M(x) = \{p_M(x)\}$;*

ii) *If P is demi-compact and u.(K)s.c. at every $x \in \mathrm{Dom}(P)$ and $\mathrm{Dom}(P)$ is closed then $\lim s_n = x - p_M(x)$ where $P_M(x) = \{p_M(x)\}$.*

iii) *If $\mathrm{Dom}(P)$ is sequentially weakly closed and P is ω—w.u.(K)s.c. at every $x \in \mathrm{Dom}(P)$, then w—$\lim s_n = x - p_M(x)$ where $P_M(x) = \{p_M(x)\}$.*

Proof. The assumptions on X and M imply $P_M(z) = \{p_M(z)\}$ for all $z \in X$. Let us first note that since $\{s_n\}_{n=0}^{\infty}$ satisfies (3.4) and P is orthogonal, by Lemma 2.13 we have

$$\lim p(s_n) = 0 \tag{3.9}$$

i) We have $s_n - p_M(s_n) = x - p_M(x)$, hence $\mathrm{dist}\ (s_n, M) = \|x - p_M(x)\| > 0$ for all $n = 0, 1, 2, \ldots$, since $x \notin M$. By $P \geqslant P_M$, $\|s_n\| \leqslant \|x\|$ for all n, and (3.9) we obtain $\lim p_M(s_n) = 0$, whence $\lim s_n = x - p_M(x)$.

ii) P being demi-compact, by (3.9) there exists a convergent subsequence of $\{s_n\}_{n=0}^{\infty}$, say, $\lim s_{n_i} = s \in \mathrm{Dom}(P)$, since $\mathrm{Dom}(P)$ is closed. Because P is u.(K)s.c. at s, by (3.9) we obtain $s \in P^{-1}(0)$. On the other hand (3.8) holds, whence since $\{s_{n_i}\}_{i=0}^{\infty}$ is convergent, we have $\lim m_{n_i} = m \in M$ and so $s = x - m$. Now we have $x - m \in P^{-1}(0) \setminus M$, and P being on M it follows $x - m \in P_M^{-1}(0)$ and so $m = p_M(x)$. Hence $\lim s_{n_i} = x - p_M(x)$. Since $\{s_{n_i}\}_{i=0}^{\infty}$ was an arbitrary convergent subsequence, the sequence $\{s_n\}_{n=0}^{\infty}$ converges to $x - p_M(x)$.

iii) We have (3.8) $\{m_n\}_{n=0}^{\infty}$ is bounded, so it has a weakly convergent subsequence, say, w—$\lim m_{n_i} = m \in M$. Thus w—$\lim s_{n_i} = x - m \in \mathrm{Dom}(P)$. Hence by (3.9) and since P is ω—w.u.(K)s.c. at $x - m$ it follows $0 \in P(x - m)$. Hence $0 \in P_M(x - m)$, and we have $m = p_M(x)$ and w—$\lim s_{n_i} = x - p_M(x)$. Since $\{m_{n_i}\}_{i=0}^{\infty}$ was an arbitrary weakly convergent subsequence of $\{m_n\}$, it follows w—$\lim s_n = x - p_M(x)$, which completes the proof.

We remark that under the hypotheses of Theorem 3.5 i) or ii), the B-set-valued mapping, $\overline{P}$ defined above, equals P_M at each $x \in (\mathrm{Dom}(\overline{P})) \setminus (M \setminus \{0\})$.

4. Examples and applications

If P_1, P_2 are two B-set-valued mappings, we shall denote by $c_x^{(i)}$, $i = 1, 2$ the corresponding c_x given by Definition 2.1 for them. We shall define another B-set-valued mapping P for which the notation c_x is maintained.

We recall (see e.g., [3]) that a normed linear space X is called *smooth* if for each $x \in X \setminus \{0\}$ there exists a unique $f \in X^*$, $\|f\| = 1$ such that $f(x) = \|x\|$.

4.1. PROPOSITION. *Let X be a normed linear space and let P_1, P_2 be B-set-valued mappings with $\mathrm{Dom}(P_2) = X$ and such that for each $x \in \mathrm{Dom}(P_1)$ $c_{x-p_1(x)}^{(2)} = b_x$ for all $p_1(x) \in P_1(x)$. Then the set-valued mapping P with $\mathrm{Dom}(P) = \mathrm{Dom}(P_1)$ defined by $P(x) = \{p_1(x) + P_2(x - p_1(x)) \mid p_1(x) \in P_1(x)\}$ for $x \in \mathrm{Dom}(P_1)$, is a B-set-valued mapping. If P_1 and P_2 are on M_1 and M_2 respectively, and X is smooth, then P is on $M = M_1 + M_2$.*

Proof. For $x \in \mathrm{Dom}(P)$ let $c_x = b_x$. Then for $p_1(x) \in P_1(x)$ we have

$$0 \leqslant c_x = c_{x-p_1(x)}^{(2)} \leqslant \|x - p_1(x)\| \leqslant \|x\| \tag{4.1}$$

For $p(x) \in P(x)$, there are $p_1(x) \in P_1(x)$ and $p_2(x - p_1(x)) \in P_2(x - p_1(x))$ such that $p(x) = p_1(x) + p_2(x - p_1(x))$. We have:

$$\|x - p(x)\| = \|x - p_1(x) - p_2(x))\| = c_{x-p_1(x)}^{(2)}(x) = b_x = c_x$$

Since $p(x) \in P(x)$ was arbitrary, condition 1) of Definition 2.1 holds. Suppose now $c_x = \|x\|$. Then by (4.1) it follows $\|x - p_1(x)\| = \|x\|$ and so $0 \in P_1(x)$. Hence $P_2(x) \subset P(x)$ and we have $c_x^{(2)} = b_x = c_x = \|x\|$, whence $0 \in P_2(x) \subset P(x)$, and condition 2 of Definition 2.1 is satisfied. Note that if $c_x = \|x\|$, then $0 \in P_1(x) \cap P_2(x)$.

Suppose now X smooth and P_i on M_i, $i = 1, 2$. Let $x \notin M$ and $x \in P_M^{-1}(0)$. Then $x \notin M_i$, and $x \in P_{M_i}^{-1}(0)$, $i = 1, 2$, whence since P_i are on M_i it follows $0 \in P_1(x) \cap P_2(x)$, and by the definition of P, $0 \in P(x)$. Conversely, if $0 \in P(x)$ for $x \notin M$, then as we have remarked above $0 \in P_1(x) \cap P_2(x)$, hence $x \in P_{M_1}^{-1}(0) \cap P_{M_2}^{-1}(0) = P_M^{-1}(0)$ since X is smooth (see e.g., [6]).

For $x \in X$ and $r \geqslant 0$ we denote $B(x, r) = \{y \in X \mid \|y - x\| \leqslant r\}$ and $S(x, r) = \{y \in X \mid \|y - x\| = r\}$.

4.2. LEMMA. *Let X be a strictly convex normed linear space and $\lambda \in R$, $0 < \lambda < 1$. Let $V : X \to 2^X$ be a set-valued mapping such that*

$$V(x) \subset \left(\frac{1-\lambda}{\lambda} S(-x, r_y)\right) \cap B(0, \|x\|) \qquad (x \in X) \tag{4.2}$$

where $r_x \geq 0$. *Then* $P(x) = \lambda(x - V(x))$, $x \in X$, *is a B-set-valued mapping with* $\text{Dom}(P) = \text{Dom}(V)$, *and we have:*

$$P^{-1}(0) = \{x \in \text{Dom}(V) \,|\, V(x) = \{x\}\} \tag{4.3}$$

Proof. Let $x \in \text{Dom}(P)$ and $p(x) \in P(x)$. Then $p(x) = \lambda(x - v(x))$ for some $v(x) \in V(x)$. By (4.2), there exists $z \in S(-x, r_x)$ with $v(x) = \frac{1-\lambda}{\lambda} z$. We have:

$$\|x - p(x)\| = \|(1-\lambda)x + \lambda v(x)\| = (1-\lambda)\|x + z\| = (1-\lambda)r_x \tag{4.4}$$

and

$$(1-\lambda)r_x = \|(1-\lambda)x + \lambda v(x)\| \leq (1-\lambda)\|x\| + \lambda\|v(x)\| \leq \|x\| \tag{4.5}$$

So, for $c_x = (1-\lambda)r_x$ we have $0 \leq c_x \leq \|x\|$, and by (4.4), P satisfies condition 1 of Definition 2.1. If $c_x = \|x\|$, then by (4.5) we obtain $\|x\| = \|v(x)\| = \|(1-\lambda)x + \lambda v(x)\|$, hence since X is strictly convex we get $x = v(x)$ and so $0 \in P(x)$. Therefore P is a B-set-valued mapping.

If $x \in P^{-1}(0)$, then $c_x = \|x\|$, and as we have seen above $v(x) = x$ for each $v(x) \in V(x)$, which proves the inclusion $\subset$ in (4.3). Since the other inclusions in obvious, this completes the proof.

4.3. THEOREM. *Let X be a uniformly convex normed linear space* $\lambda \in R$, $0 < \lambda < 1$, *and* V, P *as in Lemma 4.2. Suppose* $\text{Dom}(V) = X$, $I - V$ *is demi-compact, where I is the identity operator on X, and V is u.(K)s.c. at every* $x \in X$. *Let* $x \in \text{Dom}(P')$ *and suppose* $Y_n \neq \emptyset$, $n = 1, 2, \ldots$ *where* Y_n *are defined by* (3.1) *where we replace P by P'. If* $s_n \in Y_n$, $n = 0, 1, 2, \ldots$, *satisfy* $\lim \|s_n\| = \lim \|s_n - p'(s_n)\|$, $p'(s_n) \in P'(s_n)$, *then there exists a convergent subsequence* $\{s_{n_i}\}_{i=0}^{\infty} \subset \{s_n\}_{n=0}^{\infty}$ *with* $\lim s_{n_i} = s \in P^{-1}(0)$, *i.e.,* $V(s) = \{s\}$.

Proof. Since $P'(s_n) \neq 0$ for all $n = 0, 1, 2, \ldots$, let $p'(s_n) \in P'(s_n)$, $p'(s_n) = \lambda_n p(s_n)$, $p(s_n) \in P(s_n)$, $\lambda_n \in R$. By Lemma 4.2, P is a B-set-valued mapping and so we have for all n, $\|s_n - p'(s_n)\| \leq \|s_n - p(s_n)\| \leq \|s_n\|$. Hence by the assumption on $\{s_n\}_{n=0}^{\infty}$, we get $\lim \|s_n\| = \lim \|s_n - p(s_n)\| = \lim \text{dist}\,(s_n, [p(s_n)])$, whence by Lemma 2.13, it follows $\lim p(s_n) = 0$. Now P is dmi-compact (since so is $I - V$), hence $\{s_n\}_{n=0}^{\infty}$ being bounded, there exists a convergent subsequence of $\{s_n\}_{n=0}^{\infty}$, say, $\lim s_{n_i} = s$. Because V is u. (K) s.c. at s, P is also u. (K) s.c. at s, hence $s \in P^{-1}(0)$. By (4.3) it follows $V(s) = \{s\}$, which completes the proof.

For a set-valued mapping $V: X \to 2^X$, let us put:

$$M = \overline{\text{sp}}\{y - V(y) \,|\, y \in \text{Dom}\,(V)\} \tag{4.6}$$

We denote by $M^{\perp} = \{f \in X^* \,|\, f(m) = 0 \text{ for all } m \in M\}$ and by $V|\text{Dom}(V)$, the restriction of V to $\text{Dom}(V)$.

4.4 LEMMA. *Let X be a strictly convex normed linear space and* $V: X \to 2^X$ *a set-valued mapping such that* $V(x) \subset B(0, \|x\|)$ *for all* $x \in \text{Dom}(V)$. *Then we have:*

$$P_M^{-1}(0) \subset (X \setminus \text{Dom}(V)) \cup \{x \in \text{Dom}(V) \,|\, V(x) = \{x\}\} \tag{4.7}$$

where M is defined by (4.6).

Proof. Let $V_1: X \to 2^X$ be defined by $V_1(x) = V(x)$ for $x \in \text{Dom}(V)$ and $V_1(x) = \{x\}$ othervise. Then for M defined by (4.6) we have $M = \overline{\text{sp}}\{y - V_1(x) \,|\, y \in X\}$. By the assumption on V and the definition of V_1 we notice that $V_1(0) = \{0\}$, and we always have $0 \in P_M^{-1}(0)$. So, let $x \in P_M^{-1}(0) \setminus \{0\}$. Then by [10], Chapter I, Theorem 1.1, there exists $f \in M^{\perp}$, $\|f\| = 1$ such that $f(x) = \|x\| = 1$ such that $f(x) = \|x\|$. Let $v_1(x) \in V_1(x)$. Then $f(x - v_1(x)) = 0$ and so $\|x\| = f(x) = f(v_1(x)) \leq \|v_1(x)\| \leq \|x\|$.

Hence f attains its norm at x and $v_1(x)$, and since X is strictly convex we have $x = v_1(x)$. Since $v_1(x) \in V_1(x)$ was arbitrary, it follows $V_1(x) = \{x\}$ and so $P_M^{-1}(0) \subset \{x \in X \,|\, V_1(x) = \{x\}\}$, whence (4.7) follows

4.5. Remark. Under the assumptions of Lemma 4.4 and in addition $\text{Dom}(V) = X$ and M is reflexive, then $M = x$ if

$$\{x \in X \,|\, V(x) = \{x\}\} = \{0\}.$$

4.6. LEMMA. *Let X be a smooth normed linear space, and* $V: X \to 2^X$ *a set-valued mapping, its domain,* $\text{Dom}(V)$, *being a linear subspace of X, and such that* $V(x) \subset B(0, \|x\|)$ *for all* $x \in \text{Dom}(V)$, *and for each* $y \in \text{Dom}(V)$ *and each* $v_0(y) \in V(y)$ *there exists a linear selection for* $V|\text{Dom}(V)$, *say,* $v(x) \in V(x) (x \in \text{Dom}(V))$ *with* $v(y) = v_0(y)$. *Then:*

$$\{x \in \text{Dom}(V) \,|\, V(x) = \{x\}\} \subset P_M^{-1}(0) \tag{4.8}$$

where M is defined by (4.6).

Proof. Since $V(0) = \{0\}$ and $0 \in P_M^{-1}(0)$, let $x \in \text{Dom}(V)$, $x \neq 0$ such that $V(x) = \{x\}$. Choose $y \in \text{Dom}(V)$ and $v_0(y) \in V(y)$. Let now $v(z) \in V(z)$, $z \in \text{Dom}(V)$ such that v is a linear selection for $V|\text{Dom}(V)$, with $v(y) = v_0(y)$. We have $v(x) = x$. Since X is smooth, let $f_x \in X^*$ be the unique norm-one linear functional with $f_x(x) = \|x\|$. We define $\varphi \in (\text{Dom}(V))^*$ by $\varphi(z) = f_x(v(z))$, $z \in \text{Dom}(V)$. Then clearly φ is linear and $\|\varphi\| = 1$ since $|\varphi(z)| = |f_x(v(z))| \leq \|v(z)\| \leq \|z\|$ and $\varphi(x) = f_x(v(x)) = f_x(x) = \|x\|$. Let f be a norm-preserving extension of φ to X. Since $\|f\| = 1, f(x) = \|x\|$ and X being smooth, it follows $f_x = f$. We have

$$f_x(y - v_0(y)) = f_x(y - v(y)) = f_x(y) - f_x(v(y)) = f_x(y) - f(y) = 0$$

Since $y \in \text{Dom}(V)$ and $y_0(v) \in V(y)$ were arbitrary, it follows $f_x \in M^{\perp}$, whence again by [10], Chapter I, Theorem 1.1, $x \in P_M^{-1}(0)$.

4.7. R e m a r k. Under the assumption of Lemma 4.6, if $\{x \in \mathrm{Dom}(V) \mid V(x) = \{x\}\} \neq \{0\}$ then $M \neq X$.

4.8. E x a m p l e. Let $\{T_\alpha\}_{\alpha \in A}$ be a family of linear operators $T_\alpha : X \to X$ with $\| T_\alpha \| \leq 1$ and define for $x \in X$, $V(x) = \{T_\alpha(x) \mid \alpha \in A\}$. Then $V : X \to 2^X$ satisfies all the assumption of Lemma 4.6.

4.9. COROLLARY. *Let X be a strictly convex and smooth normed linear space and $V : X \to 2^X$ a set-valued mapping with $\mathrm{Dom}(V) = X$, satisfying all the assumptions of Lemma 4.6. Then*

$$P_M^{-1}(0) = \{x \in X \mid V(x) = \{x\}\} \tag{4.9}$$

where M is defined by (4.6). Moreover, if M is Chebyshev, then P_M is linear.

P r o o f. By Lemmas 4.4 and 4.6 we obtain (4.9). Suppose M is Chebyshev. We show that $\{x \in X \mid V(x) = \{x\}\}$ is a linear subspace of X, whence by (4.9) and [10], P_M is linear. Let $x_1, x_2 \in X$ such that $V(x_i) = \{x_i\}$, $i = 1, 2$, $\lambda_i \in R$, $i = 1, 2$ and $v_0(\lambda_1 x_1 + \lambda_2 x_2) \in V(\lambda_1 x_1 + \lambda_2 x_2)$. Let $v(x) \in V(x)$ $(x \in X)$ be a linear selection with $v(\lambda_1 x_1 + \lambda_2 x_2) = v_0(\lambda_1 x_1 + \lambda_2 x_2)$. Then $v_0(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 v(x_1) + \lambda_2 v(x_2) = \lambda_1 x_1 + \lambda_2 x_2$ and so $V(\lambda_1 x_1 + \lambda_2 x_2) = \{\lambda_1 x_1 + \lambda_2 x_2\}$, which completes the proof.

Let us note that in [12], Section 4, Lemma 4, the assumption on X to be reflexive is superfluous.

4.10. THEOREM *Let X be a strictly convex and smooth normed linear space and $\lambda \in R$, $0 < \lambda < 1$. Let $V : X \to 2^X$ be a set-valued mapping which satisfies (4.2) and all the assumptions of Lemma 4.6. Then $P(x) = \lambda(x - V(x))$, $x \in X$, is a B-set-valued mapping on M, where M is defined by (4.6). P is always 1.(K)s.c. and ω — w.1.(K)s.c. at every $x \in X$. Moreover P is u.(K)s.c., respectively u.s.c., respectively ω — w.u.(K)s.c. at $x \in X$, if and only if V has the corresponding semi-continuity property at x.*

P r o o f. By Lemma 4.2, P is a B-set-valued mapping. Now $P(X) \subset M$ and P is on M by Lemma 4.2 and Corollary 4.9.

We show now that P is 1.(K)s.c. and ω — w.1.(K)s.c. at every $x \in X$. Let $x_n \in X$, $\lim x_n = x$, respectively w-$\lim x_n = x$, and $p(x) \in P(x)$. Then $p(x) = \lambda(x - v_0(x))$ for some $v_0(x) \in V(x)$. Let $v(z) \in V(z)$, $z \in X$, be a linear selection with $v(x) = v_0(x)$.

If $\lim x_n = x$, then $\| v(x_n) - v(x) \| = \| v(x_n - x) \| \leq \| x_n - x \|$ and so $\lim v(x_n) = v(x) = v_0(x)$. Hence for $p(x_n) = \lambda(x_n - v(x)) \in P(x_n)$ we have $\lim p(x_n) = p(x)$.

If w-$\lim x_n = x$ and w-$\lim v(x) \neq v(x)$, then there is $f_0 \in X^*$ and a subsequence $\{v(x_{n_i})\}_{i=1}^{\infty} \subset \{v(x_n)\}_{n=1}^{\infty}$ such that

$$|f_0(v(x_{n_i}) - v(x))| \geq \alpha > 0 \text{ for all } i. \tag{4.10}$$

Let $f \in X^*$ be defined by $f(z) = f_0(v(z))$, $z \in X$. We have for all n that $f(x_n - x) = f_0(v(x_n - x)) = f_0(v(x_n) - v(x))$. Since w-$\lim x_n = x$ it follows

$0 = \lim |f(x_n - x)| = \lim |f_0(v(x_n) - v(x))|$ contradicting (4.10). Therefore w-$\lim v(x_n) = v(x)$ and so for $p(x_n) = \lambda(x_n - v(x_n)) \in P(x_n)$ we have w-$\lim p(x_n) = p(x)$.

The last statements of the theorem are obvious.

4.11. R e m a r k. The above proof shows that in a normed linear space X, any set-valued mapping $V : X \to 2^X$ satisfying the assumptions of Lemma 4.6, is always 1.(K)s.c. and ω — w.1.(K)s.c. at every $x \in \mathrm{Dom}(V)$.

The following example shows that there exist X and set-valued mappings $V : X \to 2^X$ satisfying all the assumptions of Theorem 4.10 (for single-valued mappings this is clear).

4.12. E x a m p l e. Let X be a Hilbert space and $T : X \to X$ a normal operator, $\| T \| = 1$. Let $0 < \lambda < 1$ and define $V(x) = \{Tx, T^*x\}$, $x \in X$. It is easy to show that $\left\| \frac{\lambda}{1-\lambda} Tx + x \right\| = \left\| \frac{\lambda}{1-\lambda} T^*x + x \right\|$, whence V satisfies (4.2) for $r_x = \left\| \frac{\lambda}{1-\lambda} Tx + x \right\|$ and also the other conditions required in Theorem 4.10.

4.13. PROPOSITION. *Under the assumptions of Theorem 4.10, if M is reflexive and V ω — w.u.(K)s.c. at every $x \in X$, then if $\{x_n\}_{n=1}^{\infty} \subset X$ is a bounded sequence and for all n there exists $p(x_n) \in P(x_n)$, where P is defined as in Theorem 4.10, such that $\lim p(x_n) = 0$, then w-$\lim p_M(x_n) = 0$, where $P_M(x_n) = \{p_M(x_n)\}$. In particular, if $\dim M < \infty$, then $P \gg P_M$.*

P r o o f. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, $\{p_M(x_n)\}_{n=1}^{\infty}$ is bounded, and we may assume w-$\lim p_M(x_n) = m \in M$. We have $p(x_n) = \lambda(x_n - v(x_n))$, where $v(x_n) \in V(x_n)$, and $\lim p(x_n) = 0$ by hypothesis. Since $x_n - p_M(x_n) \in P_M^{-1}(0)$, by Corollary 4.9 we have

$$V(x_n - p_M(x_n)) = \{x_n - p_M(x_n)\}, \quad n = 1, 2, \ldots. \tag{4.11}$$

Let v_n, $n = 1, 2, \ldots$, be linear selections $v_n(z) \in V(z)$ $(z \in X)$, with $v_n(x_n) = v(x_n)$. Hence, by (4.11) we get

$$p_M(x_n) = x_n - v(x_n) + v_n(p_M(x_n)) \quad n = 1, 2, \ldots \tag{4.12}$$

where $v_n(p_M(x_n)) \in V(p_M(x_n))$. Since $\lim p(x_n) = 0$ we have $\lim (x_n - v(x_n)) = 0$, and since w-$\lim p_M(x_n) = m$ by (4.12) it follows w-$\lim v_n(p_M(x_n)) = m$. By the assumption on V to be ω — w.u.(K)s.c. at m we have $m \in V(m)$ (hence $0 \in P^{-1}(0)$), whence by (4.3) and (4.8) it follows $m \in P_M^{-1}(0)$. But $m \in M$, and so we have $m = 0$. Since each weakly convergent subsequence of $\{p_M(x_n)\}_{n=1}^{\infty}$ converges weakly to 0, we have w-$\lim p_M(x_n) = 0$. The last statement follows now immediately by the definition of $P \gg P_M$.

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