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ON THE DENSE DIVERGENCE OF LAGRANGE INTERPOLATION IN A COMPLEX DOMAIN

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1. In this paper we present some results referring to the divergence of Lagrange interpolation methods in a domain of complex plane as an addition at the results obtained by S. Y. A'LPER [1] D. L. BERMAN [2], A. H. GERMAN [4] and P. VÉRTESI [7] and as an extension of the theorems for Lagrange interpolation on a real interval, that are already known from the works of I. MUNTEAN and S. COBZAŞ [3], [6].

2. In the complex plane $\mathbb{C}$ we consider a bounded domain D with simple connected complement and rectificable boundary. Denote by $A(D)$ the space of all continuous functions $x:\bar{D}\to\mathbb{C}$ which are analytical on D; endowed with the uniform norm

$$||x|| = \max\{|x(t)| : t\in\bar{D}\},\quad x\in A(D)$$

$A(D)$ is a Banach space. We use the notations $D_1=\{t\in\mathbb{C} : |t|<1\}$ and $\Gamma=\{t\in\mathbb{C} : |t|=1\}$.

Let M be an infinite triangular matrix:

$$M=\{t_n=(t_n^1,\ t_n^2,\ \ldots,\ t_n^n)\in\mathbb{C}^n :\ n\in\mathbb{N}\} \tag{1}$$

where $t_n^i\in\bar{D}$, $i=1,2,\ldots,n$, $n\in\mathbb{N}$ and $t_n^i\neq t_n^j$ for $i\neq j$. We associate with any line t_n of this matrix the *Lagrange interpolating operator*, L_n : $A(D)\to A(D)$ defined by

$$L_n(x;\ t)=\sum_{k=1}^{n}\frac{(t-t_n^1)\ldots(t-t_n^{k-1})(t-t_n^{k+1})\ldots(t-t_n^n)}{(t_n^k-t_n^1)\ldots(t_n^k-t_n^{k-1})(t_n^k-t_n^{k+1})\ldots(t_n^k-t_n^n)}\,x(t_n^k)$$

for $x\in A(D)$, $t\in\bar{D}$.

As in the real case, we can study the uniform or pointwise convergence of the sequence $(L_n(x\,;\ .))_{n\in\mathbf{N}}$ to $x \in A(D)$.

S. Y. AL'PER [1] proved that if the nodes of the matrix M satisfy:

(2) $$|t_n^i| = 1,\ i = 1, 2, \ldots, n \text{ and } n \in \mathbf{N},$$

then there exists $x_0 \in A(D_1)$ such that the sequence $(L_n(x_0,\,;\ .))_{n\in\mathbf{N}}$ does not converge uniformly to x_0. An extension of this result is the following theorem obtained by D. L. BERMAN [2]:

THEOREM 2.1. *For every matrix of type* (1), *there exists* $x_0 \in A(D)$ *such that*

$$\overline{\lim_{n\to\infty}}\ \|L_n(x_0\,;\ .) - x_0\| = +\infty$$

A. H. GERMAN [4] studies the pointwise convergence of the sequence $(L_n(x\,;\ .))_{n\in\mathbf{N}}$ to $x \in A(D_1)$ when the nodes of M satisfy (2) and proves that for a special class of such matrices there exists $x_0 \in A(D_1)$ such that

$$\overline{\lim_{n\to\infty}}\ |L_n(x_0\,;\ t)| = +\infty \quad \text{a.e. on } \Gamma,.$$

This result is completed by P. VÉRTESI [7] as follows:

THEOREM 2.2. *For every matrix of type* (1) *which satisfies* (2), *there exists* $x_0 \in A(D_1)$ *such that*

$$\overline{\lim_{n\to\infty}}\ |L_n(x_0\,;\ t)| = +\infty \text{ a.e. on } \Gamma.$$

3. In the following we determine the topological structure of the set of all functions $x \in A(D)$ for which the sequence $(L_n(x\,;\ .))_{n\in\mathbf{N}}$ does not converge uniformly or pointwise to x.

To this end we need some preliminary results. We recall that a subset of a topological space X is said to be *superdense* if it is noncountable, dense and of G_δ-type (see [3], [5]) and it is said to be *of first Baire category* if it can be written as the union of a countable family of nowhere dense sets in X.

From [5, Th. 2.4] we deduce:

LEMMA 3.1. *If X is a nonzero Banach space and $\mathcal{A}$ is a family of continuous mappings $A: X \to X$ satisfying the following conditions:*

a) $\|A(x + y)\| \leqslant \|A(x)\| + \|A(y)\|$ *and* $\|A(x)\| = \|A(-x)\|$ *for each* $A \in \mathcal{A}$ *and* $x, y \in X$;

b) *there exists* $x_0 \in X$ *such that* $\sup\{\|A(x_0)\| : A \in \mathcal{A}\} = +\infty$, *then the set*

$$X_1 = \{x \in X : \sup\{\|A(x)\| : A \in \mathcal{A}\} = +\infty\}$$

is superdense in X.

From [5, Th. 2.3] we have:

LEMMA 3.2. *Let X be a nonzero Banach space, T a complete, separable,*

nonvoid metric space, without isolated points and let $\mathcal{A}$ be a family of mappings $A: X \times T \to \mathbf{C}$ satisfying the following conditions:

a) $A(.\,;\ t): X \to \mathbf{C}$ *is continuous,* $|A(x + y\,;\ t)| \leqslant |A(x\,;\ t)| + |A(y\,;\ t)|$ *and* $|A(x\,;\ t)| = |A(-x\,;\ t)|$ *for each* $A \in \mathcal{A}$, $t \in T$ *and* $x, y \in X$;

b) *there exists a dense subset T_0 of T and $x_0 \in X$ such that* $\sup\{|A(x_0\,;\ t)| : A \in \mathcal{A}\} = +\infty$ *for each* $t \in T_0$.

Then there exists a superdense subset X_2 of X such that the set

$$\{t \in T : \sup\{|A(x\,;\ t)| : A \in \mathcal{A}\} = +\infty\}$$

is superdense in T for every $x \in X_2$.

The following result is a completion to the theorem 2.1:

THEOREM 3.3. *For every matrix M of type* (1) *the set*

$$X_1 = \{x \in A(D) : \sup\{\|L_n(x\,;\ .) - x\| : n \in \mathbf{N}\} = +\infty\}$$

is superdense in $A(D)$.

Proof. For each $n \in \mathbf{N}$ define $A_n : A(D) \to A(D)$ by $A_n(x)(t) = L_n(x\,;\ t)$, $x \in A(D)$, $t \in \bar{D}$. We can apply lemma 3.1 with $X = A(D)$. $\mathcal{A} = \{A_n - I : n \in \mathbf{N}\}$ where I is the identity mapping on $A(D)$ and x_0 is found by theorem 2.1. It follows that the set

$$X_1 = \{x \in A(D) : \sup\{\|(A_n - I)(x)\| : n \in \mathbf{N}\} = +\infty\} =$$
$$= \{x \in A(D) : \sup\{\|L_n(x\,;\ .) - x\| : n \in \mathbf{N}\} = +\infty\}$$

is superdense in $A(D)$.

Concerning the theorem 2.2 we obtain:

THEOREM 3.4. *If the nodes of matrix M of type* (1) *satisfy* (2) *then there exists a superdense subset X_2 of $A(D_1)$ such that the set*

$$U = \{t \in \Gamma : \sup\{|L_n(x\,;\ t)| : n \in \mathbf{N}\} = +\infty\}$$

is superdense in Γ for every $x \in X_2$.

Proof. We apply lemma 3.2 taking $X = A(D_1)$, $T = \Gamma$ with the topology induced by the metric

$$\rho(t_1, t_2) = |t_1 - t_2|$$

for $t_1, t_2 \in \Gamma$ and $\mathcal{A} = \{L_n(.\,;\ .) : n \in \mathbf{N}\}$. Then we apply the theorem 2.2; it follows that there exists a superdense subset X_2 of X such that U is a superdense set in Γ for every $x \in X_2$.

4. *Remarks.* (i) For every matrix M of type (1) the set of all functions $x \in A(D)$ for which $L_n(x\,;\ .) \to x(n \to +\infty)$ is of first Baire category in $A(D)$.

Indeed, if T is a topological space and S is a subset of T with $S \subset T \setminus S'$ where $S' \subset T$ is of G_δ-type and dense in T, then S is of first

Baire category in T. Now, from

$$\{x \in A(D) : L_n(x\,;\ .) \to x,\ n \to +\infty\} \subset A(D) \setminus \{x \in A(D) : \sup\{||L_n(x\,;\ .) - x|| : n \in \mathbf{N}\} = +\infty\}$$

and by theorem **3.3** we obtain (i).

(ii) For every matrix M of type (1) checking (2), the set of all functions $x \in A(D_1)$ for which $L_n(x\,;\ t) \to x(t)$ $(n \to +\infty)$ for any $t \in \bar{D}_1$, is of first Baire category in $A(D_1)$.

To prove this, observe that:

$$\{x \in A(D_1) : L_n(x\,;\ t) \to x(t),\ n \to +\infty \text{ for any } t \in D_1\} \subset$$

$$\subset \{x \in A(D_1) : L_n(x\,;\ t) \to x(t),\ n \to +\infty \text{ for any } t \in \Gamma\} \subset$$

$$\subset A(D_1) \setminus X_2$$

where X_2 is given by theorem 3.4; after that we proceed as in (i).

(iii) The sets X_1 and X_2 in the theorems 3.3 and 3.4 are *of second Baire cateogory* (they are not of first Baire category).

Indeed, the complements of X_1 and X_2 are of first Baire category in $A(D)$, respectively in $A(D_1)$. Since in a Banach space the complement of a set of first Baire category is of second Baire category, we obtain (iii).

By the same argument we have:

(iv) The set U in the theorem 3.4 is of second Baire category.

This result is obtained in [7] in some other way.

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