MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 12, Nº 2, 1983, pp. 125—129

ON THE CONVERGENCE OF THE MONOTONE SEQUENCES OF OPERATOR-VALUED FUNCTIONS

by

A. TIDJANE DIALLO
(Cluj-Napoca)

1. In this paper, we shall extend some results of [4] to the monotone sequences of operator-valued functions.

In what follows, we shall denote by X a real Banach space X endowed with a cone K and with the order relation defined by this cone.

We denote by $L(X)$ the Banach algebra of linear continuous operators $U : X \to X$.

Let $T : [a, b] \to L(X)$ be an operator-valued function and $x \in X$. In [1] the following notions have been introduced:

DEFINITION 1.1. *The expression* $[t_1, t_2, \ldots, t_{n+1}; T(\cdot)x]$ *defined by the relations*

$$[t_1;\ T(\cdot)x] = T(t_1)x$$

$$[t_1, t_2, \ldots, t_{n+1};\ T(\cdot)x] = \frac{[t_2, t_3, \ldots t_{n+1};\ T(\cdot)x] - [t_1, t_2, \ldots, t_n;\ T(\cdot)x]}{t_{n+1} - t_1}$$

is called the generalized divided difference of order n, associated to the point x, of the *operator-valued function* T, *for the distinct points* $t_1, t_2, \ldots, t_{n+1} \in [a, b]$.

DEFINITION 1.2. *We say that an operator-valued function* $T : [a, b] \to L(X)$ *has a bounded divided difference of order* n *on* $[a, b]$ *with respect to a subset* $M \subset X$, *if for any* $x \in M$, *we have*

$$\sup_{\substack{t_1, t_2, \ldots, t_{n+1} \\ \text{distinct} \in [a, b]}} \| [t_1, t_2, \ldots, t_{n+1};\ T(\cdot)x] \| < +\infty$$

DEFINITION 1.3. *An operator-valued function* $T : [a, b] \to L(X)$ *is called non-concave of order* n, *on the interval* $[a, b]$, *if for any system of* $n + 2$

distinct points of $[a, b]$, *the following condition is fulfilled:*

$$\forall u \in K \Rightarrow [t_1, t_2, \ldots, t_{n+2};\ T(\cdot)u] \in K$$

Let $T_1 : [a, b] \to L(X)$ and $T_2 : [a, b] \to L(X)$ be two operator-valued functions. Denote by K a generating cone in X.

DEFINITION 1.4. *We say that* $T_1 \leqq T_2$ *if for any* $t \in [a, b]$ *and for any* $u \in K$ *the following condition is satisfied:*

$$T_2(t)u - T_1(t)u \in K$$

This is an order relation in the real linear space $F([a, b], L(X))$ of all operator-valued functions defined on $[a, b]$.

2. Consider a monotone sequence $(T_m)_{m=1}^{\infty}$ of operator-valued functions $T_m : [a, b] \to L(X)$, strong continuous on $[a, b]$:

$$T_1 \leqq T_2 \leqq \ldots \leqq T_m \leqq T_{m+1} \leqq \ldots$$

LEMMA 2.1. *Assume that the cone* $K \subset X$ *is generating and normal. If there is an operator-valued function* $T : [a, b] \to L(X)$, *strong continuous on* $[a, b]$ *such that the sequence* $(T_m)_{m=1}^{\infty}$ *converges to* T *in the strong operator topology, then the sequence* $(T_m)_{m=1}^{\infty}$ *converges to* T *in the strong operator topology, uniformly with respect* $t_0 t \in [a, b]$.

Proof: Let $u \in K$; $\lim_{m\to\infty} ||T(t)u - T(t)u|| = 0$. Consider a point $t_0 \in [a, b]$. For any $\varepsilon > 0$, there exists a natural number $r(t_0)$ such that, for every $m \geqq r(t_0)$, we have

$$||T(t_0)u - T(t_0)u|| < \frac{\varepsilon}{3}.$$

Since T and $T_{r_{(t_0)}}$ are strong continuous on $[a, b]$, there exists $\delta(t_0)$ such that for any $t \in [a, b]$, for which $|t - t_0| < \delta(t_0)$, we have $||T(t)u - T(t_0)u|| < \frac{\delta}{3}$ and

$$||T_{r_{(t_0)}}(t)u - T_{r_{(t_0)}}(t_0)u|| < \frac{\varepsilon}{3}.$$

Hence $||T(t)u - T_{r_{(t_0)}}(t)u|| < \varepsilon$.

Set $V(t_0) = (t - \delta(t_0), t_0 + \delta(t_0))$ for $t_0 \in (a, b)$ and $V(a) = [a, a + \delta(a))$, $V(b) = (b - \delta(b), b]$. For any $t \in V(t_0)$ we have $||T(t)u - T_{r_{(t_0)}}(t)u|| < \varepsilon$. Since $[a, b]$ is compact there exist $t_1, t_2, \ldots, t_j \in [a, b]$ such that $[a, b] = \bigcup_{i=1}^{j} V(t_i)$. Denote by $r_0 = \max_{1 \leqslant i \leqslant j} r(t_i)$. For $t \in V(t_i)$, $u \in K$ and $m \geqq r_0$, we have:

$$\theta \underset{K}{\leqq} T(t)u - T_m(t)u \underset{K}{\leqq} T(t)u - T_{r_0}(t)u \underset{K}{\leqq} T(t)u - T_{r_{(t_i)}}(t)u.$$

Since the cone K is normal, there exists a constant $d > 0$ such that:

$$||T(t)u - T_m(t)u|| \leqq d\,||T(t)u - T_{r_0}(t)u|| \leqq d^2\,||T(t)u - T_{r_{(t_i)}}(t)u|| < d^2\varepsilon$$

Since the cone K is generating, from the preceding inequalities we obtain:

$\forall x \in X$, $\lim_{m\to\infty} ||T_m(t)x - T(t)x|| = 0$ uniformly with respect to $t \in [a, b]$.

Let $(T_m)_{m=1}^{\infty}$ be a sequence of operator-valued functions $T_m : [a, b] \to L(X)$ with bounded difference of order n on $[a, b]$ with respect to X, $n \geqq 1$.

LEMMA 2.2. *Assume that:*

(*i*) $\forall x \in X,\ \exists C_x > 0 : \forall m \in \mathbf{N},\ \sup_{\substack{t_1, t_2, \ldots, t_{n+1} \\ \text{distinct} \in [a, b]}} ||[t_1, t_2, \ldots, t_{n+1}; T_m(\cdot)x]|| \leqq C_x$

(*ii*) *there exists an operator-valued function* $T : [a, b] \to L(X)$ *such that the sequence* $(T_m)_{m=1}^{\infty}$ *converges to* T *in the strong operator topology, then* T *has a bounded divided difference of order* n *on* $[a, b]$ *with respect to* X.

Denote by $H_{-1,n} = \{T \mid T : [a, b] \to L(X)$ non-negative and non-concave of order n on $[a, b]\}$ $n = 0, 1, 2, \ldots$

Consider two operator-valued functions $T_1 : [a, b] \to L(X)$ and $T_2 : [a, b] \to L(X)$.

DEFINITION 2.1. *It is said that* $T_1 \underset{H_{-1,n}}{\leqq} T_2$ *is the following conditions are fulfilled:*

a) $\forall u \in K$, $\forall t \in [a, b] \Rightarrow T_1(t)u \underset{K}{\leqq} T_2(t)u$

b) $\forall u \in K$, $\forall t_1, t_2, \ldots, t_{n+2}$ *distinct in* $[a, b] \Rightarrow [t_1, t_2, \ldots, t_{n+2};\ T_1(\cdot)u] \underset{K}{\leqq} [t_1, t_2, \ldots, t_{n+2};\ T_2(\cdot)u]$.

Consider a sequence $(T_m)_{m=1}^{\infty}$ of operator-valued functions, strong continuous on $[a, b]$.

THEOREM 2.1. *Assume that the cone* $K \subset X$ *is generating and regular. If the following conditions are fulfilled:*

(*i*) *the sequence* $(T_m)_{m=1}^{\infty}$ *is monotone with respect to the cone* $H_{-1,n}(n \geqq 0)$, *that is*

$$T_1 \underset{H_{-1,n}}{\leqq} T_2 \underset{H_{-1,n}}{\leqq} \ldots \underset{H_{-1,n}}{\leqq} T_m \underset{H_{-1,n}}{\leqq} T_{m+1} \underset{H_{-1,n}}{\leqq} \ldots,$$

(*ii*) *there exists an operator-valued function* $T : [a, b] \to L(X)$, *strong continuous on* $[a, b]$, *such that*

$$T_m \underset{H_{-1,n}}{\leqq} T \qquad m = 1, 2, \ldots$$

(*iii*) $\forall x \in X,\ \exists C_x > 0;\ \forall m \in \mathbf{N},\ \sup_{\substack{t_1, t_2, \ldots, t_{n+2} \\ \text{distinct} \in [a, b]}} ||[t_1, t_2, \ldots, t_{n+2}; T_m(\cdot)x]|| \leqq C_x$

then there exists an operator-valued function $S: [a, b] \to L(X)$, *strong continuous on* $[a, b]$, *such that* $(T_m)_{m=1}^{\infty}$ *converges to* S, *in the strong operator topology, uniformly with respect to* $t \in [a, b]$.

Proof. Let $t \in [a, b]$ and $u \in K$. From (*i*) and (*ii*) it follows that:

$$T_1(t)u \underset{K}{\leqq} T_2(t)u \underset{K}{\leqq} \ldots \underset{K}{\leqq} T_m(t)u \underset{K}{\leqq} T_{m+1}(t)u \underset{K}{\leqq} \ldots \underset{K}{\leqq} T(t)u$$

Since the cone K is regular, there exists an element $y_t(u) \in X$ such that

$$\lim_{m\to\infty} \| T_m(t)u - y_t(u) \| = 0$$

Define a mapping $S(t): K \to X$ by

$$S(t)u = y_t(u).$$

Since the operators $T_m(t)$ are linear, $S(t)$ is a positively homogeneous mapping of K into X. $S(t)$ is also additive on K.

Since the cone K is generating, $S(t)$ can be extended to the whole space X according to a well-known method [3], by setting for $x \in X$

$$x = u - v, \qquad u, v \in K$$

$$S(t)x = S(t)u - S(t)v$$

$S(t)$ is independent of the choice of $u, v \in K$ such that $x = u - v$.

Observe that $S(t): X \to X$ is a linear operator.

From the definition of $S(t)$ it follows that for any $x \in X$, we have $\lim_{m\to\infty} \| T_m(t)x - S(t)x \| = 0.$

According to the Banach—Steinhaus Theorem, the operator $S(t): X \to X$ is continuous. Hence $S(t) \in L(X)$.

From the conditions (*i*) and (*ii*) it follows that for $u \in K$ and for any system of $n + 2$ distinct points $t_1, t_2, \ldots, t_{n+2} \in [a, b]$, the sequence $([t_1, t_2, \ldots, t_{n+2}; T_m(\cdot)u])_{m=1}^{\infty}$ converges and the limit is $[t_1, t_2, \ldots, t_{n+2}; S(\cdot)u])$.

Since the cone K is generating, for every $x \in X$ and for every system of $n + 2$ distinct points $t_1, t_2, \ldots, t_{n+2} \in [a, b]$

$$\lim_{m\to\infty} [t_1, t_2, \ldots, t_{n+2}; T_m(\cdot)x] = [t_1, t_2, \ldots, t_{n+2}; S(\cdot)x].$$

From (*iii*) and from Lemma 2.2, it follows that $S: [a, b] \to L(X)$ has a bounded divided difference of order $n + 1$ on $[a, b]$, with respect to X. From Corollary 3.4 [1], it follows that S is continuous on $[a, b]$, in the strong operator topology. Since a regular cone is normal [2], it follows by Lemma 2.1 that the sequence $(T_m)_{m=1}^{\infty}$ converges to S in the strong operator topology, uniformly with respect to $t \in [a, b]$.

For $X = \mathbf{R}$ we find the result established by Elena Popoviciu—Moldovan in the paper [4].

REFERENCES

[1] Diallo, A. T., *Une notion de convexité pour les functions opératorielles*, L'Analyse Numérique et la Théorie de l'Approximation, **10**, 2, 155—178 (1981).

[2] Krasnosselskij M. A., *Positive Solutions of Operator Equations*, Moscow, 1962 (in Russian).

[6] Peressini, A. L., *Ordered Topological Vector Spaces, Harper and Row Publishers* New York, Evanston and London 1967.

[4] Popoviciu-Moldovan, Elena, *Sur la convergence de certaines suites de fonctions*, Mathematica (Cluj), **7**, (30), 327—331 (1965)

Received 25.VIII.1982.

Facultatea de matematică
Universitatea Babeş-Bolyai
Str. Kogălniceanu 1
3400 *Cluj-Napoca*