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ON COMMON FIXED POINT IN UNIFORMIZABLE SPACES

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In this paper we shall prove a generalization of a common fixed point theorem [2] in uniformizable spaces and of a fixed point theorem from [9]. As an application we shall obtain a common fixed point theorem in probabilistic locally convex space $(S, \{\mathcal{F}^i\}_{i\in I}, \min)$ and a generalization of Krasnoseljski's fixed point theorem.

Let X be an arbitrary set. A mapping $d: X\times X \to \mathbb{R}^+ = [0, \infty)$ is called a *pseudometric* if for every x, y, z $\in$ X:

1. $d(x, y) \geqslant 0$, $d(x, x) = 0$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leqslant d(x, z) + d(z, y)$.

A pair $(X, \{d_i\}_{i\in I})$, where d_i is a pseudometric for every $i\in I$, is called a uniformizable space. The convergence of the sequences in $(X, \{d_i\}_{i\in I})$ is defined by:

$$x_n \to x \quad (x_n,\ x\in X),\ \text{if}\ n\to\infty \Leftrightarrow \lim_{n\to\infty} d_i(x_n - x) = 0,\ i\in I$$

and the notions of Cauchy sequence and completness is introduced in the usual way. We say that $(X, \{d_i\}_{i\in I})$ is Hausdorff if and only if:

$$d_i(x, y) = 0, \text{ for every } i\in I \Leftrightarrow x = y.$$

Now, we shall pove a common fixed point theorem in uniformizable space.

THEOREM 1. *Let $(X, \{d_i\}_{i\in I})$ be a complete Hausdorff uniformizable space, $f: I\to I$, S and T be continuous mappings from X into X, $A: X \to SX \cap TX$ be continuous so that A commutes with S and T and the following conditions are satisfied:*

1. *For every* $i \in I$, *there exists* $q_i : \mathbf{R}^+ \to [0, 1]$ *which is a nondecreasing function, for which is* $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(t) < 1$ *for every* $i \in I$ *and every* $t \in \mathbf{R}^+$ *and for every* $i \in I$ *and every* $x, y \in X$:

$$d_i(Ax, Ay) \leqslant q_i(d_{f(i)}(Sx, Ty))d_{f(i)}(Sx, Ty).$$

2. *There exists* $x_0 \in X$ *and* $x_1 \in X$ *so that* $Tx_1 = Ax_0$ *and for every* $i \in I$, $\sup_{n\in\mathbf{N}} d_{f^n(i)}(Ax_0, Ax_1) \leqslant p_i$, $p_i \in \mathbf{R}^+$.

Then there exists $z \in X$ *so that* $Az = Sz = Tz$. *If, in addition, for every* $i \in I$, $\sup_{n\in\mathbf{N}} d_{f^n(i)}(A^j x_1, A^2 x_0) \leqslant M_i$ $M_i \in \mathbf{R}^+ (j \in \{2, 3\})$ *then there exists one and only one element* $w \in X$ *such that* $w = Aw = Sw = Tw$ *and*:

$$\sup_{n\in\mathbf{N}} d_{f^n(i)}(w, A^2x_0) \leqslant N_i,\ N_i \in \mathbf{R}^+, \text{ for every } i \in I.$$

Proof: As in [2] let $\{x_n\}_{n\in\mathbf{N}}$ be such a sequence from X that $Sx_{2k} = Ax_{2k-1}$ and $Tx_{2k+1} = Ax_{2k}$, for every $k \in \mathbf{N}$ where x_0 and x_1 are from the condition 2. of the Theorem. Then for every $i \in I$ and every $k \in \mathbf{N}$ we have:

$$d_i(Ax_{2k}, Ax_{2k-1}) \leqslant q_i(d_{f(i)}(Sx_{2k}, Tx_{2k-1}))d_{f(i)}(Sx_{2k}, Tx_{2k-1}) =$$
$$= q_i(d_{f(i)}(Ax_{2k-1}, Ax_{2k-2}))\, d_{f(i)}(Ax_{2k-1}, Ax_{2k-2})) \leqslant$$
$$\leqslant q_i(d_{f(i)}(Ax_{2k-1}, Ax_{2k-2}))\, q_{f(i)}(d_{f^2(i)}(Sx_{2k-2}, Tx_{2k-1})) \times$$
$$\times d_{f^2(i)}(Sx_{2k-2}, Tx_{2k-1}) \leqslant \ldots \leqslant \prod_{s=0}^{2k-2} q_{f^s(i)}(d_{f^{s+1}(i)}(Ax_{2k-s-1}, Ax_{2k-s-2}))$$
$$\times d_{f^{2k-1}(i)}(Ax_0, Ax_1)$$

and similarly:

$$d_i(Ax_{2k+1}, Ax_{2k}) \leqslant q_i(d_{f(i)}(Ax_{2k}, Ax_{2k-1})d_{f(i)}(Ax_{2k-1}, Ax_{2k}) \leqslant$$
$$\leqslant q_i(d_{f(i)}(Ax_{2k}, Ax_{2k-1})) \prod_{s=0}^{2k-2} q_{f^{s+1}(i)}(d_{f^{s+2}(i)}(Ax_{2k-s-1}, Ax_{2k-s-2})) \times$$
$$\times d_{f^{2k}(i)}(Ax_0, Ax_1).$$

Since $q_i(t) \leqslant 1$, for every $i \in I$ and every $t \in \mathbf{R}^+$ it follows that for every $i \in I$ and every $n \in \mathbf{N}$:

$$d_j(Ax_n, Ax_{n-1}) \leqslant P_i, \text{ for every } j \in 0(i, f) = \{f^s(i) \mid s \in \mathbf{N} \cup \{0\}\}$$

and so:

$$d_i(Ax_{2k}, Ax_{2k-1}) \leqslant \prod_{s=0}^{2k-2} q_{f^s(i)}(P_i)P_i$$

$$d_i(Ax_{2k+1}, Ax_{2k}) \leqslant \prod_{s=0}^{2k-1} q_{f^s(i)}(P_i)P_i.$$

Let $n_i \in \mathbf{N}$ be such that:

$$q_{f^n(i)}(P_i) \leqslant Q_i < 1, \text{ for every } n > n_i (i \in I).$$

Such number n_i exists since $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(P_i) < 1$. Let $2k - 2 > n_i$.

Then

$$d_i(Ax_{2k}, Ax_{2k-1}) \leqslant \prod_{s=0}^{n_i} q_{f^s(i)}(P(i))(Q_i)^{2k-2-n_i} P_i = \frac{\prod_{s=0}^{n_i} q_{f^s(i)}(P_i)}{(Q_i)^{n_i+2}} (Q_i)^{2k} P_i (i \in I).$$

So it is obvious that there exists $R_i > 0$ such that for every $n > n_i$ $d_i(Ax_n, Ax_{n-1}) \leqslant R_i(Q_i)^n$ $i \in I$, which implies that $\{Ax_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence. Let $z = \lim_{n\to\infty} Ax_n$. As in [2] it follows that

$$Az = Sz = Tz.$$

Now, we shall prove that:

$$\text{(1)} \qquad \sup_{n\in\mathbf{N}} d_{j^n(i)}(A^3x_1, A^2x_0) \leqslant M_i (i \in I)$$

implies that $w = Aw = Sw = Tw$ for $w = Az$. First, we shall prove that (1) implies that:

$$\text{(2)} \qquad \sup_{j\in 0(i, f)} d_j(A^2z, Az) \leqslant M_i (i \in I).$$

Since $d_j(A^2z, Az) = \lim_{n\to\infty} d_j(A^2 Ax_{2n+1}, AAx_{2n})$ it is enough to prove that:

$d_j(A^3x_{2n+1}, Ax_{2n}) \leqslant M_i$, for every $n \in \mathbf{N}$ and $j \in (i, f)$ $(i \in I)$. It follows that:

$$d_j(A^3x_{2n+1}, A^2x_{2n}) \leqslant q_j(d_{f(j)}(S(Ax_{2n}), TA^2x_{2n+1}))d_{f(j)}(S(Ax_{2n}), TA^2x_{2n+1}) \leqslant$$
$$\leqslant d_{f(j)}(ASx_{2n}, A^2Tx_{2n+1}) = d_{f(j)}(A^2x_{2n-1}, A^3x_{2n}) \leqslant \ldots \leqslant$$
$$\leqslant d_{f^{2n}(j)}(A^3x_1, A^2x_0) \leqslant M_i, \text{ for every } n \in \mathbf{N},\ j \in 0(i, f)\ (i \in I)$$

and so (2) is proved. Now, let us prove that $d_i(A^2z, Az) = 0$, for every $i \in I$. This follows from the inequalities:

$$d_i(A^2z, Az) \leqslant q_i(d_{f(i)}(SAz, Tz))\, d_{f(i)}(SAz, Tz) =$$
$$= q_i(d_{f(i)}(A^2z, Az))\, d_{f(i)}(A^2z, Az) \leqslant \ldots \leqslant$$
$$\leqslant q_i(d_{f(i)}(A^2z, Az))\, q_{f(i)}(d_{f^2(i)}(A^2z, Az)) \ldots q_{f^n(i)}(d_{f^{n+1}(i)}(Az^2, Az)) \times$$
$$\times d_{f^{n+1}(i)}(A^2z, Az) \leqslant q_i(M_i)\, q_{f(i)}(M_i) \ldots q_{f^n(i)}(M_i)\, M_i$$

for every $i \in I$. Since $\overline{\lim_{n\to\infty}}\, q_{f^n(i)}(M_i) < 1$ it follows that $d_i(A^2z, Az) = 0$, for every $i \in I$ which implies that $A^2z = Az$. Now, from $Az = Tz = Sz$

it follows that $A^2z = SAz = TAz = Az$ and so $w = Az$ is the common fixed point for A, S and T.

Let $M = \{x | x \in X,\ x = Ax = Sx = Tx,$ there exists $\{T_i\}_{i\in I}(T_i \in \mathbb{R}^+,\ i \in I)$ so that for every $i \in I : d_j(x, A^2x_0) \leqslant T_i$, for every $j \in 0(i, f)\}$ We shall prove that:

$$d_j(A^2x_1, A^2x_0) \leqslant M_i,\ j \in 0(i, f), \text{ for every } i \in I$$

implies $M = \{w\}$. First, we shall prove that the cardinal number of M is less or equal to 1. Suppose that $u, v \in M$ and prove that $u = v$. If $u, v \in M$ then $u = Au = Tu = Su$, $v = Av = Tv = Sv$ and:

$$d_j(u, A^2x_0) \leqslant T_i,\ d_j(v, A^2x_0) \leqslant T'_i, \text{ for every } j \in 0(i, f)(i \in I).$$

Then we have:

$$d_i(u, v) = d_i(Au, Av) \leqslant q_i(d_{f(i)}(Su, Tv))d_{f(i)}(Su, Tv) =$$

$$= q_i(d_{f(i)}(u, v))p_{f(i)}(u, v) \leqslant \ldots \leqslant$$

$$\leqslant q_i(d_{f(i)}(u, v))\, q_{f(i)}(d_{f^2(i)}(u, v)) \ldots q_{f^n(i)}(d_{f^{n+1}(i)}(u, v)) \times$$

$$\times d_{f^{n+1}(i)}(u, v) \leqslant d_{f^{n+1}(i)}(u, v)$$

Further for every $j \in 0(i, f)$ it follows that:

$$d_j(u, v) \leqslant T_i + T'_i \text{ (for every } i \in I)$$

which implies that:

$d_i(u, v) \leqslant q_i(T_i + T'_i)q_{f(i)}(T_i + T'_i) \ldots q_{f^n(i)}(T_i + T'_i)(T_i + T'_i)$, for every $i \in I$. Since $\overline{\lim\limits_{n\to\infty}}\, q_{f^n(i)}(T_i + T'_i) < 1$ it follows that $d_i(u, v) = 0$, for every $i \in I$ and so $u = v$. We shall prove that $w \in M$. This means that there exists $\{T_i\}_{i\in I}(T_i \in \mathbb{R}^+,\ i \in I)$ so that

$$d_j(w, A^2x_0) \leqslant T_i,\ j \in 0(i, f), \text{ for every } i \in I.$$

Since $w = Az$ and $z = \lim\limits_{n\to\infty} Ax_{2n}$ it follows that:

$$d_i(w, A^2x_0) = \lim_{n\to\infty} d_i(AAx_{2n}, A^2x_0)$$

and we shall prove that there exists $\{T_i\}_{i\in I}(T_i \in \mathbb{R}^+,\ i \in I)$ so that $d_j(AAx_{2n}, AAx_0) \leqslant T_i$, for every $i \in I$ and every $j \in 0(i, f)$ $n \in \mathbb{N}$. Further, we have:

$$d_i(AAx_{2n}, AAx_{2n-1}) \leqslant q_i(d_{f(i)}(A^2x_{2n-1}, A^2x_{2n-2})) \times$$

$$\times q_{f(i)}(d_{f^2(i)}(A^2x_{2n-2}, A^2x_{2n-3}) \ldots q_{f^{2n-2}(i)}(d_{f^{2n-2}(i)}(A^2x_1, A^2x_0)) \times$$

$$\times d_{f^{2n-2}(i)}(A^2x_1, A^2x_0)$$

and since

$$d_j(A^2x_1, A^2x_0) \leqslant M_i, \text{ for every } i \in I,\ j \in 0(i, f)$$

it is easy to see that:

$$d_j(AAx_{2n}, AAx_{2n-1}) \leqslant q_j(M_i) \ldots q_{f^{2n-2}(j)}(M_i)M_i \qquad j \in 0(i, f)$$

and similarly:

$$d_j(A^2x_{2n-1}, A^2x_{2n-2}) \leqslant q_j(M_i) \ldots q_{f^{2n-3}(j)}(M_i)M_i.$$

So we have:

$$d_j(AAx_{2n}, AAx_0) \leqslant d_j(AAx_{2n}, AAx_{2n-1}) + \ldots + d_j(A^2x_1, A^2x_0) \leqslant$$

$$\leqslant M_i(1 + q_j(M_i) + \ldots + q_j(M_i) \ldots q_{f^{2n-2}(j)}(M_i))$$

and since $\overline{\lim\limits_{n\to\infty}}\, q_{f^n(i)}(M_i) < 1$ it follows that there exists $\{T_i\}_{i\in I}$ so that:

$$p_j(AAx_{2n}, AAx_0) \leqslant T_i, \text{ for every } n \in \mathbb{N},\ j \in 0(i, f),\ i \in I.$$

COROLLARY [9] *Let $(X, \{d_i\}_{i\in I})$ be a complete Hausdorff uniformizable space, A be a mapping from X in X satisfying the following conditions:*

a) *For each $i \in I$ there exists a nondecreasing function $q_i : \mathbb{R}^+ \to [0,1]$ so that:*

$$d_i(Ax, Ay) \leqslant q_i(d_{f(i)}(x, y))d_{f(i)}(x, y)$$

for every $x, y \in X$.

b) *For each $i \in I$ and $t \in \mathbb{R}^+$, $\overline{\lim\limits_{n\to\infty}}\, q_{f^n(i)}(t) < 1$.*

c) *There is $x_0 \in X$ such that for each $i \in I$:*

$$\sup_{n\in\mathbb{N}} d_{f^n(i)}(x_0, Ax_0) = K_i,\ K_i \in \mathbb{R}^+.$$

Then there exists a unique $x^* \in X$ such that $x^* = Ax^*$ and for each $i \in I$, $\sup\limits_{n\in\mathbb{N}} d_{f^n(i)}(x_0, x^*) = S_i,\ S_i \in \mathbb{R}^+$.

Proof: We shall prove that all the conditions of Theorem 1 are satisfied. The condition 1. follows from a) and b) if $S = T = Id$. Let us prove that the condition 2. is satisfied. Since:

$$d_{f^n(i)}(Ax_0, Ax_1) \leqslant q_{f^n(i)}(d_{f^{n+1}(i)}(x_0, x_1))d_{f^{n+1}(i)}(x_0, x_1)$$

and $x_1 = Ax_0$ it follows that $\sup\limits_{n\in\mathbb{N}} d_{f^n(i)}(Ax_0, Ax_1) \leqslant P_i,\ P_i \in \mathbb{R}^+$. Let us prove that c) implies that for every $i \in I$,

$$\text{(3)} \qquad \sup_{n\in\mathbb{N}} d_{f^n(i)}(A^jx_1, A^2x_0) \leqslant M_i,\ M_i \in \mathbb{R}^+(j \in \{2, 3\}).$$

Since for every $i \in I$ and every $n \in \mathbb{N}$:

$$d_{f^n(i)}(A^2x_1, A^2x_0) \leqslant d_{f^{n+1}(i)}(x_1, x_0)$$

it follows that (3) is satisfied for $i = 2$. Further, for every $i \in I$ and every $n \in \mathbf{N}$

$$d_{f^n(i)}(A^3x_1,\ A^2x_0) \leqslant d_{f^{n+2}(i)}(Ax_1,\ x_0) \leqslant d_{f^{n+2}(i)}(Ax_1,\ Ax_0) + $$
$$+\ d_{f^{n+2}(i)}(Ax_0,\ x_0) \leqslant d_{f^{n+3}(i)}(x_1,\ x_0) + d_{f^{n+2}(i)}(x_1,\ x_0)$$

and so (3) is satisfied for $i = 3$.
From Theorem 1. it follows that there exists one and only one element $w \in X$ such that $Aw = w$ and:

$$\text{(4)} \qquad \sup_{n \in \mathbf{N}} d_{f^n(i)}(w,\ A^2x_0) \leqslant N_i,\ N_i \in \mathbf{R}^+.$$

This implies that there exists one and only one element $w \in X$ such that $Aw = w$ and:

$$\text{(5)} \qquad \sup_{n \in \mathbf{N}} d_{f^n(i)}(w,\ x_0) = S_i,\ S_i \in \mathbf{R}^+$$

since (5) implies (4).

From Theorem 1 we obtain the Corollary on the existence of the common fixed point in probabilistic locally convex spaces.

DEFINITION [7] *Let X be a linear space, I a set and $\mathfrak{F}^i : X \to \Delta^+$ for every $i \in I$, where Δ^+ is a family of distribution functions such that $F(0)=0$ for every $F \in \Delta^+$. The triplet $(X, \{\mathfrak{F}^i\}_{i \in I}, t)$ is a probabilistic locally convex space if t is a T-norm and for every $i \in I$ the following conditions are satisfied ($\mathfrak{F}^i(x)$ is denoted by F^i_x):*

I. $F^i_x(u) = H(u)$, *for every $i \in I$ and $u \in \mathbf{R} \Leftrightarrow x = 0$.*

II. $F^i_{sx}(u) = F^i_x\left(\frac{u}{|s|}\right)$, *for every $i \in I$, $x \in X$ and $\neq\ s \in \mathfrak{K}$ ($\mathfrak{K}$ is the scalar field).*

III. *For every $i \in I$, every $u_1,\ u_2 \in \mathbf{R}$ and every $x, y \in X$*

$$F^i_{x+y}(u_1 + u_2) \geqslant t(F^i_x(u_1),\ F^i_y(u_2)).$$

In X is introduced the so called (ε, λ)-topology ($\varepsilon \in \mathbf{R}^+$, $\lambda \in (0, 1)$) in the following way:

The fundamental system of neighborhoods of zero in X is defined by $\{U^i_{\varepsilon,\lambda}\}_{\varepsilon>0,\ \lambda \in (0,1)}$ $i \in I$ where $U^i_{\varepsilon,\lambda} = \{x | F^i_x(\varepsilon) > 1 - \lambda\}$ and if T-norm t continuous X is, in the (ε, λ)-topology, a topological vector space which is Hausdorff.

COROLLARY 2. *Let $(X, \{\mathfrak{F}^i\}_{i \in I}, \min)$ be a complete probabilistic locally convex space, $S, T : X \to X$ be continuous mappings, $A : X \to SX \cap TX$ be continuous mapping which commutes with S and T and the following conditions are satisfied:*

A) *For every $i \in I$, there exists $q_i : \mathbf{R}^+ \to [0, 1]$ which is a nondecreasing function continuous from the right such that for every $i \in I$ and every*

$t \in \mathbf{R}^+$, $\overline{\lim}_{n \to \infty} q_{f^n(i)}(t) < 1$ and for every $i \in I$, every $x, y \in X$ and every $t \in \mathbf{R}^+$:

$$F^i_{Ax-Ay}(q_i(t)\,t) \geqslant F^{f(i)}_{Sx-Ty}(t)$$

B) *There exists x_0 and x_1 from X such that $Tx_1 = Ax_0$ and for every $i \in I$:*

$$\overline{\lim_{t \to \infty}}\ F^j_{Ax_0 - Ax_1}(t) = 1,\ \textit{uniformly in}\ j \in 0(i, f).$$

Then there exists $z \in X$ so that $Az = Sz = Tz$. If, in addition, for every $i \in I$:

$$\overline{\lim_{t \to \infty}}\ F^j_{A^sx_1 - A^2x_0}(t) = 1,\ \textit{uniformly in}\ j \in (i, f)\ \ (s \in \{2, 3\}),$$

then there exists one and only one element $w \in X$ such that $w = Aw = Sw = Tw$ and:

$$\text{(6)} \qquad \overline{\lim_{t \to \infty}}\ F^j_{w - A^2x_0}(t) = 1,\ \textit{for every}\ i \in I,\ \textit{uniformly in}\ j \in 0(i, f).$$

Proff: The space X becomes a Hausdorff uniformizable space with the family of pseudometrics $d_{\alpha,i}(x, y) = \sup\{t | F^i_{x-y}(t) \leqslant 1 - \alpha\}$ where $\alpha \in (0, 1)$, $i \in I$. Let $q_{(\alpha,i)} = q_i$, for every $\alpha \in (0, 1)$ and $i \in I$ and $f(\alpha, i) = (\alpha, f(i))$, for every $\alpha \in (0, 1)$ and $i \in I$. Let us prove that

$$\text{(7)} \qquad d_{(\alpha,i)}(Ax,\ Ay) \leqslant q_{(\alpha,i)}(d_{(\alpha,f(i))}(Sx,\ Ty))\,d_{(\alpha,f(i))}(Sx,\ Ty)$$

for every $x, y \in X$, every $i \in I$ and every $\alpha \in (0, 1)$. The proof is similar to the proof of Theorem 8 from [9]. Suppose that (7) is not satisfied which means that for some $\alpha \in (0, 1)$, $i \in I$ and some $x, y \in X$

$$d_{(\alpha,i)}(Ax,\ Ay) > q_{(\alpha,i)}(d_{(\alpha,f(i))}(Sx,\ Ty))\,d_{(\alpha,f(i))}(Sx,\ Ty).$$

Since the mapping $q_{(\alpha,i)}$ is continuous from the right there exists $t > d_{(\alpha,\ (i))}(Sx,\ Ty)$ such that:

$$\text{(8)} \qquad d_{(\alpha,i)}(Ax,\ Ay) < q_i(t)\,t$$

and so, since F^i_x is a distribution function, for every $x \in X$ (8) implies that:

$$\text{(9)} \qquad F^i_{Ax-Ay}(d_{(\alpha,i)}(Ax,\ Ay)) \geqslant F^i_{Ax-Ay}(q_i(t)t) \geqslant F^{f(i)}_{Sx-Ty}(t)$$

Since $t > d_{(\alpha,f(i))}(Sx,\ Ty)$ implies that $F^{f(i)}_{Sx-Ty}(t) > 1 - \alpha$ from (9) it follows that $F^i_{Ax-Ay}(d_{(\alpha,i)}(Ax,\ Ay)) > 1 - \alpha$ which is in the contradiction with the definition of the pseudometric $d_{(\alpha,i)}$. As in [8] it is easy to see that B) implies 2, from Theorem 1 and so all the conditions Theorem 1 are satisfied which implies that there exists one and only one element $w \in X$, $w = Aw = Sw = Tw$ such that $d_{(\alpha, f^n(i))}(w,\ A^2x_0) \leqslant N_i$, $N_i \in \mathbf{R}^+$, for every $i \in I$ and every $n \in \mathbb{N}$. Similary as in [8] it follows from this inequality that (6) is satisfied.

Now, we shall prove a generalization of the Krasnoseljski fixed point theorem.

In the next theorem we shall suppose that X is a locally convex topological vector space in which the topology is defined by the family of seminorms $\{p_i\}_{i \in I}$. Then X is uniformizable space, where $d_i(x, y) = p_i(x - y)$, for every $x, y \in X$ $(i \in I)$.

THEOREM 2. *Let $(X, \{p_i\}_{i \in I})$ be a complete Hausdorff locally convex topological vector space, $K = \overline{co}\, K$ be a subset of X, S and T be additive continuous mappings so that $SK \cap TK$ is bounded, $F: K \to X$ be compact mapping so that $AK + FK \subseteq SK \cap TK \subseteq K$ and $S|F(K) = T|F(K) = Id|K$. Suppose that the condition 1. in Theorem 1 is satisfied for every $x, y \in K$. For every $i \in I$ there exists $g(i) \in I$ so that:*

$$p_{f^n(i)}(x) \leqslant m_i p_{g(i)}(x), \tag{10}$$

for every $n \in \mathbb{N}$ and every $x \in U - U$ where $U = SK \cap TK$. Then there exists $x \in K$ so that

$$x = Ax + Fx = Sx = Tx. \tag{11}$$

Proof: It is easy to see that for every $u \in K$ the mapping A_u defined by $A_u x = Ax + Fu$ $(x, u \in K)$ satisfies all the conditions of Theorem 1 and so there exists $Ru \in SK \cap TK$, for every $u \in K$ so that $Ru = ARu + Fu = SRu = TRu$. We shall prove that the mapping R is continuous and RK is relatively compact which implies that there exists u^* so that $u^* = Ru^*$. For this u^* we then have that (11) holds for $x = u^*$. Let $u_1, u_2 \in K$. Then $Ru_1 = ARu_1 + Fu_1 = SRu_1 = TRu_1$ and $Ru_2 = ARu_2 + Fu_2 = SRu_2 = TRu_2$. So, for every $i \in I$ we have:

$$\begin{aligned} p_i(Ru_1 - Ru_2) &= p_i(ARu_1 - ARu_2) + p_i(Fu_1 - Fu_2) \leqslant q_i(p_{f(i)})(SRu_1 - \\ &- TRu_2)) \times p_{f(i)}(SRu_1 - TRu_2) + p_i(Fu_1 - Fu_2) = q_i(p_{f(i)}(Ru_1 - \\ &- Ru_2))p_{f(i)}(Ru_1 - Ru_2) + p_i(Fu_1 - Fu_2) \leqslant q_i(p_{f(i)}(Ru_1 - \\ &- Ru_2))[q_{f(i)}(p_{f^2(i)}(Ru_1 - Ru_2)) \times p_{f^2(i)}(Ru_1 - Ru_2) + p_{f(i)}(Fu_1 - \\ &- Fu_2)] + p_i(Fu_1 - Fu_2) = q_i(p_{f(i)}(Ru_1 - Ru_2))q_{f(i)}(p_{f^2(i)}(Ru_1 - \\ &- Ru_2))p_{f^2(i)}(Ru_1 - Ru_2) + q_i(p_{f(i)}(Ru_1 - Ru_2))p_{f(i)}(Fu_1 - Fu_2) + \\ &+ p_i(Fu_1 - Fu_2) \leqslant \ldots \leqslant p_i(Fu_1 - Fu_2) + \ldots + q_i(p_{f(i)}(Ru_1 - \\ &- Ru_2))q_{f(i)}(p_{f^2(i)}(Ru_1 - Ru_2)) \ldots \times q_{f^n(i)}(p_{f^{n+1}(i)}(Ru_1 - \\ &- Ru_2))p_{f^{n+1}(i)}(Fu_1 - Fu_2) + q_i(p_{f(i)})(Ru_1 - Ru_2))q_{f(i)}(p_{f^2(i)}(Ru_1 - \\ &- Ru_2)) \ldots q_{f^{n+1}(i)}p_{f^{n+2}(i)}(Ru_1 - Ru_2)p_{f^{n+2}(if)}(Ru_1 - Ru_2). \end{aligned}$$

Further, from (10) it follows that for every $n \in \mathbb{N}$, $i \in I$, $p_{f^n(i)}(Ru_1 - Ru_2) \leqslant m_i p_{g(i)}(Ru_1 - Ru_2)$ and since $SK \cap TK$ is bounded and $Ru \in SK \cap TK$, for every $u \in K$ it follows that:

$$m_i p_{g(i)}(Ru_1 - Ru_2) \leqslant B_i, \text{ for every } u_1, u_2 \in K$$

which implies that $p_{f^n(i)}(Ru_1 - Ru_2) \leqslant B_i$, for every $i \in I$ and for every $n \in N$.

Now, we have:

$$p_i(Ru_1 - Ru_2) \leqslant p_i(Fu_1 - Fu_2) + \ldots + q_i(B_i)q_{f(i)}(B_i) \ldots q_{f^n(i)}(B_i)p_{f^{n+1}(i)}$$
$$(Fu_1 - Fu_2) + q_i(B_i)q_{f(i)}(B_i) \ldots q_{f^{n+1}(i)}(B_i)B_i$$

and since (10) holds for every $y \in FK - FK = Ax + FK - (Ax + FK) \in U - U (x \in K)$ it follows that:

$$p_i(Ru_1 - Ru_2) \leqslant m_i p_{g(i)}(Fu_1 - Fu_2)[1 + q_i(B_i) + q_i(B_i)q_{f(i)}(B_i) + \ldots +$$
$$+ q_i(B_i)q_{f(i)}(B_i) + \ldots q_{f^n(i)}(B_i) + \ldots]$$

since $\overline{\lim\limits_{n \to \infty}}\, q_{f^n(i)}(B_i) < 1$, and so it is obvious that there exists $M_i \in \mathbb{R}^+$ such that:

$$p_i(Ru_1 - Ru_2) \leqslant M_i\, p_{g(i)}(Fu_1 - Fu_2), \tag{12}$$

for every $u_1, u_2 \in K$ and for every $i \in I$. Since F is continuous it follows that R is continuous. Since $\overline{FK}$ is compact similarly as in [6] from (12) it follows that $\overline{RK}$ is compact and so R on K satisfies all the conditions of Tihonov's fixed point theorem which implies the existence of an element $x \in K$ such that $Rx = x$ and so $x = Ax + Fx = Sx = Tx$.

Remark: It is easy to see that in Theorem 1. and 2. we can also suppose that for every $i \in I$ the mapping q_i is a bounded function such that $\overline{\lim\limits_{n \to \infty}}\, q_{f^n(i)}(t) \leqslant Q_i < 1$, for any $t \in \mathbb{R}^+$, $i \in I$.

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