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GENERAL SOLUTION OF THE ARCTANGENT FUNCTIONAL EQUATION

by

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Abstract

We shall show that every solution $f: \mathbf{R} \to \mathbf{R}$ of the functional equation

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) \text{ for all } x, y \in \mathbf{R} \text{ with } xy < 1,$$

which is bounded or measurable on an interval of positive length, possesses a finite derivative at the point $x = 0$ and has the form $f(x) = f'(0) \cdot \arctan x$, $x \in \mathbf{R}$. Nonmeasurable solutions of this equation are exhibited. The general solutions for functional equations considered by I. Stamate and N. Ghircoiaşiu and by H. Kiesewetter are derived from our results under weaker hypotheses on unknown functions.

1. Introduction

There are many methods to define what is commonly called arctangent function. The methods of Euclidean geometry first introduce the direct trigonometric functions sine, cosine and tangent by quotients of lengths of some adequate straight line segments, and then the arctangent function f_1 by inversion of the restriction to the interval $\left]-\frac{\pi_1}{2}, +\frac{\pi_1}{2}\right[$ of the tangent function. The method of definite integrals generates the values of the arctangent f_2 by

$$f_2(x) = \int_0^x \frac{dt}{1+t^2} \quad \text{for } x \in \mathbf{R}$$

(see [6], p. 389, and [4]). The method of recurrent sequences, proposed by A. Hurwitz [7] and subsequently developed in [9], pp. 33—36, and [10], pp. 20—27, introduces the arctangent function f_3 by

$$f_3(x) = \lim_{n\to\infty} 2^n \cdot z_n(x) \quad \text{for } x \in \mathbf{R},$$

where

$$z_0(x) = x \text{ and } z_{n+1} = \frac{z_n(x)}{1+\sqrt{1+(z_n(x))^2}} \quad \text{for } n \geqslant 0.$$

Other methods to define the arctangent function are reviewed in [10], pp. 2—5.

The number π, defined in the Euclidean geometry as the quotient π_1 of the length of any circle of positive radius by its diameter, appears into later methods through the formulae $\pi_2 = 4 \cdot f_2(1)$ and $\pi_3 = 2 \cdot \lim_{x\to+\infty} f_3(x)$, respectively.

It is to be expected, of course, that the enumerated methods are nothing else but different proceedings to introduce one and the same function, that is the following equlities

$$(1.1) \qquad f_1(x) = f_2(x) = f_3(x) \quad \text{for all } x \in \mathbf{R},$$

are true, which then imply $\pi_1 = \pi_2 = \pi_3$. Each of the functions f_1, f_2 and f_3 satisfies the same conditional functional equation

$$(1.2) \qquad f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) \quad \text{for all } x, y \in \mathbf{R} \text{ with } xy < 1$$

(see [6], pp. 390—391, for f_2, and [7] for f_3), the strong condition of differentiability on $\mathbf{R}$ and the equalities

$$(1.3) \qquad f_1'(0) = f_2'(0) = f_3'(0) = 1.$$

We intend to derive (1.1) from (1.2) and (1.3) and some weaker regularity conditions on the functions f_1, f_2 and f_3.

Denote by arctan one of the functions f_1, f_2 and f_3 and by π the corresponding number. The functional equation in (1.2) has been considered by W. Alt [2] even in a more general setting, but without taking into account of its conditional character. When the condition $xy < 1$ in (1.2) is replaced by $xy > 1$, the function $f(x) = \arctan x$, $x \in \mathbf{R}$, satisfies another functional equation:

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) + \pi \cdot \operatorname{sign} x \quad \text{for all } x,\ y \in \mathbf{R} \text{ with } xy > 1;$$

further, when the condition $xy < 1$ in (1.2) is replaced by $xy \neq 1$, H. Kiesewetter [8] showed that the null function $f(x) = 0$, $x \in \mathbf{R}$ is the single continuous solution of the functional equation

$$(1.4) \qquad f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) \quad \text{for all } x,\ y \in \mathbf{R} \text{ with } xy \neq 1.$$

The proceeding in [1], pp. 59—62, would allow the reduction of the functional equation (1.2) to a Cauchy functional equation for the function $t \mapsto f(\tan t)$ on a restricted domain. Avoiding the difficulties concerning the formulation and integration of the last equation, we shall directly prove in this paper that every solution $f\colon \mathbf{R} \to \mathbf{R}$ of the functional equation (1.2), which satisfies one of the following regularity conditions:

1. f is differentiable at a point in $\mathbf{R}$,
2. f is continuous at a point in $\mathbf{R}$,
3. f is bounded on an interval of positive length,
4. f is monotone on an interval of positive length,
5. f is measurable on an interval of positive length,

possesses a finite derivative at the point $x = 0$ and has the form

$$(1.5) \qquad f(x) = f'(0) \cdot \arctan x, \quad x \in \mathbf{R}.$$

As an application of these results, the general solutions for functional equations considered by I. Stamate and N. Ghircoiaşiu [12] and by H. Kiesewetter [8] are derived under general hypotheses of boundedness or measurability of unknown functions.

2. Differentiable solutions and continuous solutions

Let us first remark that for a function $f\colon \mathbf{R} \to \mathbf{R}$ satisfying (1.2) we have

$$(2.1) \qquad f(0) = 0$$

(put $x = y = 0$ into (1.2)), and

$$(2.2) \qquad f(-x) = -f(x) \quad \text{for } x \in \mathbf{R}$$

(put $y = -x$ into (1.2) and use (2.1)); from (1.2) with $y = x$ we also obtain

$$(2.3) \qquad 2f(x) = f\left(\frac{2x}{1-x^2}\right) \quad \text{for } |x| < 1.$$

Under the differentiability condition of f on $\mathbf{R}$, the following theorem is essentially due to G. H. Hardy [6], p. 360:

THEOREM 2.1. *If $f\colon \mathbf{R} \to \mathbf{R}$ is a differentiable function at the point $x = 0$ and satisfies the functional equation* (1.2), *then* (1.5) *holds.*

Proof. We shall show that f is differentiable at each point $x \in \mathbf{R}$. For all numbers $h \in \mathbf{R}$ having its absolute value sufficiently small we obtain

$-x(x+h) < 1$, whence by (2.2), (1.2), (2.1) and the differentiability of f at 0 we derive

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\cdot f\left(\frac{h}{1+x(x+h)}\right) = \frac{1}{1+x^2}\cdot\lim_{k\to 0}\frac{f(k)-f(0)}{k} = \frac{f'(0)}{1+x^2}.$$

Thus, f is differentiable at each point $x \in \mathbf{R}$ and

$$f'(x) = \frac{f'(0)}{1+x^2},$$

hence there is a $c \in \mathbf{R}$ such that $f(x) = f'(0)\cdot\arctan x + c$, $x \in \mathbf{R}$. The equalities (2.1) and $\arctan 0 = 0$ yield $f(x) = f'(0)\cdot\arctan x$, $x \in \mathbf{R}$.

In the case of continuous solutions we have:

THEOREM 2.2. *If $f: \mathbf{R}\to\mathbf{R}$ is a continuous function at the point $x = 0$ and satisfies the functional equation* (1.2), *then f is differentiable at this point and* (1.5) *holds.*

Proof. We shall first transfer the continuity of f from $x = 0$ to any point $x \in \mathbf{R}$. For all $h \in \mathbf{R}$ sufficiently close to x we have $-xh < 1$ and by (1.2) we obtain

$$f(x) + f(-h) = f\left(\frac{x-h}{1+xh}\right).$$

Using (2.2), (2.1) and the continuity of f at 0 we get

$$\lim_{h\to x} f(h) = f(x) - \lim_{k\to 0} f(k) = f(x).$$

To prove the differentiability of f at $x = 0$, choose a positive number a such that $a < 1$, integrate (1.2) with respect to y in $[0, 1]$ and obtain

$$(2.4)\qquad f(x) = b + \int_0^1 f\left(\frac{x+y}{1-xy}\right)dy, \qquad x \in [-a, a]$$

where b is a constant. After the change of variable

$$z = \frac{x+y}{1-xy}, \qquad y \in [0, 1],$$

we have

$$z \in \left[x, \frac{1+x}{1-x}\right] \subset \left[-a, \frac{1+a}{1-a}\right] \text{ and } 1 + xz \geqslant \frac{1}{1+a} > 0$$

and (2.4) becomes

$$f(x) = b + (1+x^2)\int_x^{\frac{1+x}{1-x}} \frac{f(z)}{(1+xz)^2}\,dz, \quad x \in [-a, a].$$

The theorem on differentiation of integrals with respect to a parameter ensures the differentiation of f at $x = 0$ and Theorem 2.1 applies.

3. Bounded solutions and monotone solutions

THEOREM 3.1. *If $f: \mathbf{R}\to\mathbf{R}$ is a bounded function on an interval of positive length and satisfies the functional equation* (1.2), *then f is differentiable at $x = 0$ and* (1.5) *holds.*

Proof. Let $x_0 \in \mathbf{R}$, $a > 0$ and suppose that there is an $L > 0$ such that $|f(y)| \leqslant L$ for all $y \in [x_0 - a, x_0 + a]$. Choose $b > 0$ with $b|x_0| < 1$ and $b(1 + x_0^2) \leqslant a(1 - b|x_0|)$. If $x \in [-b, b]$, then $xx_0 \leqslant b|x_0| < 1$ and for

$$y = \frac{x + x_0}{1 - xx_0}$$

we have

$$|y - x_0| = |x|\cdot\frac{1+x_0^2}{1-xx_0} \leqslant b\,\frac{1+x_0^2}{1-b|x_0|} \leqslant a$$

and so (1.2) yields

$$(3.1)\quad |f(x)| = \left|-f(x_0) + f\left(\frac{x+x_0}{1-xx_0}\right)\right| \leqslant |f(x_0)| + |f(y)| \leqslant M, \ x \in [-b, b],$$

where $M = |f(x_0)| + L$.

Now, we assert that f is continuous at $x = 0$. Supposing the contrary, there exists an $\varepsilon > 0$ such that for each integer $n \geqslant 1$ we can indicate a number $x_n \in \left[-\frac{1}{n}, \frac{1}{n}\right]$ with $|f(x_n)| \geqslant \varepsilon$. It follows from (3.1) that there is a subsequence (denoted in the same manner) of the sequence $(x_n)_{n\geqslant 1}$ and there is a number c such that $|c| \geqslant \varepsilon$ and

$$(3.2)\qquad \lim_{n\to\infty} x_n = 0 \quad\text{and}\quad \lim_{n\to\infty} f(x_n) = c.$$

By induction we associate with each integer $k \geqslant 0$ a sequence $(x_n^k)_{n\geqslant n_k}$ having the properties

$$(3.3)\qquad \lim_{n\to\infty} x_n^k = 0 \text{ and } \lim_{n\to\infty} f(x_n^k) = 2^k\cdot c.$$

Namely, for $k = 0$ we put $n_0 = 1$ and $x_n^0 = x_n$ for all $n \geqslant 1$, and we see that the equalities (3.3) revert to (3.2). Admitting that the sequence $(x_n^k)_{n\geqslant n_k}$ with the properties (3.3) is constructed, we first determine an index n_{k+1} so large as $|x_n^k| < 1$ for all $n \geqslant n_{k+1}$ and then we put

$$x_n^{k+1} = \frac{2x_n^k}{1-(x_n^k)^2} \text{ for } n \geqslant n_{k+1}.$$

Then $\lim\limits_{n\to\infty} x_n^{k+1} = 0$ and $f(x_n^{k+1}) = 2f(x_n^k)$ (see (2.3)), hence

$$\lim_{n\to\infty} f(x_n^{k+1}) = 2\cdot 2^k\cdot c = 2^{k+1}\cdot c.$$

Now choose an integer $k \geq 0$ such that $2^k \cdot \varepsilon \geq M + 1$. It follows from (3.3) that there exists an $n \geq n_k$ so large that $|x_n^k| \leq b$ and $|f(x_n^k)| > 2^k|c| - 1 \geq 2^k\varepsilon - 1$ whence, in virtue of (3.1), we arrive at the contradiction $M \geq |f(x_n^k)| > 2^k\varepsilon - 1 \geq M$. Consequently, f must be continuous at $x = 0$ and Theorem 2.2 applies.

COROLLARY 3.2. *If $f: \mathbf{R} \to \mathbf{R}$ is a monotone function on an interval of positive length and satisfies the functional equation (1.2), then f is differentiable at $x = 0$ and (1.5) holds.*

COROLLARY 3.3. *If $f: \mathbf{R} \to \mathbf{R}$ satisfies the functional equation (1.2) and there is an interval I of positive length, $0 \in I$, such that $xf(x) \geq 0$ for all $x \in I$ (or $xf(x) \leq 0$ for all $x \in I$), then f is differentiable at $x = 0$ and (1.5) holds.*

Proof. Using (2.2) we may admit that I has the form $I = [0, a]$, where $a > 0$. We shall prove that if $xf(x) \geq 0$ for all $x \in I$, then f is increasing on I. Let $x, z \in I$ with $x < z$. Then $y = \frac{z - x}{1 + xz} > 0$, $y \leq z$, $xy < 1$, and $f(y) \geq 0$, hence

$$f(z) = f\left(\frac{x + y}{1 - xy}\right) = f(x) + f(y) \geq f(x).$$

Similarly, if $xf(x) \leq 0$ for all $x \in I$, then f is decreasing on I. In both cases Corollary 3.2 applies.

COROLLARY 3.4. *If $f: \mathbf{R} \to \mathbf{R}$ is a differentiable (or continuous) function at a point in $\mathbf{R}$ and satisfies the functional equation (1.2), then f is differentiable at $x = 0$ and (1.5) holds.*

4. Measurable solutions and nonmeasurable solutions

Let $p, q, x \in \mathbf{R}$ with $p \neq q$, $|p| \leq q$ and $q|x| < 1$. The homographic function $h = h_x: [p, q] \to \left[\frac{x + p}{1 - xp}, \frac{x + q}{1 - xq}\right]$, defined by

$$h(y) = \frac{x + y}{1 - xy}, \quad y \in [p, q], \tag{4.1}$$

is strictly increasing and continuous together with its inverse. Consequently, h maps open (closed) sets into open (closed, respectively) sets and the image by h of an open interval $]a, b[\subset [p, q]$, $a < b$, is the open interval

$$h(]a, b[) =]c, d[, \text{ where } c = \frac{x + a}{1 - ax} \text{ and } d = \frac{x + b}{1 - bx}. \tag{4.2}$$

Moreover,

$$\frac{1}{(1 + q|x|)^2}(b - a) \leq d - c \leq \frac{1 + x^2}{(1 - q|x|)^2}(b - a). \tag{4.3}$$

LEMMA 4.1. *The image by h of any measurable set $E \subset]p, q[$ is measurable and satisfies*

$$\text{mes } h(E) \geq \frac{1}{(1 + q|x|)^2} \cdot \text{mes } E. \tag{4.4}$$

Proof. We use the following well-known characterization of measurable sets in $\mathbf{R}$ (cf. [11], pp. 73–76): a set M in $\mathbf{R}$ is measurable if and only if for each $\varepsilon > 0$ there exist an open set $G \subset \mathbf{R}$ and a closed set $F \subset \mathbf{R}$ such that $F \subset M \subset G$ and mes $(G \setminus F) < \varepsilon$.

Let $\varepsilon > 0$. Since E is measurable, there exist an open set $G \subset]p, q[$ and a closed set $F \subset \mathbf{R}$ such that $F \subset E \subset G$ and

$$\text{mes } (G \setminus F) < \frac{(1 - q|x|)^2}{1 + x^2} \cdot \varepsilon. \tag{4.5}$$

The open set $G \setminus F$ can be represented as union of a countable family of mutually disjoint open intervals: $G \setminus F = \bigcup \{I_k : k \geq 1\}$ (cf. [11], p. 54), and so the measure of $G \setminus F$ is given by

$$\text{mes } (G \setminus F) = \sum_{k=1}^{\infty} \text{mes } I_k. \tag{4.6}$$

The images $G_1 = h(G)$ and $F_1 = h(F)$ are open and closed, respectively, and verify $F_1 \subset h(E) \subset G_1$. From (4.3), (4.6) and (4.5) we derive

$$\text{mes } (G_1 \setminus F_1) \leq \text{mes } h(G \setminus F) \leq \text{mes } (\bigcup \{h(I_k) : k \geq 1\}) \leq \tag{4.7}$$

$$\leq \sum_{k=1}^{\infty} \text{mes } h(I_k) \leq \frac{1 + x^2}{(1 - q|x|)^2} \cdot \sum_{k=1}^{\infty} \text{mes } I_k = \frac{1 + x^2}{(1 - q|x|)^2} \cdot \text{mes } (G \setminus F) < \varepsilon,$$

i.e. the measurability of $h(E)$ is proved.

The inequality (4.4) can be deduced from a known result (cf. [11], pp. 228–229). However, for the sake of completeness we present here a direct proof of (4.4). The open set G_1 can also be written as union of a countable family of mutually disjoint open intervals: $G_1 = \bigcup \{J_k : k \geq 1\}$, where $J_k =]c_k, d_k[$. Denote by $a_k = a$ and $b_k = b$ the numbers obtained from the last two equalities in (4.2) for $c = c_k$ and $d = d_k$. Clearly, $J_k = h(L_k)$, where $L_k =]a_k, b_k[$. Moreover,

$$G \subset \bigcup \{L_k : k \geq 1\}. \tag{4.8}$$

Indeed, supposing the contrary, there exists an $y \in G$ with $y \notin \bigcup \{L_k : k \geq 1\}$. For the number $z = h(y) \in h(G) = G_1$ it must exist an integer $k \geq 1$ such that $z \in J_k = h(L_k)$, hence there is an $y' \in L_k$ with $z = h(y')$. Now, the injectivity of h leads to the contradiction $y = y' \in L_k$.

Using (4.3) and (4.8) we obtain

$$\text{mes } G_1 = \sum_{k=1}^{\infty} \text{mes } J_k \geq \frac{1}{(1 + q|x|)^2} \cdot \sum_{k=1}^{\infty} \text{mes } L_k \geq \frac{\text{mes } E}{(1 + q|x|)^2},$$

which together with $G_1 = h(E) \cup (G_1 \setminus h(E)) \subset h(E) \cup (G_1 \setminus F_1)$ and (4.7) yield

$$\text{mes } h(E) \geqslant \text{mes } G_1 - \text{mes } (G_1 \setminus F_1) > \frac{\text{mes } E}{(1 + q|x|)^2} - \varepsilon.$$

Since ε is an arbitrary positive number, (4.4) is proved.

THEOREM 4.2. *If $f: \mathbf{R} \to \mathbf{R}$ is a measurable function on an interval of positive length and satisfies the functional equation* (1.2), *then f is differentiable at $x = 0$ and* (1.5) *holds.*

Proof. Our argument is based on the Banach's method [3] for the integration of Cauchy functional equation.

Suppose that f is measurable on the interval $[x_0 - a, x_0 + a]$, where $x_0 \geqslant 0$ and $a > 0$. In virtue of Theorem 2.2 our theorem will be proved whenever we show that f is continuous at the point $x = 0$. By the well-known Lusin's approximation theorem (cf. [11], pp. 118–119) there is a continuous function $g: [x_0 - a, x_0 + a] \to \mathbf{R}$ such that

(4.9) $\quad \text{mes } H < \frac{a}{4}$, where $H = \{x \in]x_0 - a, x_0 + a[: f(x) \neq g(x)\}$.

Let ε be a positive number. Since g is uniformly continuous, there is a $\delta > 0$ with

(4.10) $\quad |g(u) - g(v)| < \varepsilon$ for $u, v \in [x_0 - a, x_0 + a]$, $|u - v| < \delta$.

Put $p = x_0 - \frac{a}{2}$, $q = x_0 + \frac{a}{2}$ and

$$\eta = \min\left\{\frac{1}{2(x_0 + a)}, \frac{a}{2[1 + q(x_0 + a)]}, \frac{\delta}{2(1 + q^2)}\right\}. \tag{4.11}$$

Let $x \in \mathbf{R}$ with $|x| < \eta$. Clearly, $|p| \leqslant q$, $p \neq q$ and $q|x| < 1$. According to (4.11) the function $h = h_x$ defined in (4.1) fulfils $h(p) > x_0 - a$ and $h(q) < x_0 + a$, hence

$$h(]p, q[) \subset]x_0 - a, x_0 + a[. \tag{4.12}$$

By (4.9), the measurable set $E =]p, q[\setminus H$ satisfies $\text{mes} E > a - \frac{a}{4} = \frac{3}{4} a$. In virtue of Lemma 4.1 we have

$$\text{mes } h(E) \geqslant \frac{\text{mes } E}{(1 + q|x|)^2} > \frac{\text{mes } E}{(1 + q\eta)^2} > \frac{1}{3} \text{mes } E > \frac{a}{4} \tag{4.13}$$

since $q\eta < \sqrt{3} - 1$.

The inequalities in (4.9) and (4.13) show that the set $h(E) \setminus H$ is nonvoid, hence by (4.12) there is a number

$$z \in h(E) \subset h(]p, q[) \subset]x_0 - a, x_0 + a[$$

with $z \notin H$, and so there exists an $y \in E$ such that $\frac{x + y}{1 - xy} = h(y) = z$. Therefore, $y \notin H$ and $\frac{x + y}{1 - xy} \notin H$. The last relations can be written in the form

$$f(y) = g(y) \quad \text{and} \quad f\left(\frac{x + y}{1 - xy}\right) = g\left(\frac{x + y}{1 - xy}\right) = g(z). \tag{4.14}$$

By (4.11) we obtain

$$|z - y| = |x| \cdot \frac{1 + y^2}{1 - xy} < \eta \cdot \frac{1 + q^2}{1 - q\eta} < 2\eta(1 + q^2) \leqslant \delta,$$

hence from (1.2), (4.14) and (4.10) we derive

$$|f(x) - f(0)| = |f(x)| = \left|f\left(\frac{x + y}{1 - xy}\right) - f(y)\right| = |g(z) - g(y)| < \varepsilon$$

and the continuity of f at $x = 0$ is proved.

REMARK 4.3. G. Hamel [5] constructed a discontinuous solution $g: \mathbf{R} \to \mathbf{R}$ of the Cauchy functional equation

$$g(u + v) = g(u) + g(v) \text{ for all } u, v \in \mathbf{R}. \tag{4.15}$$

We shall prove that the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = g(\arctan x)$ is an unbounded and nonmeasurable solution of (1.2) on any interval of positive length.

The function f satisfies (1.2) since, if $x, y \in \mathbf{R}$ and $xy < 1$, from (4.15) we derive

$$\begin{aligned} f(x) + f(y) &= g(\arctan x) + g(\arctan y) = \\ &= g(\arctan x + \arctan y) = g\left(\arctan \frac{x + y}{1 - xy}\right) = f\left(\frac{x + y}{1 - xy}\right). \end{aligned} \tag{4.16}$$

Now, f is neither bounded nor measurable on any interval of positive length. Indeed, supposing the contrary, (4.16) together with Theorems 3.1 and 4.2 would imply the continuity of f at $x = 0$. But $g(y) = f(\tan y)$, $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, so g would be continuous at $y = 0$ and the obtained contradiction achieves the proof of our assertion.

5. Some applications

Under the differentiability condition of f on $\mathbb{R}$ the following corollary has been proved by I. Stamate and N. Ghircoiaşiu [12]:

COROLLARY 5.1. *Let $f, g, h: \mathbf{R} \to \mathbf{R}$ be given functions which satisfy the conditional functional equation*

$$f(x) + g(y) = h\left(\frac{x + y}{1 - xy}\right) \textit{ for all } x, y \in \mathbf{R} \textit{ with } xy < 1. \tag{5.1}$$

If f is bounded (or measurable) on an interval of positive length, then the functions, f, g and h are differentiable at $x = 0$ and they are given by the formulae $f(x) = f'(0) \arctan x + f(0)$, $g(x) = f'(0) \arctan x + g(0)$ *and* $h(x) = f'(0) \arctan x + f(0) + g(0)$ *for all* $x \in \mathbf{R}$.

Proof. Using (5.1) with $x = 0$ and then with $y = 0$, we find

(5.2) $$h(z) = f(0) + g(z) \text{ and } h(z) = f(z) + g(0) \text{ for all } z \in \mathbf{R},$$

whence

(5.3) $$g(z) = f(z) + g(0) - f(0) \text{ for all } z \in \mathbf{R}.$$

Define the function $\varphi : \mathbf{R} \to \mathbf{R}$ by $\varphi(x) = f(x) - f(0)$ and remark that, according to (5.2) and (5.3), the equation (5.1) takes the form

$$\varphi(x) + \varphi(y) = \varphi\left(\frac{x+y}{1-xy}\right) \text{ for all } x, y \in \mathbf{R} \text{ with } xy < 1.$$

Now, from Theorems 3.1 and 4.2 it follows that φ is differentiable at $x = 0$ and it is given by $\varphi(x) = \varphi'(0) \cdot \arctan x$, $x \in \mathbf{R}$, so the conclusion of Corollary 5.1 is immediate.

When the function f is continuous on $\mathbf{R}$ the following result has been established by H. Kiesewetter [8]:

COROLLARY 5.2. *If* $f : \mathbf{R} \to \mathbf{R}$ *is a bounded (or measurable) function on an interval of positive length and satisfies the conditional functional equation* (1.4), *then* $f(x) = 0$ *for all* $x \in \mathbf{R}$.

Proof. Since f satisfies the functional equation (1.2) too, from Theorems 3.1 and 4.2 it follows that f is differentiable at $x = 0$ and it is expressed by (1.5). From (1.4) and (2.2) we get $2f(\sqrt{2} - 1) = f(1)$, $2f(\sqrt{2} + 1) = f(-1) = -f(1)$ and

$$f(1) + f(\sqrt{2} - 1) = f(\sqrt{2} + 1) = -\frac{1}{2} f(1),$$

whence $f(1) = 0$. Using (1.5) with $x = 0$, we obtain $0 = f(1) = f'(0) \cdot \arctan 1 = f'(0) \cdot \frac{\pi}{4}$, hence $f(x) = 0$ for all $x \in \mathbf{R}$.

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