MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 12, Nº 2, 1983, pp. 187—192

A HIERARCHY OF CONVEXITY FOR SEQUENCES

by
GH. TOADER
(Cluj-Napoca)

An interesting property, called in [4] "hierarchy of convexity", was proved, for functions, by A. M. BRÜCKNER and E. OSTROW in [3]. The main aim of this paper is to prove that this hierarchy is also valid in the case of sequences.

We begin by the definitions of sequence classes which we consider in what follows. We also prove representation theorems for some of this classes.

DEFINITION 1. *A sequence* $(a_n)_{n=0}^{\infty}$ *is called convex if its second order differences:*

(1) $$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n$$

are positive for any $n \geqslant 0$.

Although we have given in [7] a general representation theorem, for making a minor change in the formulation of the result, we prefer, in this particular case, to deduce it from the following:

LEMMA 1. *If the sequence* $(a_n)_{n=0}^{\infty}$ *is given by:*

(2) $$a_n = \sum_{k=0}^{n} (n - k + 1)\, b_k$$

then:

(3) $$\Delta^2 a_n = b_{n+2}.$$

The proof follows by a simple computation, hence it is omitted. Because the relation (2) is equivalent with:

(2′) $$b_0 = a_0,\ b_n = a_n - \sum_{k=0}^{n-1} (n - k + 1)\, b_k \quad \text{for} \quad n \geqslant 1$$

any sequence may be represented in this form and from lemma 1 we deduce:

LEMMA 2. *The sequence* $(a_n)_{n=0}^{\infty}$ *is convex if and only if* $b_n \geqslant 0$ *for* $n \geqslant 2$ *in the representation* (2).

DEFINITION 2. *A sequence* $(a_n)_{n=0}^{\infty}$ *is called starshaped if it satisfies:*

$$(4) \qquad \frac{a_{n-1} - a_0}{n-1} \leqslant \frac{a_n - a_0}{n} \text{ for any } n \geqslant 2.$$

REMARK 1. As it was proved by N. OZEKI (see [5]), a convex sequence $(a_n)_{n=0}^{\infty}$, with $a_0 = 0$, has the property:

$$(4') \qquad \frac{a_{n-1}}{n-1} \leqslant \frac{a_n}{n}.$$

Although this property may be easily put in connection with the similar property of functios, the definition of starshaped sequences we have not found neither in [5] nor elsewhere. We prefer the relation (4) instead of (4') to allow $a_0 \neq 0$.

LEMMA 3. *The sequence* $(a_n)_{n=0}^{\infty}$ *is starshaped if and only if it may be represented by:*

$$(5) \qquad a_n = n \sum_{k=1}^{n} \frac{c_k}{k} - (n-1) \cdot c_0$$

with $c_k \geqslant 0$ *for* $k \geqslant 2$.

Proof. We denote $c_0 = a_0$ and $c_1 = a_1$. From (4), for $n = 2$, we have:

$$a_2 \geqslant 2a_1 - a_0 = 2c_1 - c_0$$

that is, there exists a number $c_2 \geqslant 0$ such that:

$$a_2 = 2c_1 - c_0 + c_2 = 2(c_1 + c_2/2) - c_0.$$

Suppose that (5) is valid for a natural n. From (4), for $n+1$, we have:

$$a_{n+1} \geqslant \frac{n+1}{n} a_n - \frac{1}{n} a_0$$

that is, for some $c_{n+1} \geqslant 0$:

$$a_{n+1} = c_{n+1} + \frac{n+1}{n} a_n - \frac{1}{n} a_0 = c_{n+1} + (n+1) \sum_{k=1}^{n} \frac{c_k}{k} -$$

$$- \left(\frac{n^2-1}{n} + \frac{1}{n}\right) c_0 = (n+1) \sum_{k=1}^{n+1} \frac{c_k}{k} - n \cdot c_0.$$

So, the lemma is proved by induction.

LEMMA 4. *If the sequence* $(a_n)_{n=0}^{\infty}$ *is represented by* (5), *then:*

$$(6) \qquad \Delta^2 a_n = c_{n+2} - \frac{n}{n+1} c_{n+1}.$$

DEFINITION 3. *The sequence* $(a_n)_{n=0}^{\infty}$ *is called superadditive if it verifies:*

$$(7) \qquad a_{n+m} + a_0 \geqslant a_n + a_m, \text{ for any } n, m \in \mathbf{N}.$$

REMARK 2. As it is done in [2] for functions, we added the term a_0 in the first side of the relation (7) to avoid the restriction: $a_0 \leqslant 0$. As a matter of fact, this change is unimportant since from (7) follows that the sequence $(a'_n)_{n=0}^{\infty}$ given by $a'_n = a_n - a_0$, satisfies the usual relation:

$$(7') \qquad a'_{n+m} \geqslant a'_n + a'_m.$$

The following result, deduced from [6], is easily to check up:

LEMMA 5. *The sequence* $(a_n)_{n=0}^{\infty}$ *is superadditive if it may be respresented by:*

$$(8) \qquad a_0 = d_0,\ a_n = d_0 + \sum_{k=1}^{n} \left[\frac{n}{k}\right] d_k, \text{ for } n \geqslant 1$$

with $d_k \geqslant 0$ for $k \geqslant 1$, where $[x]$ denotes the integer part of x.

REMARK 3. Any sequence $(a_n)_{n=0}^{\infty}$ may be represented by (8). It is superadditive if and only if every d_n verifies:

$$(9) \qquad d_n \geqslant - \min_{p=1,\ldots,[n/2]} \sum_{k=2}^{n-1} \left(\left[\frac{n}{k}\right] - \left[\frac{p}{k}\right] - \left[\frac{n-p}{k}\right]\right) d_k$$

but (9) becomes $d_n \geqslant 0$ only for prime values of n.

DEFINITION 4. *The sequence* $(a_n)_{n=0}^{\infty}$ *has the property "P" in the mean, if the sequence* $(A_n)_{n=0}^{\infty}$ *has the property, "P", where:*

$$(10) \qquad A_n = \frac{a_0 + \ldots + a_n}{n+1}.$$

LEMMA 6. *The sequence* $(a_n)_{n=0}^{\infty}$ *is mean — convex if and only if it may be represented by:*

$$(11) \qquad a_n = \sum_{k=0}^{n} (2n - k + 1) \cdot e_k$$

with $e_k \geqslant 0$ for $k \geqslant 2$.

Proof. By lemma 2, the sequence $(a_n)_{n=0}^{\infty}$ is mean-convex if and only if the sequence $(A_n)_{n=0}^{\infty}$ may be represented under the form:

$$(12) \qquad A_n = \sum_{k=0}^{n} (n - k + 1) \cdot e_k$$

with $e_k \geqslant 0$ for $k \geqslant 2$. From (10) we have:

$$(13) \qquad a_0 = A_0,\ a_n = (n+1) \cdot A_n - n \cdot A_{n-1}, \text{ for } n \geqslant 1.$$

Combining (12) and (13), by a simple calculation we get (11).

LEMMA 7. *If the sequence* $(a_n)_{n=0}^{\infty}$ *is represented by means of* (11), *then:*

(14) $$\Delta^2 a_n = (n+3) \cdot e_{n+2} - n \cdot e_{n+1}.$$

LEMMA 8. *The sequence* $(a_n)_{n=0}^{\infty}$ *is mean — starshaped if and only if it may be represented by:*

(15) $$a_n = (n+1) f_n + 2n \sum_{k=1}^{n-1} \frac{f_k}{k} - (2n-1) \cdot f_0$$

where $f_k \geq 0$ *for* $k \geq 2$.

The proof is based, like that of lemma 6, on the relation (13), and uses for A_n the representation (5).

In what follows we denote by S_1, S_2, S_3, S_4, S_5 and S_6 the sets of convex, mean — convex, starshaped, superadditive, mean — starshaped, respectively mean — superadditive sequences. The main result, similar to that of [3], is given by the following:

THEOREM. *The following inclusions:*

(16) $$S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6$$

hold, each of them being strictly.

Proof. (*i*) Let us suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented as in (2) and also as in (11). Then, from (3) and (14) we deduce:

(17) $$b_{n+2} = (n+3) \cdot e_{n+2} - n \cdot e_{n+1}$$

that is:

$$b_2 = 3 \cdot e_2 \text{ and } (n+3) \cdot e_{n+2} = b_{n+2} + n \cdot e_{n+1}.$$

So, if $b_n \geq 0$ for $n \geq 2$, then $e_n \geq 0$ for $n \geq 2$. By lemmas 2 and 6, if the sequence $(a_n)_{n=0}^{\infty}$ is convex, it is mean — convex, i.e. $S_1 \subset S_2$. The inclusion is strictly because we have, for exemple, $b_3 = 4e_3 - e_2$, and so $e_2 = 1$ and $e_3 = 0$ give us $b_3 = -1 < 0$.

(*ii*) Let us represent the sequence $(a_n)_{n=0}^{\infty}$ under the forms (11) and (5). From (14) and (6) we have:

(18) $$(n+3) \cdot e_{n+3} - n \cdot e_{n+1} = c_{n+2} - \frac{n}{n+1} \cdot c_{n+1}$$

that is:

$$c_2 = 3e_2, \ c_3 = 4 \cdot e_3 + 1/2 \cdot e_2$$

and:

$$c_n = (n+1) e_n + \frac{1}{n-1} \sum_{k=2}^{n-1} (k-1) \cdot e_k$$

what may be proved by induction. So, $e_n \geq 0$ for $n \geq 2$ implies $c_n \geq 0$ for $n \geq 2$, i.e. by lemmas 3 and 6, $S_2 \subset S_3$. On the other hand, for $c_2 = 3$ and $c_3 = 0$, we have $e_3 = -1/8 < 0$, that is the above inclusion is strictly.

(*iii*) Let us suppose that the sequence $(a_n)_{n=0}^{\infty}$ is in S_3. Then, on the basis of the representation given by the lemma 3:

$$a_{n+m} + a_0 - a_n - a_m = n \sum_{k=n+1}^{n+m} \frac{c_k}{k} + m \sum_{k=m+1}^{m+n} \frac{c_k}{k} \geq 0$$

that is $(a_n)_{n=0}^{\infty}$ is in S_4. The inclusion $S_3 \subset S_4$ is strictly because the sequence with the general term $a_n = [n/2]$ is, by lemma 5, in S_4 but:

$$\frac{a_3 - a_0}{3} - \frac{a_2 - a_0}{2} = -\frac{1}{6} < 0$$

so that it is not in S_3.

(*iv*) Let the sequence $(a_n)_{n=0}^{\infty}$ be in S_4. Then:

$$a_n + a_0 \geq a_k + a_{n-k}, \text{ for } k = 1, \ldots, n-1$$

that is:

$$(n-1)(a_n + a_0) \geq 2 \sum_{k=1}^{n-1} a_k$$

or:

$$a_n \geq \frac{2}{n-1} \sum_{k=1}^{n-1} a_k - a_0.$$

So:

$$\frac{A_n - A_0}{n} = \frac{\sum_{k=1}^{n} a_k - n a_0}{n(n+1)} \geq \frac{\left(1 + \frac{2}{n-1}\right) \sum_{k=1}^{n-1} a_k - (n+1) a_0}{n(n+1)} =$$

$$= \frac{\sum_{k=1}^{n-1} a_k - (n-1) a_0}{n(n-1)} = \frac{A_{n-1} - A_0}{n-1}$$

i.e. $(a_n)_{n=0}^{\infty}$ is in S_5. The inclusion $S_4 \subset S_5$ is, in his turn, strictly because if $(a_n)_{n=0}^{\infty}$ is represented through (5) we have:

$$a_4 + a_0 - a_3 - a_1 = 5c_4 - \frac{4}{3} c_3 + c_2 < 0$$

for $c_4 = c_2 = 0$, $c_3 = 1$.

(*v*) The inclusion $S_5 \subset S_6$ follows from (*iii*). His strictness also follows by taking $A_n = [n/2]$, that is:

$$a_n = (n+1) \left[\frac{n}{2}\right] - n \left[\frac{n-1}{2}\right]$$

which gives a sequence in S_6 but not in S_5.

REMARK 4. As follows from [5], N. OZEKI has proved, by other means, the inclusion $S_1 \subset S_2$, and, in the case $a_0 = 0$, $S_1 \subset S_3$.

REMARK 5. If we set the sequence $(a_n)_{n=0}^{\infty}$ in the form (15), we have:

$$a_{n+m} + a_0 - a_n - a_m = n(f_{n+m} - f_n) + m(f_{n+m} - f_m) +$$

$$+ f_{n+m} + f_n + f_m + 2n \sum_{k=n+1}^{n+m-1} \frac{f_k}{k} + 2m \sum_{k=m+1}^{m+n-1} \frac{f_k}{k}.$$

Taking into account the inclusion $S_4 \subset S_5$, this means that in order to get a superadditive sequence $(a_n)_{n=0}^{\infty}$ it is necessary to use in (15) a sequence $(f_n)_{n=0}^{\infty}$ with $f_n \geqslant 0$ for $n \geqslant 2$, and it is sufficiently that the sequence $(f_n)_{n=1}^{\infty}$ be increasing. In spite of this result and that given in the remark 3, we have unfortunately no satisfactory formula for the respresentation of superadditive sequences.

REMARK 6. The theorem may be used to simplify some of the proofs from [3].

REFERENCES

[1] Beckenbach, E. F., *Superadditivity inequalities*. Pacific J. Math. **14** (1964), 421–438.

[2] Bruckner, A. M., *Some relationships between locally superadditive functions*. Proc. Amer. Math. Soc. **15** (1964), 61–65.

[3] Bruckner, A. M., Ostrow, E., *Some function classes related to the class of convex functions*. Pacific J. Math. **12**(1962), 1203–1215.

[4] Mitrinović, D. S., (In cooperation with P. M. Vasić), *Analytic Inequalities*. Springer – Verlag, 1970.

[5] Mitrinović, D. S., Lacković, I. B., Stanković, M. S., *On some convex sequences connected with N. Ozeki's results*. Univ. Beograd. Publik. Elektrotechn. Fak., Ser. Mat. Fiz. 634–677 (1979), 3–24.

[6] Toader, Gh., *A divisibility problem* (Romanian). Gazeta Matem. **4**(1978), 150–151.

[7] Toader, Gh., *The representation of n-convex sequences*. L'anal. Num. et la Th. de l'Approx. **10**(1981), 113–118.

Received 5.III.1983.

Institutul Politehnic
Catedra de Matematică

I. P. CLUJ, Municipiul Cluj-Napoca 3068/1983