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AN IMPROVEMENT FOR THE AREA OF CONVERGENCE OF THE ACCELERATED OVERRELAXATION ITERATIVE METHOD

by

M. MADALENA MARTINS
(Coimbra)

1. Introduction

After the introduction of the Accelerated Overrelaxation Iterative Method (ADR), by Hadjidimos, in 1979

$$x^{(i+1)} = (I - rE)^{-1}[(1 - w)I + (w - r)E + wF]x^{(i)} + w(I - rE)^{-1}b \tag{1.1}$$
$$i = 0,\ 1,\ 2,\ \ldots$$

many improvements on the corresponding convergence have appeared. As, for different values of the parameters w, r, this method includes other known methods (Jacobi method for $r = 0$, $w = 1$; Gauss-Seidel method for $r = w = 1$, Successive Overrelaxation method (SOR) for $r = w$ and Simultaneous Overrelaxation (JOR) for $r = 0$) it is of large use for computing the solution of a linear system

$$Ax = b \tag{1.2}$$

Here, $A = I - E - F$ is a real $n \times n$ matrix, b a real, known, n column vector and x the unknown n column vector.

Papers [1], [2], [3] give some results on the convergence of this iterative method. Such results have been improved for strictly diagonally dominant matrices in [5], [6].

Later, we took the idea of generalized diagonal dominance (see definition 3 of [4]) and improved the last results for various types of matrices (see [7]). In this paper we generalize the theorem 2 of [5]: "If A of

(1.2) is a strictly diagonally dominant matrix and $w > r > 0$, then a sufficient condition for the convergence of the (AOR) method is

$$0 < w < \frac{2}{1 + \max_i (e_i + f_i)}.$$

With this generalization we improve the results of [5], [6] and [7].

2. Convergence's Conditions of the AOR method

THEOREM 1. — *If A of (1.2) is a strictly diagonally dominant $n \times n$ matrix and $w < r$, then the* AOR *method is convergent, for:*

(2.1) (*i*) $0 < r < q$ *and* $f(r) < w < 1$

or

(2.2) (*ii*) $1 < w < m$ *and* $w < r < s(w)$,

where

$$q = \min_i \frac{1 + e_i - f_i}{2e_i}, \quad f(r) = \max_i \frac{2e_i r}{1 + e_i - f_i},$$

$$m = \min_i \frac{2 - 2e_i}{1 - e_i + f_i} \quad \text{and} \quad s(w) = \min_i \frac{2 - w(1 - e_i + f_i)}{2e_i}$$

where e_i and f_i are respectively the i-row sums of the moduli of the entries of E and F, respectively

Proof. Bearing in mind Theo. 1 of [5] and considering $w < r$, we can define the function:

$$g(\lambda) = (r - \lambda)e_i + \lambda f_i + |1 - \lambda| + re_i$$

If $0 < \lambda < 1$, $g(\lambda)$ is a decreasing function and $g(0) = 2re_i + 1 > 1$ with $g(\lambda) < 1$ if $\lambda > \frac{2e_i r}{1 + e_i - f_i}$. As $\lambda \leqslant 1$ we see that $\frac{2e_i r}{1 + e_i - f_i} < 1$ or equivalently $r < \frac{1 + e_i - f_i}{2e_i}$.

For $\lambda > 1$, we have $g(\lambda) = (r - \lambda)e_i + \lambda f_i + \lambda - 1 + re_i$. Now, $g(\lambda)$ is an increasing function and $g(\lambda) < 1$ if $r < \frac{1}{e_i} - \frac{\lambda[1 - e_i + f_i]}{2e_i}$.

As $r > \lambda > 1$, we must have $\frac{1}{e_i} - \frac{\lambda(1 - e_i + f_i)}{2e_i} > 1$ or $\lambda < \frac{2 - 2e_i}{1 - e_i + f_i}$.

With this conditions we conclude that the AOR method is convergent for w and r given by (2.1) and (2.2).

THEOREM 2. *If A of* (1.2) *is a strictly diagonally dominant matrix, then the* AOR *method is convergent, $\rho(L_{r,w}) < 1$, for:*

(*i*) $0 < r < q$ *and* $f(r) < w < 1$

or

(*ii*) $1 < w < m$ *and* $w < r < s(w)$

if $w < r$

or

(*iii*) $0 < r < w$ *and* $0 < w < t$

with

$$t = \frac{2}{1 + \max_i (e_i + f_i)}$$

Proof. This result comes immediately from the preceding theorem and from theorem 2 of [5], and is a generalization of this one.

We give a geometric intepretation of th. 2 of [5] (fig. 1) and the theorem 2 (fig. 2). We can see that the area of convergence given by (fig. 2) is larger than that which is given by fig. 1.

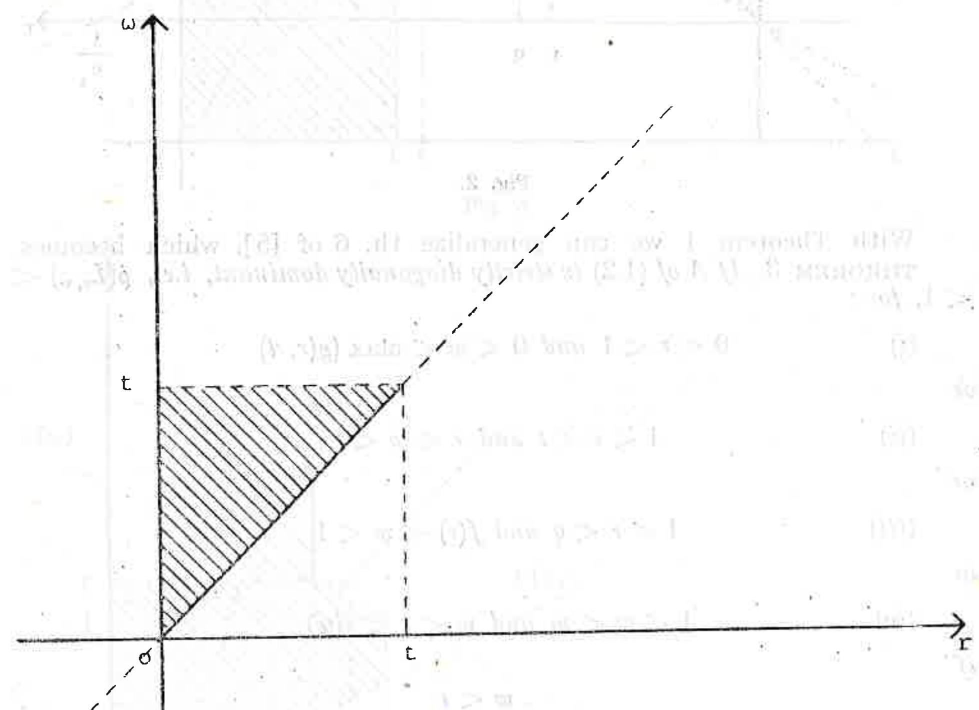

Fig. 1.

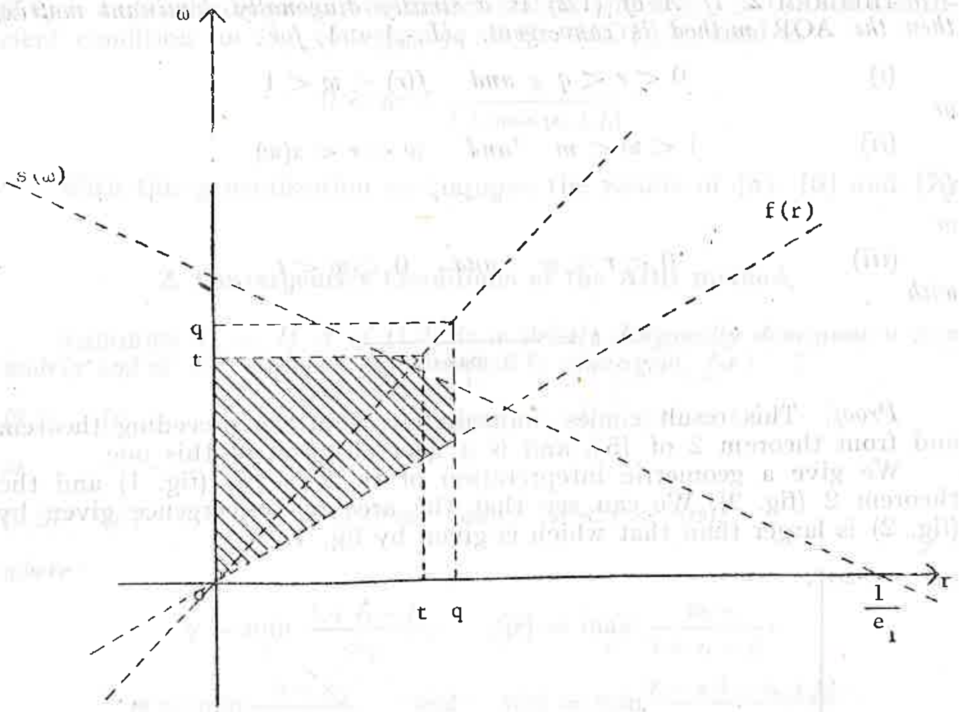

Fig. 2.

With Theorem 1 we can generalize th. 6 of [5], which becomes.

THEOREM 3. *If A of* (1.2) *is strictly diagonally dominant, i.e.,* $\rho(L_{r,w}) < 1$, *for :*

(*i*) $0 < r < 1$ *and* $0 < w < \max(g(r,\ t)$

or

(*ii*) $1 < r < t$ *and* $r < w < t$

or

(*iii*) $1 < r < q$ *and* $f(r) < w < 1$

or

(*iv*) $1 < w < m$ *and* $w < r < s(w)$

if

$$w < r$$

with $q(r) = \dfrac{2}{1 + \rho(L_{r,r})}$.

It is evidente that this result is an improvement on th. 6 of [5], as we can see from the fig. 4. for the theorem 3 and from fig. 3 for the th. 6 of [5].

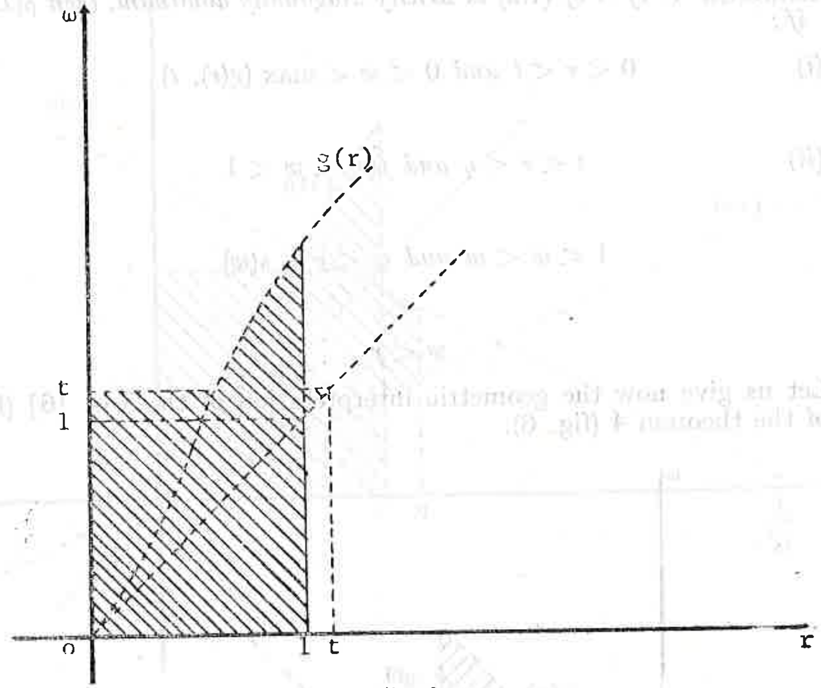

Fig. 3.

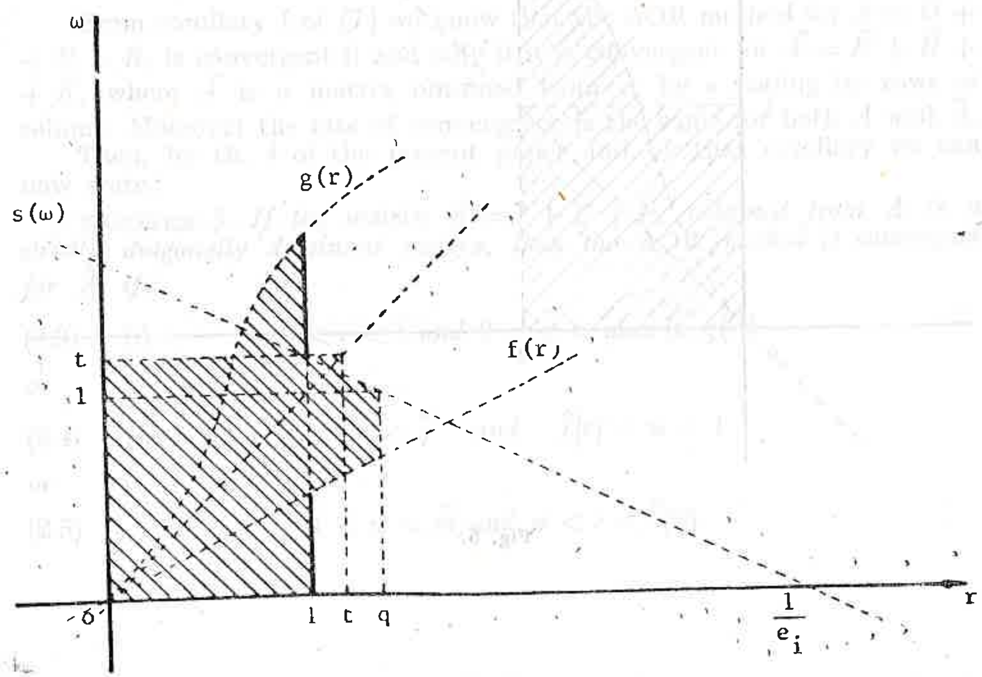

Fig. 4.

The Theorem 6 of [5] as improved by theorem 5 of [6], which can be stated now, in a generalized form.

THEOREM 4. *If A of* (1.2) *is strictly diagonally dominant, then* $\rho(L_{r,w}) < 1$, *if:*

$$(i) \qquad 0 < r < t \text{ and } 0 < w < \max(g(r),\ t)$$

or

$$(ii) \qquad t < r < q \text{ and } f(r) < w < 1$$

or

$$1 < w < m \text{ and } w < r < s(w)$$

if

$$w < r.$$

Let us give now the geometric interpretation of th. 5 of [6] (fig. 5) and of the theorem 4 (fig. 6).

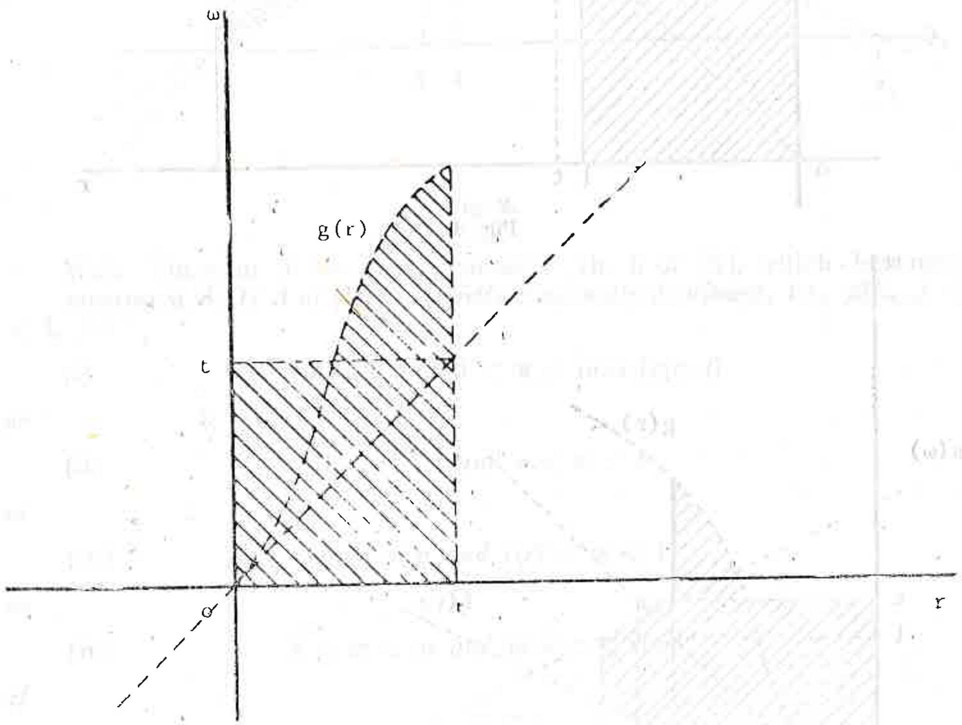

Fig. 5.

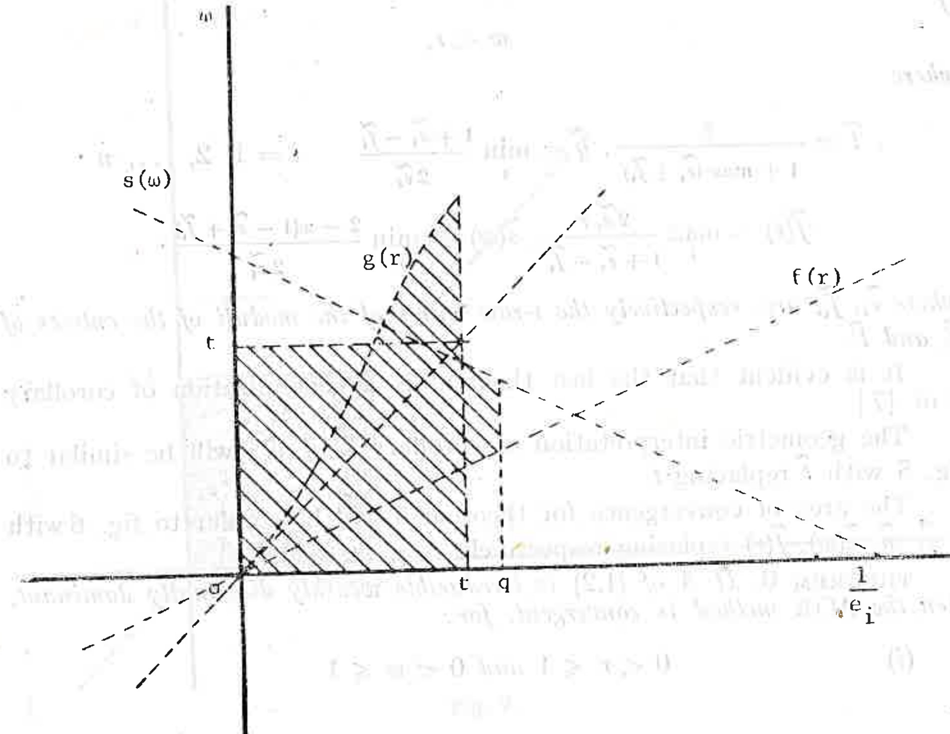

Fig. 6.

From corollary 1 of [7] we know that the AOR method for $A = D + H + K$, is convergent if and only if it is convergent for $\tilde{A} = \tilde{D} + \tilde{H} + \tilde{K}$, where $\tilde{A}$ is a matrix obtained from A by a scaling by rows or colums. Moreover the rate of convergence is the same for both A and $\tilde{A}$.

Then, by th. 4 of the present paper and by that corollary we can now state:

THEOREM 5. *If the matrix $\tilde{A} = I + \tilde{E} + \tilde{F}$, obtained from A is a strictly diagonally dominant matrix, then the* AOR *method is convergent for $\tilde{A}$, if:*

(2.3) $\quad (i) \qquad 0 < r < \tilde{t}$ *and* $0 < w < \max(\tilde{t},\ g(r))$

or

(2.4) $\quad (ii) \qquad \tilde{t} \leqslant r < \tilde{q}$ *and* $\tilde{f}(r) < w < 1$

or

(2.5) $\qquad 1 < w < \tilde{m}$ *and* $w < r < \tilde{s}(w)$

if

$$w < r,$$

where

$$\tilde{t} = \frac{2}{1 + \max (\tilde{e}_i + \tilde{f}_i)}, \quad \tilde{q} = \min_i \frac{1 + \tilde{e}_i - \tilde{f}_i}{2 \tilde{e}_i} \qquad i = 1, 2, \ldots, n$$

$$\tilde{f}(r) = \max_i \frac{2 \tilde{e}_i r}{1 + \tilde{e}_i - \tilde{f}_i}, \quad \tilde{s}(w) = \min_i \frac{2 - w(1 - \tilde{e}_i + \tilde{f}_i)}{2 \tilde{e}_i},$$

where $\tilde{e}_i$, $\tilde{f}_i$, *are, respectively the i-row sums of the moduli of the entries of* $\tilde{E}$ *and* $\tilde{F}$.

It is evident that the last theorem is a generalization of corollary 2 of [7].

The geometric interpretation of corollary 2 of [7] will be similar to fig. 5 with $\tilde{t}$ replacing t.

The area of convergence for theorem 5 will be similar to fig. 6 with $\tilde{t}$, $\tilde{q}$, $\tilde{m}$, $\tilde{s}(w)$, $\tilde{f}(r)$ replacing respectively t, q, m, $s(w)$, $f(r)$.

THEOREM 6. *If A of* (1.2) *is irreducible wearkly diagonally dominant, then the* AOR *method is convergent, for:*

(*i*) $$0 < r \leqslant 1 \text{ and } 0 < w \leqslant 1$$

or

(*ii*) $$1 < r \leqslant q \text{ and } f(r) \leqslant w \leqslant 1$$

or

$$1 < w < m \text{ and } w < r < s(w)$$

if

$$w < r.$$

Proof. This result comes from th. 1 and Corollary 1 of [6] and from last th. 1, applied to this type of matrices.

If we consider the geometrical meaning of the Corollary of [6] (Fig. 7) we see that its area of convergence is larger than that which is given by theo. 6 (Fig. 8).

THEOREM 7. *If A of* (1.2) *is an irreducible weakly diagonally dominant matrix, then the* AOR *method is convergent for w and r given by* (2.3), (2.4) *and* (2.5).

Proof. This result is obtained from the theorem 6 and from the considerations of Walter [10] about irreducible weakly diagonally dominant matrices.

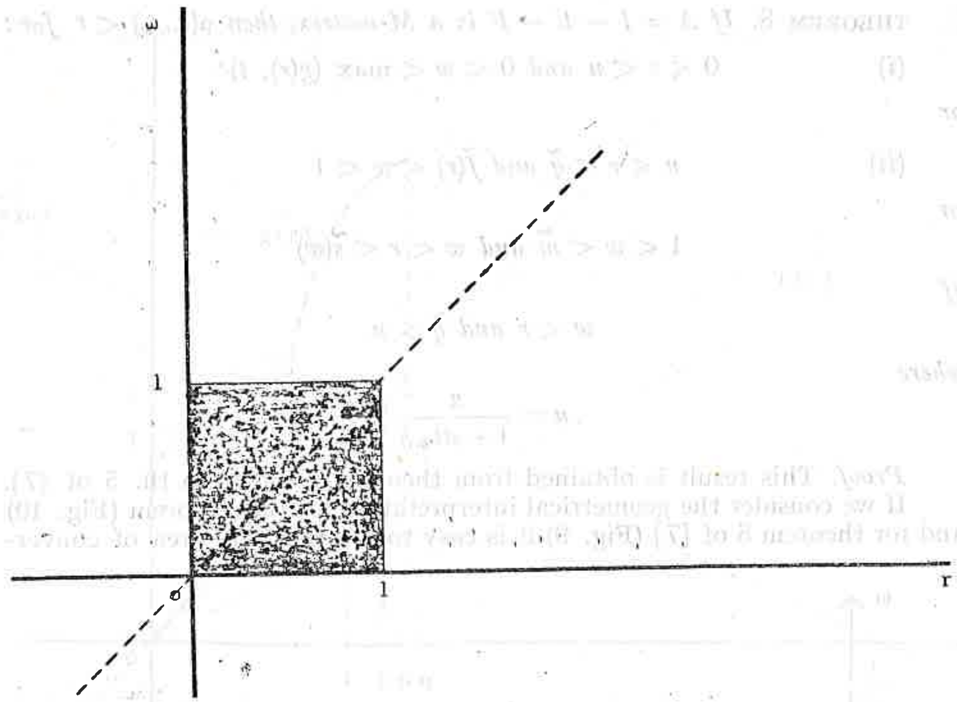

Fig. 7.

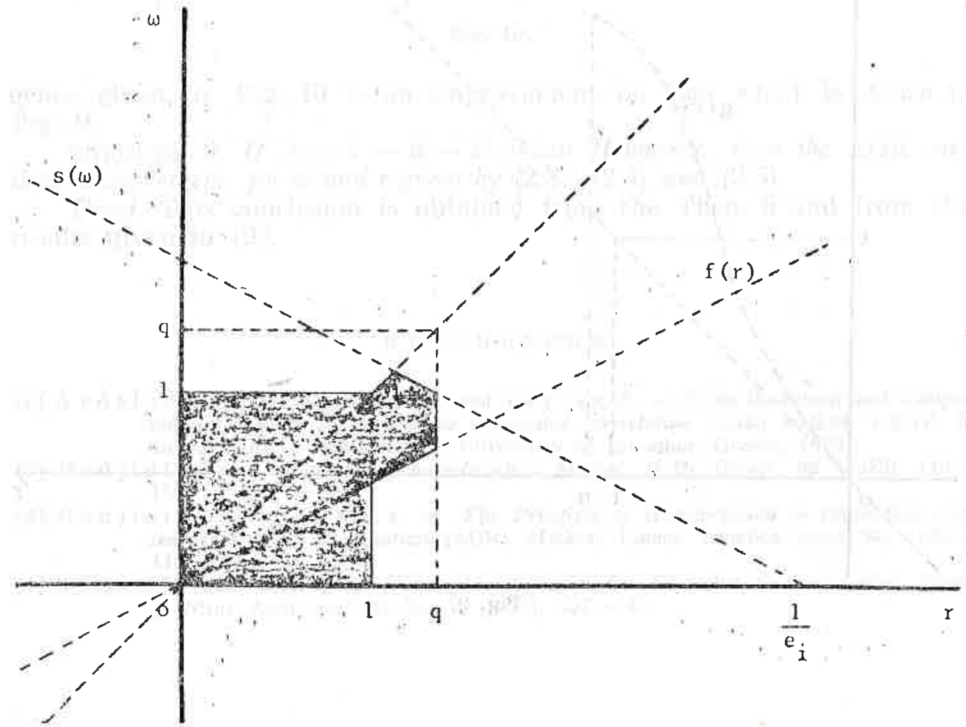

Fig. 8.

THEOREM 8. *If* $A = I - E - F$ *is a M-matrix, then* $\rho(L_{r,w}) < r$ *for*:

(*i*) $0 \leqslant r < n$ *and* $0 < w < \max\ (g(r),\ t)$

or

(*ii*) $n \leqslant r < \tilde{q}$ *and* $\tilde{f}(r) < w < 1$

or

$1 < w < \tilde{m}$ *and* $w < r < \tilde{s}(w)$

if

$w < r$ *and* $\tilde{q} > n$.

where

$$n = \frac{2}{1 + \rho(L_{0,1})}$$

Proof. This result is obtained from theorem 5 and from th. 5 of [7]. If we consider the geometrical interpretation for this theorem (Fig. 10) and for theorem 5 of [7] (Fig. 9) it is easy to see that the area of conver-

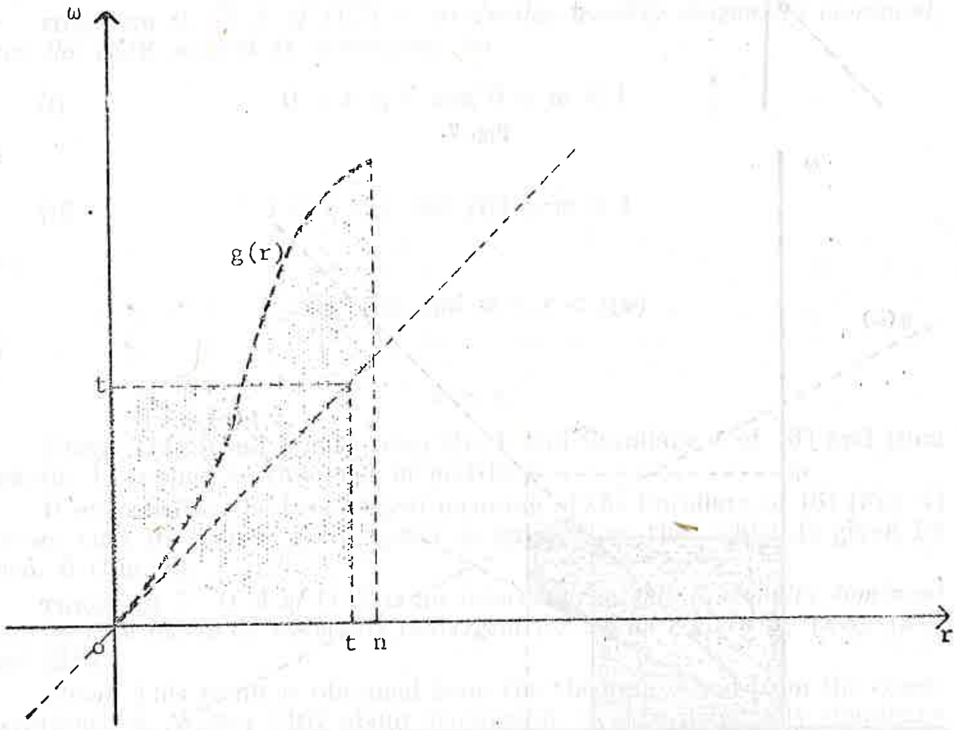

Fig. 9.

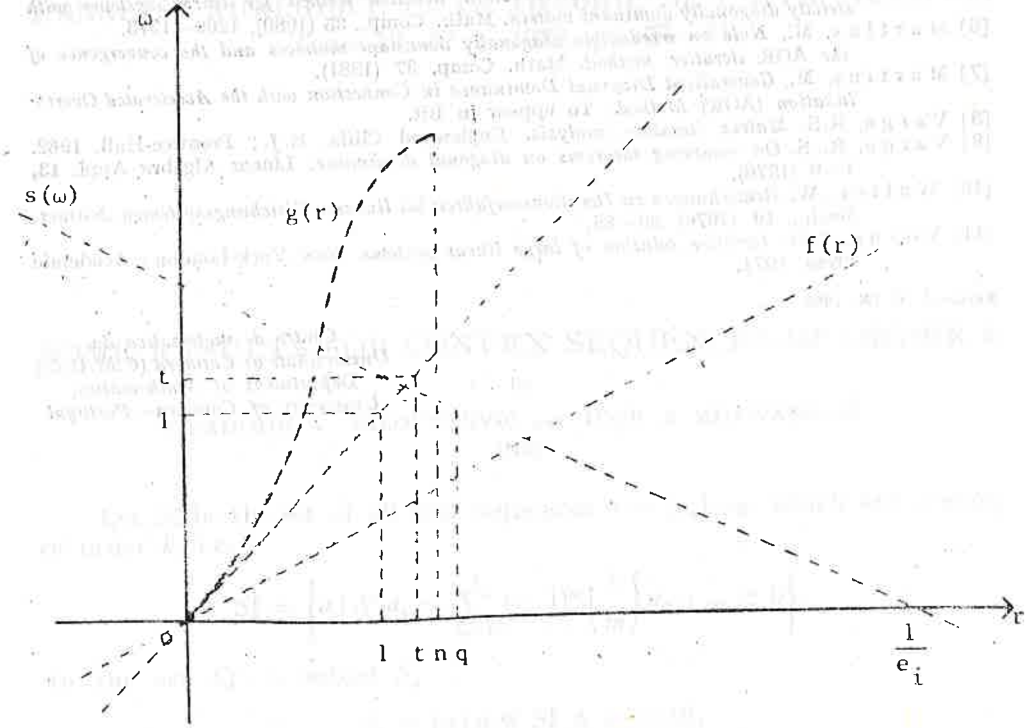

Fig. 10.

gence given by Fig. 10 is an improvement on that which is shewn in Fig. 9.

THEOREM 9. *If* $A = I - E - F$ *is an H-matrix, then the* AOR *method is convergent for w and r given by* (2.3), (2.4) *and* (2.5).

Proof. This conclusion is obtained from the Theo. 5 and from the results given in [9].

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Centro de matematica da Universidate de Coimbra (*C.M.U.C.*)
Department of Mathematics,
University of Coimbra — Portugal