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A PRINCIPLE OF CONDENSATION OF SINGULARITIES FOR SET-VALUED FUNCTIONS

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1. Introduction. Many books on functional analysis consider the following theorem, called *the principle of uniform boundedness,* as one of the most important results in the theory of real or complex normed linear spaces.

THEOREM 1.1. *Let X be a Banach space, Y a normed linear space, and* $\mathfrak{F}$ *a family of continuous linear mappings from X to Y such that*

$$\sup \{\|f(x)\| : f \in \mathfrak{F}\} < +\infty \text{ for all } x \in X.$$

Then holds

$$\sup \{\|f\| : f \in \mathfrak{F}\} < +\infty.$$

But an application of this theorem to the problem of the pointwise convergence of sequences of continuous linear mappings already reveals a disadvantage. Indeed, if $(f_n)_{n \in N}$ is a sequence of continuous linear mappings from the Banach space X to the normed linear space Y such that

$$\sup \{\|f_n\| : n \in N\} = +\infty,$$

then Theorem 1.1 implies the existence of at least one point $x \in X$ for which

$$\sup \{\|f_n(x)\| : n \in N\} = +\infty.$$

In other words, by applying the principle of uniform boundedness we can conclude only that there exists at least one point in X at which the sequence $(f_n)_{n \in N}$ diverges. By far more informations about the set of those points of X at which $(f_n)_{n \in N}$ diverges are obtained, if one applies the following theorem instead of the uniform boundedness principle.

THEOREM 1.2. *Let X be a Banach space, Y a normed linear space, and $\mathcal{F}$ a family of continuous linear mappings from X to Y with*

$$\sup \{\|f\| : f \in \mathcal{F}\} = +\infty.$$

Then the set $S_{\mathcal{F}}$ of all $x \in X$ for which

$$\sup \{\|f(x)\| : f \in \mathcal{F}\} = +\infty$$

is residual.

This classical result, which generalizes Theorem 1.1, is known as *the principle of condensation of singularities.* It has been found to be very useful in proving the dense divergence of many well-known approximation methods.

Both the above-mentioned theorems are based on the same result, namely Baire's Category Theorem (see for their proofs [10, pp. 134—136]). They have been generalized by numerous authors and in several directions. In some papers, instead of the normed linear spaces X and Y, more general linear spaces have been considered (for instance topological linear spaces, barrelled spaces, ultrabarrelled spaces). In other papers the family $\mathcal{F}$ of continuous linear mappings is replaced by a family of nonlinear functions of a certain kind (for instance convex functions, rationally s-convex functions, preconvex functions). For detailed information we refer the reader to the original papers by COBZAŞ Ş. and MUNTEAN I. [7], KOSMOL P. [12], BRECKNER W. W. [4], KOLUMBÁN I. [11].

Unlike the previously mentioned investigations, the main aim of the present paper is to state a principle of condensation of singularities of a new type, which does not require any algebraic structure of the considered spaces and no assumptions as to the shape of the functions that are concerned. The principle we shall prove is valid in the very general setting of topological spaces, more precisely for lower semicontinuous functions defined on a topological space and taking values in the power set of a topological space. By particularizing it yields several known principles of condensation of singularities stated until now by using individual methods, among them also Theorem 1.2.

2. Residual Sets. In this section we shall briefly review some notions from general topology and results concerning residual sets which we shall need. The terminology used is that of [8].

Let X be a topological space. We assume that X as well as all topological spaces which will occur in our paper are nonempty. For any subset A of X we denote by int A its interior and by cl A its closure.

A subset of X is said to be *rare* (or *nowhere dense*) if its closure has an empty interior. If A is a dense open subset of X, then it is easily seen that $X \setminus A$ is rare.

The union of any finite family of rare sets is rare, but the union of a countable family of rare sets is not always rare.

A subset of X is said to be of *the first category* (or *meagre*) if it can be written as the union of a countable family of rare subsets of X. Any

subset of X which is not of the first category is said to be of *the second category* (or *nonmeagre*). A subset of X is called *residual* if its complement is of the first category.

A characterization of residual sets is given by the following theorem.

THEOREM 2.1. *A subset A of a topological space X is residual if and only if there exists a nonempty countable family $\{G_n : n \in N\}$ of dense open subsets of X such that*

$$\bigcap_{n \in N} G_n \subseteq A. \tag{2.1}$$

If each residual subset of the topological space X is dense in X, then X is said to be a *Baire space.* A number of useful characterizations of Baire spaces are collected in the next theorem.

THEOREM 2.2. *The following properties of a topological space X are equivalent:*

1° *X is a Baire space.*

2° *The intersection of every nonempty countable family of dense open subsets of X is dense in X.*

3° *Every nonempty open subset of X is of the second category.*

4° *Each subset of X which is of the first category has an empty interior.*

Theorem 2.1 and Theorem 2.2 imply the following characterization of residual sets in Baire spaces.

Corollary 2.3. *Let X be a Baire space. A subset A of X is residual if and only if there is a dense G_δ-set G in X such that $G \subseteq A$.*

By Theorem 2.2 any Baire space is of the second category. Moreover, it can be shown that a real or complex topological linear space is a Baire space if and only if it is of the second category.

One of the most important results concerning Baire spaces is the famous Baire Category Theorem [8, p. 390]. It asserts that a complete pseudometric space, and hence also a complete metric space, is a Baire space. Consequently any Banach space is a Baire space. But it should be noted that there exist normed linear spaces which are Baire spaces, even though they are not complete (see for instance [9, Exercise 6.23, p. 461]).

PROPOSITION 2.4. *Let X be a topological space of the second category, and A a residual subset of X. Then A is of the second category and hence nonempty.*

Proof. Assume that A is of the first category. Then X must be also of the first category, since it is the union of the two sets of the first category A and $X \setminus A$. ∎

A more relevant result about the cardinal number of a residual subset of the topological space X can be obtained if X is a Baire space satisfying the separation axiom T_1 and without isolated points. Before deriving this result, we state the following elementary lemma.

LEMMA 2.5. *Let A be a finite subset of a T_1-space X. Then A is closed, and for every $x \in X$ there exists a neighbourhood V of x such that*

$$V \cap A \subseteq \{x\}.$$

Proof. Since X is a T_1-space, each subset of X consisting of a single point is closed. A is the union of a finite family of such sets, and therefore it must be closed.

Let x be any point of X. The set $A \setminus \{x\}$ is finite, and hence closed. Since it does not contain x, there exists a neighbourhood V of x such that $V \cap (A \setminus \{x\}) = \emptyset$. Thus we have $V \cap A \subseteq \{x\}$. ■

By applying this lemma we get the following proposition.

PROPOSITION 2.6. *Let X be a Baire space satisfying the separation axiom T_1 and without isolated points, and let A be a residual subset of X. Then A is of the second category and uncountable.*

Proof. By Proposition 2.4 A is of the second category. So it remains only to be shown that A is uncountable.

Suppose that A is countable. Since it can not be empty, there is a sequence $(a_n)_{n \in N}$ in X such that $A = \{a_n : n \in N\}$. On the other hand, by Theorem 2.1 there exists a nonempty countable family $\{G_n : n \in N\}$ of dense open subsets of X such that (2.1) holds. Put now

$$E_n = G_n \setminus \{a_1, \ldots, a_n\} \text{ for every } n \in N.$$

For each $n \in N$ the finite set $\{a_1, \ldots, a_n\}$ is closed in view of Lemma 2.5, and therefore E_n is open. We claim that all the sets E_n, $n \in N$, are dense in X.

To prove this, we assume that there is a positive integer m such that E_m is not dense in X. Take any point $x \in X \setminus \text{cl}\, E_m$, and let U be a neighbourhood of x such that $U \cap E_m = \emptyset$. Hence we have

$$U \cap G_m \subseteq \{a_1, \ldots, a_m\}. \tag{2.2}$$

But according to Lemma 2.5, there exists a neighbourhood V of x for which $V \cap \{a_1, \ldots, a_m\} \subseteq \{x\}$. From (2.2) we then obtain

$$U \cap V \cap G_m \subseteq \{x\}. \tag{2.3}$$

On the other hand, taking into account that $x \in X = \text{cl}\, G_m$, we conclude that $U \cap V \cap G_m \neq \emptyset$. Thus (2.3) implies $U \cap V \cap G_m = \{x\}$. Since G_m is open, this equality shows that x is an isolated point of X. Hence we have arrived at a contradiction. So all sets E_n, $n \in N$, must be dense in X, as claimed.

Applying now Theorem 2.2, it results that the set

$$E = \bigcap_{n \in N} E_n$$

is dense in X. In consequence, E must be nonempty. Let x_0 denote any point of E. Then we have

$$x_0 \in \bigcap_{n \in N} G_n \quad \text{and} \quad x_0 \notin A.$$

But this contradicts (2.1). Thus A must be uncountable. ■

3. Lower Semicontinuous Set-Valued Functions. Set-valued function occur in diverse fields of mathematics, such as optimization theory, control theory, calculus of variations, perturbation theory for ordinary differential equations. A survey of the principal topological properties of set-valued functions has been given by SMITHSON R. E. [13]. Concerning set-valued functions we recall here only the notion of lower semicontinuity which has been introduced by BERGE C. [1, p. 114] and which will be fundamental in the investigations of the next section.

Let X and Y be topological spaces, 2^Y the set consisting of all subsets of Y, F a function from X to 2^Y, and x_0 a point of X. We say that F is *lower semicontinuous at* x_0 if, for every open subset Y_0 of Y with $F(x_0) \cap Y_0 \neq \emptyset$, there exists a neighbourhood V of x_0 such that

$$F(x) \cap Y_0 \neq \emptyset \text{ for all } x \in V.$$

If F is lower semicontinuous at each point of X, then we say that F is *lower semicontinuous on* X.

PROPOSITION 3.1. *Let X and Y be topological spaces, x_0 a point o X, and $\mathfrak{F}$ a family of functions from X to Y which are continuous at x_0. Then the function $F : X \to 2^Y$ defined by*

$$F(x) = \{f(x) : f \in \mathfrak{F}\} \text{ for all } x \in X \tag{3.1}$$

s lower semicontinuous at x_0.

Proof. If $\mathfrak{F}$ is empty the assertion is trivial. In order to prove the assertion in case $\mathfrak{F} \neq \emptyset$, consider any open subset Y_0 of Y such that $F(x_0) \cap Y_0 \neq \emptyset$. Consequently there exists a function $f_0 \in \mathfrak{F}$ with $f_0(x_0) \in Y_0$. Since f_0 is continuous at x_0, there is a neighbourhood V of x_0 so that $f_0(x) \in Y_0$ for all $x \in V$. Hence we have $F(x) \cap Y_0 \neq \emptyset$ for all $x \in V$. ■

Each function f from a topological space X to a topological space Y induces a set-valued function $F : X \to 2^Y$ if we define F by particularizing (3.1), more precisely by

$$F(x) = \{f(x)\} \text{ for all } x \in X.$$

If f is continuous at a point $x_0 \in X$, then this function F is lower semicontinuous at x_0 by Proposition 3.1. Conversely, if the induced function F is lower semicontinuous at x_0, then it follows immediately that f is continuous at x_0. These remarks show that in the special case when a set-valued function is single-valued, i.e. an ordinary function, its lower semicontinuity at a point is equivalent to the usual requirement of continuity at that point.

4. Singularities of Lower Semicontinuous Set-Valued Functions. Let X and Y be topological spaces, I a nonempty set, $B : I \times N \to 2^Y$ a function whose values are closed subsets of Y, and let F be a function from X to 2^Y.

We say that F is *uniformly B-bounded* if for every $i \in I$ there exist a positive integer n and a nonempty open subset X_0 of X such that

$$F(x) \subseteq B(i, n) \text{ for all } x \in X_0.$$

F is said to be *B-bounded at a point* $x_0 \in X$ if for every $i \in I$ there exists a positive integer n such that $F(x_0) \subseteq B(i, n)$. If F is B-bounded at each point of X, then we say that F is *pointwise B-bounded on* X.

A point in X at which F is not B-bounded is said to be a *singularity* of F. The set of all singularities of F is denoted by S_F. Obviously F is pointwise B-bounded on X if and only if S_F is empty.

The following theorem constitutes the main result of our paper and points out some properties of the set S_F when the set-valued function F is not uniformly B-bounded.

THEOREM 4.1. *Let X and Y be topological spaces, and $F : X \to 2^Y$ a lower semicontinuous function on X which is not uniformly B-bounded. Then S_F is a residual set. If in addition X is of the second category, then S_F is of the second category and hence nonempty, while if X is a Baire space satisfying the separation axiom T_1 and without isolated points, then S_F is of the second category and uncountable.*

Proof. Choose $i \in I$ such that

$$[\bigcup_{x \in X_0} F(x)] \cap [Y \setminus B(i, n)] \neq \emptyset$$

for every $n \in N$ and every nonempty open subset X_0 of X. Since F is not uniformly B-bounded such an i exists. Put

$$G_n = \{x \in X : F(x) \cap [Y \setminus B(i, n)] \neq \emptyset\} \tag{4.1}$$

for all $n \in N$. We claim that all the sets G_n, $n \in N$, are open and dense in X.

Indeed, let n be any positive integer, and x_0 any point of G_n. According to (4.1) we have

$$F(x_0) \cap [Y \setminus B(i, n)] \neq \emptyset.$$

Taking now into consideration that F is lower semicontinuous at x_0 and that $Y \setminus B(i, n)$ is open, it follows that there exists a neighbourhood V of x_0 such that

$$F(x) \cap [Y \setminus B(i, n)] \neq \emptyset \text{ for all } x \in V.$$

Hence we have $V \subseteq G_n$. Therefore x_0 is an interior point of G_n. Since x_0 was arbitrary in G_n, the set G_n is open.

Suppose now that there is a positive integer n for which G_n is not dense in X. Then $X \setminus \text{cl}\, G_n$ is open and nonempty. In view of the choice of i it follows that there exists at least a point $x \in X \setminus \text{cl}\, G_n$ with

$$F(x) \cap [Y \setminus B(i, n)] \neq \emptyset. \tag{4.2}$$

But, on the other hand, we have $x \in X \setminus G_n$, hence $x \notin G_n$, because

$$X \setminus \text{cl}\, G_n \subseteq X \setminus G_n.$$

From (4.1) we then obtain

$$F(x) \cap [Y \setminus B(i, n)] = \emptyset,$$

which contradicts (4.2).

So we have shown that all the sets G_n, $n \in N$, are open and dense in X as claimed. Since

$$\bigcap_{n \in N} G_n \subseteq S_F,$$

it follows by Theorem 2.1 that S_F is residual.

The second part of the assertion results from Proposition 2.4 and Proposition 2.6. ∎

Theorem 4.1 is a principle of condensation of the singularities of a lower semicontinuous set-valued function. It yields the following corollary, which is a principle of uniform B-boundedness of a lower semicontinuous set-valued function.

Corollary 4.2. *Let X and Y be topological spaces, X of the second category, and let $F : X \to 2^Y$ be a function which is lower semicontinuous and pointwise B-bounded on X. Then F is uniformly B-bounded.*

5. Singularities of Families of Lower Semicontinuous Functions Taking Values in an Ordered Set. Let X be a topological space, Y a nonempty totally ordered set, and $\mathcal{F}$ a family of functions from X to Y. The ordering on Y is denoted by $\leqslant$, and its associated strict ordering by $<$.

We say that $\mathcal{F}$ is *locally bounded from above at a point* $x_0 \in X$ if there exist a point $y \in Y$ and a neighbourhood V of x_0 such that

$$f(x) \leqslant y \text{ for all } f \in \mathcal{F} \text{ and all } x \in V.$$

If there is no point in X at which $\mathcal{F}$ is locally bounded from above, then we say that $\mathcal{F}$ is *nowhere locally bounded from above.*

The family $\mathcal{F}$ is said to be *bounded from above at a point* $x_0 \in X$ if there exists a point $y \in Y$ such that

$$f(x_0) \leqslant y \text{ for all } f \in \mathcal{F}.$$

If $\mathcal{F}$ is bounded from above at each point of X, we say that $\mathcal{F}$ is *pointwise bounded from above on* X.

Let $S_{\mathcal{F}}$ denote the set of those points of X at which $\mathcal{F}$ is not bounded from above. Under certain additional assumptions we shall show that $S_{\mathcal{F}}$ is a residual set. One of these assumptions will require the functions of $\mathcal{F}$ to be lower semicontinuous. From [6, pp. 319–320] we recall that a function $f : X \to Y$ is said to be *lower semicontinuous at the point* $x_0 \in X$ if for each $y \in Y$, $y < f(x_0)$, there exists a neighbourhood V of x_0 such that $y < f(x)$ for all $x \in V$. We say that f is *lower semicontinuous on* X

if it is lower semicontinuous at each point of X. If we merely say that f is lower semicontinuous, we mean that it is lower semicontinuous on X.

The lower semicontinuity of a function $f: X \to Y$ can be considered as continuity in the usual sense if Y is topologized in a suitable manner. Indeed, the system consisting of Y and all intervals

$$]y, \to[= \{z \in Y : y < z\} \text{ with } y \in Y$$

is the base of a topology on Y which will be denoted by $\mathcal{T}_Y$. It is easily shown that $f: X \to Y$ is lower semicontinuous at $x_0 \in X$ if and only if f, considered as a function from the topological space X to the topological space $(Y, \mathcal{T}_Y)$, is continuous at x_0.

PROPOSITION 5.1. *Let X be a topological space, Y a nonempty totally ordered set endowed with the topology $\mathcal{T}_Y$, and let $\mathcal{F}$ be a family of lower semicontinuous functions from X to Y. Then the set-valued function $F : X \to 2^Y$ defined by (3.1) is lower semicontinuous on X.*

Proof. Apply Proposition 3.1. ∎

PROPOSITION 5.2. *Let X be a topological space, Y a nonempty totally ordered set endowed with the topology $\mathcal{T}_Y$, let $\{y_n : n \in N\}$ be a nonempty countable subset of Y which is not bounded from above, and let $\mathcal{F}$ be a family of functions from X to Y. If $F : X \to 2^Y$ is the set-valued function defined by (3.1) and if $B : \{1\} \times N \to 2^Y$ is the function defined by*

$$B(1, n) = \{y \in Y : y \leqslant y_n\} \text{ for all } n \in N,$$

then the following assertions hold:

1° *F is uniformly B-bounded if and only if there exists a point in X at which $\mathcal{F}$ is locally bounded from above.*

2° *F is B-bounded at a point $x_0 \in X$ if and only if $\mathcal{F}$ is bounded from above at x_0.*

3° *We have $S_F = S_{\mathcal{F}}$.*

The proof of this proposition is straightforward and is left to the reader.

In view of Proposition 5.1 and Proposition 5.2 the results stated in Section 4 yield:

THEOREM 5.3. *Let X be a topological space, Y a totally ordered set containing a nonempty countable subset which is not bounded from above, and let $\mathcal{F}$ be a family of lower semicontinuous functions from X to Y which is nowhere locally bounded from above. Then $S_{\mathcal{F}}$ is a residual set. If in addition X is of the second category, then $S_{\mathcal{F}}$ is of the second category and hence nonempty, while if X is a Baire space satisfying the separation axiom T_1 and without isolated points, then $S_{\mathcal{F}}$ is of the second category and uncountable.*

Corollary 5.4. *Let X be a topological space of the second category, Y a totally ordered set containing a nonempty countable subset which is not bounded from above, and let $\mathcal{F}$ be a family of lower semicontinuous functions from X to Y which is pointwise bounded from above on X. Then there exists a point in X at which $\mathcal{F}$ is locally bounded from above.*

When in Corollary 5.4 Y is taken to be the set of all real numbers, we obtain the following well-known result from topology (see for instance [6, Theorem 22 B.3, p. 384]), which is often used in functional analysis to derive uniform boundedness principles.

Corollary 5.5. *If X is a topological space of the second category, then there exist, for each family $\mathcal{F}$ of real-valued lower semicontinuous functions on X which is pointwise bounded from above on X, a nonempty open subset X_0 of X and a real number α, such that*

$$f(x) \leqslant \alpha \text{ for all } f \in \mathcal{F} \text{ and all } x \in X_0.$$

We should like to emphasize that the converse of this corollary is also true (see [6, Theorem 22 B.4, p. 385] or [5, Theorem 1]).

6. Singularities of Families of Continuous Linear Mappings. Let X and Y be topological linear spaces over the same field K, where K denotes either the field of real numbers or the field of complex numbers. Let $L(X, Y)$ denote the set of all continuous linear mappings from X to Y, and let $\mathcal{F}$ be a subset of $L(X, Y)$.

$\mathcal{F}$ is said to be *equicontinuous* (at 0) if for every neighbourhood V of the origin of Y there exists a neighbourhood U of the origin of X such that

$$\{f(x) : f \in \mathcal{F}\} \subseteq V \text{ for all } x \in U.$$

$\mathcal{F}$ is said to be *bounded at the point* $x_0 \in X$ if the set

$$\{f(x_0) : f \in \mathcal{F}\}$$

is bounded in Y, i.e. if for every neighbourhood V of the origin of Y there exists a positive integer n such that

$$\{f(x_0) : f \in \mathcal{F}\} \subseteq nV.$$

If $\mathcal{F}$ is bounded at each point of X, we say that $\mathcal{F}$ is *pointwise bounded* on X. Let $S_{\mathcal{F}}$ denote the set of those points of X at which $\mathcal{F}$ is not bounded.

PROPOSITION 6.1. *Let X and Y be topological linear spaces over K, $\mathcal{B}$ a neighbourhood base at the origin of Y composed of closed sets, $B : \mathcal{B} \times N \to 2^Y$ the function defined by*

$$B(V, n) = nV \text{ for all } (V, n) \in \mathcal{B} \times N,$$

$\mathcal{F}$ a subset of $L(X, Y)$, and $F : X \to 2^Y$ the set-valued function defined by (3.1). Then the following assertions hold:

1° *F is lower semicontinuous on X.*

2° *F is uniformly B-bounded if and only if $\mathcal{F}$ is equicontinuous.*

3° *F is B-bounded at a point $x_0 \in X$ if and only if $\mathcal{F}$ is bounded at x_0.*

4° *We have $S_F = S_{\mathcal{F}}$.*

Proof. Assertion 1° follows from Proposition 3.1. The proof of assertion 3° is immediate, while assertion 4° is a consequence of 3°. So it remains only to be shown that assertion 2° is true.

Assume that F is uniformly B-bounded. If $\mathfrak{F}$ is not equicontinuous one can choose a neighbourhood V of the origin of Y such that in every neighbourhood U of the origin of X there is at least one point u with $F(u) \cap (Y \setminus V) \neq \emptyset$. Let $V_0 \in \mathfrak{B}$ be such that $V_0 - V_0 \subseteq V$. Since F is uniformly B-bounded, there correspond to V_0 a positive integer n_0 and a nonempty open subset X_0 of X for which

$$F(x) \subseteq B(V_0, n_0) = n_0 V_0 \text{ for all } x \in X_0.$$

Let x_0 be any point of X_0, and U_0 a neighbourhood of the origin of X such that $x_0 + U_0 \subseteq X_0$. Then we have

$$F\left(\frac{1}{n_0} x\right) = F\left(\frac{1}{n_0}(x_0 + x) - \frac{1}{n_0} x_0\right) \subseteq$$

$$\subseteq \frac{1}{n_0} F(x_0 + x) - \frac{1}{n_0} F(x_0) \subseteq V_0 - V_0 \subseteq V$$

for all $x \in U_0$. Since $\frac{1}{n_0} U_0$ is a neighbourhood of the origin of X, we have arrived at a contradiction with the choice of the neighbourhood V. Therefore $\mathfrak{F}$ must be equicontinuous.

Conversely, it is obvious that F is uniformly B-bounded if $\mathfrak{F}$ is equicontinuous. This completes the proof of assertion 2°. ■

In view of this proposition the results stated in Section 4 imply:

THEOREM 6.2. *Let X and Y be topological linear spaces over K, and let $\mathfrak{F}$ be a subset of $L(X, Y)$ which is not equicontinuous. Then $S_{\mathfrak{F}}$ is a residual set. If in addition X is of the second category, then $S_{\mathfrak{F}}$ is of the second category and hence nonempty, while if X is of the second category and satisfies the separation axiom T_1, then $S_{\mathfrak{F}}$ is of the second category and uncountable.*

Corollary 6.3. *Let X be a topological linear space over K of the second category, Y a topological linear space over K, and $\mathfrak{F}$ a subset of $L(X, Y)$ which is pointwise bounded on X. Then $\mathfrak{F}$ is equicontinuous.*

It should be remarked that Theorem 6.2 includes results due to BOURBAKI N. [2, Exercise 15, p. 37], and to COBZAŞ Ş. and MUNTEAN I. [7, Theorem 3.1, (i)]. Obviously it is a generalization of Theorem 1.2, while Corollary 6.3 is a generalization of Theorem 1.1.

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