MATHEMATICA - REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Anal. Numér. Théor. Approx., Tome 12 (1983) no. 2, pp. 131–140
ictp.acad.ro/jnaat

A DIRECT METHOD FOR THE CONSTRUCTION OF GAUSSIAN QUADRATURE RULES FOR CAUCHY TYPE AND FINITE-PART INTEGRALS

by

N. I. IOAKIMIDIS

(Patras)

Abstract. It is shown how the construction of Gaussian quadrature rules for Cauchy type principal value integrals, as well as for finite-part integrals with an algebraic singularity, can be based on the theory of Gaussian quadrature rules for ordinary integrals by using nonclassical weight functions (distributions), but classical systems of orthogonal polynomials. A series of quadrature rules for the aforementioned class of integrals is derived by this new approach and this illustrates its possibilities.

1. INTRODUCTION

Several quadrature rules appeared during the last few years for Cauchy type principal value integrals [1–3, 5, 6, 8, 9, 12, 19] and for finite-part integrals [7, 10, 11, 13–18]. Among them the Gaussian quadrature rules (based on appropriate systems of orthogonal polynomials) are the most interesting in practice because of their accuracy and their convergence under mild assumptions. Furthermore, several approaches for the construction of quadrature rules have been suggested and used in the above references. Here we suggest a new approach and, to our opinion, the most direct of all: to apply directly the theory of Gaussian quadrature rules for ordinary integrals (see, e.g., [4]) to Cauchy type principal value integrals, as well as to finite-part integrals although the corresponding weight functions are not classical and can be interpreted as distributions [15]. This approach has been already used by

Kutt [15] for finite-part integrals, but the orthogonal polynomials used there (which were recently shown to be shifted Jacobi polynomials [10]) had one or some of their roots outside the integration interval corresponding to negative or complex weights. Generally, this situation is uncomfortable. On the other hand, the results of Kutt [15] can be considered equivalent to the algebraic approach to the construction of Gaussian quadrature rules.

The approach used here differs from that of Kutt in that we use orthogonal polynomials with roots inside the integration interval although we have nonclassical weight functions. We consider also both quadrature rules for Cauchy type principal values integrals (where no results analogous to Kutt's were ever presented probably because of their lack of interest) and for finite-part integrals (where the difficulties in the Gaussian quadrature rules of Kutt led both Kutt [15] and Paget [18] to use interpolatory quadrature rules with nodes either equispaced [15] or roots of the shifted Legendre polynomials [18]. Although convenient closed-form formulae for the weights exist in all these cases (in spite of the fact that in [18] the contrary is implied for the results of [15]), these interpolatory quadrature rules present the usual disadvantages of interpolatory quadrature rules: low accuracy (compared to the accuracy achieved with Gaussian quadrature rules) and weights of generally alternating signs and large absolute values.

Here we will use the above-described approach to construct a series of already known quadrature rules for the aforementioned classes of integrals with very little effort and based on the classical results for Gaussian quadrature rules [4]. Further extensions of the present results to more complicated cases are quite possible.

2. CAUCHY TYPE INTEGRALS

We consider at first Cauchy type principal value integrals of the form

$$I(x) = \int_a^b w(t)\,\frac{g(t)}{t-x}\,dt, \quad a < x < b, \tag{2.1}$$

where $[a,b]$ is a finite or infinite integration interval, $w(t)$ a nonnegative weight function and $g(t)$ the integrand, assumed possessing a continuous first derivative in $[a,b]$ [3]. For the numerical evaluation of $I(x)$, we rewrite it as

$$I(x) = \int_a^b W(t)g(t)dt, \tag{2.2}$$

where the new weight function $W(t)$ (its dependence on x not denoted explicitly) is defined by

$$W(t) = w(t)/(t-x). \tag{2.3}$$

For the construction of a Gaussian quadrature rule for $I(x)$, instead of looking for the system of orthogonal polynomials with respect to $W(t)$ (which, if they exist, most probably will not have all their roots real and inside $[a,b]$ [4]), we use the preassigned node $t = x$ (following a device suggested by Struble [20] for ordinary integrals) and, next, we apply the theory of Gaussian quadrature formulae with a preassigned node [4]. Then we have to use as nodes the roots of the polynomial $p_n(t)$ of the system of the orthogonal polynomials with respect to the weight function

$$\bar{w}(t) = w(t)v(t), \tag{2.4}$$

where

$$v(t) = t - x, \tag{2.5}$$

which, because of (2.3), coincides with $w(t)$, that is

$$\bar{w}(t) = w(t). \tag{2.6}$$

Hence, the nodes are the n roots of $p_n(t)$ plus the preassigned node $t_{n+1} = x$. As regards the weights, we easily find them on the basis of the theory reported in [4] as

$$A_i = \frac{1}{(t_i - x)p_n'(t_i)} \int_a^b w(t)\,\frac{p_n(t)}{t - t_i}\,dt = \frac{\mu_i}{t_i - x}, \quad i = 1(1)n, \tag{2.7}$$

where μ_i are the corresponding weights for the Gaussian quadrature rule (for ordinary integrals) with $w(t)$ as a weight function and the same integration interval. These weights are given by [4]

$$\mu_i = q_n(t_i)/p_n'(t_i), \quad i = 1(1)n, \tag{2.8}$$

where

$$q_n(x) = \int_a^b w(t)\,\frac{p_n(t)}{t - x}\,dt. \tag{2.9}$$

As regards the weight corresponding to the preassigned node $t_{n+1} = x$, it is determined from [4]

$$A_{n+1} = \frac{1}{p_n(x)} \int_a^b w(t)\,\frac{p_n(t)}{t - x}\,dt = \frac{q_n(x)}{p_n(x)}. \tag{2.10}$$

These results hold true if $x \neq t_i$ $(i = 1(1)n)$. Then the quadrature rule for $I(x)$ takes the form

$$I(x) = \sum_{i=1}^{n} \mu_i\,\frac{g(t_i)}{t_i - x} + \frac{q_n(x)}{p_n(x)}\,g(x) + E_n(g;x), \quad x \neq t_i, \quad i = 1(1)n, \tag{2.11}$$

where E_n is the error term. This quadrature rule was obtained for the first time in [6,12] as a generalization of the results of [1,5,19]. Finally, it is trivial to say that both the nodes t_i and the weights μ_i $(i = 1(1)n)$ depend on n. Moreover, the quadrature rule (2.11) has $n+1$ nodes (one of which is preassigned). Hence, it is exact for integrands polynomials of up to $[2(n+1)-1]-1 = 2n$ degree as is well known [6, 12]. The convergence of the quadrature rule (2.11) was considered in [2, 3].

Now, we proceed to the case where x coincides with a root t_j of $p_n(t)$. In this case, instead of assuming t_j to be a double node (working in a manner analogous to the previous one), it is more convenient to apply directly the theory of Hermite quadrature, based on the Hermite (or osculating) interpolation formula [4], with no preassigned node. Then we have the quadrature rule [4]

$$\int_a^b W(t)g(t)dt = \sum_{i=1}^{n} H_i g(t_i) + \sum_{i=1}^{n} \bar{H}_i g'(t_i) + E_n(g; t_j), \tag{2.12}$$

where $W(t)$ is given again by (2.3), t_i are the roots of $p_n(t)$, as previously, and H_i and $\bar{H}_i$ are appropriate weights to be determined.

As regards the weights $\bar{H}_i$, they are determined from the formula [4]

$$\bar{H}_i = \int_a^b W(t)(t-t_i)l_i^2(t)dt, \tag{2.13}$$

where

$$l_i(t) = p_n(t)/[(t-t_i)p_n'(t_i)]. \tag{2.14}$$

By taking into account that

$$\frac{1}{(t-x)(t-t_i)} = \frac{1}{t_i - x}\left(\frac{1}{t-t_i} - \frac{1}{t-x}\right), \quad x \neq t_i, \quad i = 1(1)n, \tag{2.15}$$

we find for $i \neq j$ from (2.13) (because of (2.14))

$$\bar{H}_i = \frac{1}{p_n'^2(t_i)(t_i - t_j)}\left[\int_a^b w(t)\frac{p_n^2(t)}{t-t_i}\,dt - \int_a^b w(t)\frac{p_n^2(t)}{t-t_j}\,dt\right] = 0, \tag{2.16}$$
$$i = 1(1)n, \quad i \neq j,$$

since we have assumed that $p_n(t)$ is the polynomial of degree n of the system of orthogonal polynomials with respect to the weight function $w(t)$ along $[a, b]$ and t_i and t_j are roots of $p_n(t)$. Moreover, the weight $\bar{H}_j$ (for $x = t_j$) is equal to

$$\bar{H}_j = \int_a^b w(t)l_j^2(t)dt = \mu_j \tag{2.17}$$

because of (2.3) and the results reported in [4]. In (2.17) μ_j is determined again from (2.8).

Now, as regards the weights H_i, they are determined from [4]

$$H_i = \int_a^b W(t) l_i^2(t) dt - 2l_i'(t_i)\bar{H}_i. \tag{2.18}$$

For $i \neq j$, by taking into account (2.8), (2.9) and (2.14) to (2.17), we directly find from (2.18) that

$$H_i = \mu_i/(t_i - t_j), \quad i = 1(1)n, \quad i \neq j. \tag{2.19}$$

Finally, for $i = j$ we use again (2.18) and the aforementioned formulae and we find after some simple calculations that

$$H_j = \left[q_n'(t_j) - \tfrac{1}{2}\mu_j p_n''(t_j)\right]/p_n'(t_j), \tag{2.20}$$

where, because of (2.9), $q_n'(t_j)$ is the Cauchy type principal value integral

$$q_n'(t_j) = \int_a^b w(t) \frac{p_n(t)}{(t - t_j)^2}\, dt \tag{2.21}$$

(t_j being a root of $p_n(t)$). On the contrary, $q_n(t_j)$ is an ordinary integral.

Now, because of (2.16), (2.17), (2.19) and (2.20), the quadrature rule (2.12) takes the form

$$I(t_j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \mu_i \frac{g(t_i)}{t_i - t_j} + \mu_j g'(t_j) + v_j g(t_j) + E_n(g; t_j) \tag{2.22}$$

with

$$v_j = \left[q_n'(t_j) - \tfrac{1}{2}\mu_j p_n''(t_j)\right]/p_n'(t_j). \tag{2.23}$$

This quadrature rule was obtained for the first time in [12] as a generalization of previous results of [5]. It was also rederived in [3], where the results of [12] were taken into account, but without reference to [12]. The convergence of (2.11) and (2.22) (which are essentially one quadrature rule) was considered in [2, 3].

3. FINITE-PART INTEGRALS

We restrict our attention to finite-part integrals of the form

$$J = \int_0^1 \frac{g(t)}{t^\lambda}\, dt, \quad 1 < \lambda < 2. \tag{3.1}$$

Generalizations of the present results to more complicated cases (with other integration intervals, weight functions or stronger algebraic singularities) are quite possible. For the construction of a Gaussian quadrature rule for the numerical evaluation of the finite-part integral J, we can be based on the definition of this integral [15, 16].

$$J = \int_0^1 \frac{g(t) - g(0)}{t^\lambda}\, dt + g(0) \int_0^1 \frac{dt}{t^\lambda}. \tag{3.2}$$

Next, we apply a Gaussian quadrature rule of the form

$$\int_0^1 t^{1-\lambda} g(t) dt = \sum_{i=1}^n \mu_i g(t_i) + E_n(g) \tag{3.3}$$

with nodes the roots of the shifted Jacobi polynomial $\bar{p}_n^{(0,1-\lambda)}(t)$ for the approximation of the first integral in the right-hand side of (3.2) (assuming that $g(t)$ possesses a continuous first derivative in the neighbourhood of $t = 0$) and we find

$$J = \sum_{i=1}^n \mu_i \frac{g(t_i)}{t_i} + \left(\frac{1}{1-\lambda} - \sum_{i=1}^n \frac{\mu_i}{t_i} \right) g(0) + E_n(g), \tag{3.4}$$

since [15, 16]

$$\int_0^1 \frac{dt}{t^\lambda} = \frac{1}{1-\lambda}. \tag{3.5}$$

The construction of quadrature rules for finite-part integrals on the basis of their definition is not recommended by Kutt [15], but the author believes that it is the best possibility because of its accuracy (when it is combined with Gaussian quadrature rules as previously) and its simplicity.

Now we will reconstruct (3.4) on the basis of the approach adopted in this paper since our aim is to illustrate this approach and not to construct new quadrature rules. To this end, we work with the weight function

$$W(t) = t^{-\lambda}, \quad 1 < \lambda < 2, \tag{3.6}$$

and we choose the preassigned node $t = 0$ to transform it to the classical weight function

$$\bar{w}(t) = W(t) v(t) = t^{-\lambda} \tag{3.7}$$

with $v(t) = t$. Next, we use the nodes t_i of the Gaussian quadrature rule corresponding to $\bar{w}(t)$ (along $[0, 1]$) plus the additional node $t_{n+1} = 0$. By applying the corresponding theoretical results reported in [4] for quadrature rules with preassigned nodes, we obtain in our case (with $p_n(t) = \bar{p}_n^{(0,1-\lambda)}(t)$)

$$A_i = \frac{1}{t_i p_n'(t_i)} \int_0^1 t^{1-\lambda} \frac{p_n(t)}{t - t_i}\, dt = \frac{\mu_i}{t_i}, \quad i = 1(1)n, \tag{3.8}$$

$$A_{n+1} = \frac{1}{p_n(0)} \int_0^1 t^{-\lambda} p_n(t) dt. \tag{3.9}$$

By applying (3.4) with $g(t) = p_n(t)$ (whence this rule is exact), we see directly that

$$\int_0^1 t^{-\lambda} p_n(t) dt = p_n(0) \left(\frac{1}{1-\lambda} - \sum_{i=1}^{n} \frac{\mu_i}{t_i} \right) \tag{3.10}$$

and the quadrature rule derived now coincides with (3.4). As regards its accuracy, it is equal to $2n$ as was the case for the corresponding quadrature rule for Cauchy type principal value integrals constructed, in a similar manner, in the previous section.

4. ORTHOGONAL POLYNOMIALS FOR NONCLASSICAL WEIGHT FUNCTIONS

In some special cases, it is possible to use classical systems of orthogonal polynomials for nonclassical weight functions. We will illustrate this possibility in this section by constructing the corresponding quadrature rules.

Consider at first the Cauchy type principal value integral

$$K = \int_{-1}^{1} \frac{g(t)}{t}\, dt. \tag{4.1}$$

In can be seen that the Legendre polynomials $P_{2m}(t) = P_n(t)$ (with $n = 2m$) are orthogonal with respect to the weight function

$$w(t) = 1/t \tag{4.2}$$

to all polynomials $\pi_k(t)$ of degree $k \le n$ along $[-1, 1]$, since

$$\int_{-1}^{1} \frac{1}{t}\, \pi_k(t) P_n(t) dt = \int_{-1}^{1} \frac{\pi_k(t) - \pi_k(0)}{t}\, P_n(t) dt + \pi_k(0) \int_{-1}^{1} \frac{P_n(t)}{t}\, dt = 0. \tag{4.3}$$

This is due to the fact that the Legendre polynomials form a system of orthogonal polynomials along $[-1, 1]$ with respect to the weight function $w_0(t) = 1$ [4]

and, moreover, $[\pi_k(t) - \pi_k(0)]/t$ is a polynomial of degree less than or equal to $n-1$. Clearly, in (4.3) n should be an even positive integer.

Based now on the classical theory of Gaussian quadrature rules [4] for the weight function (4.2) along $[-1, 1]$, we find the Gaussian quadrature rule

$$K = \sum_{i=1}^{n} A_i g(t_i) + E_n(g), \tag{4.4}$$

where t_i are the roots of $P_n(t)$ and the weights A_i are determined from [4]

$$A_i = \int_{-1}^{1} \frac{1}{t}\, l_i(t) dt = \int_{-1}^{1} \frac{1}{t}\, \frac{P_n(t)}{(t-t_i)P_n'(t_i)}\, dt = \frac{\mu_i}{t_i}, \tag{4.5}$$

where (2.8) and (2.15) were also taken into account. (In (4.5) μ_i denote simply the weights of the Gauss–Legendre quadrature rule with n nodes.) Of course, since $P_n(t)$ is a Legendre polynomial of even degree, the point $t = 0$ is not included among the nodes t_i. We can add that the quadrature rule (4.4) is due to Piessens [19].

As a final application of our developments, we consider the finite-part integral

$$L = \int_{-1}^{1} \frac{g(t)}{t^2}\, dt. \tag{4.6}$$

At first, we can evaluate this integral on the basis of its definition [15, 16] and we find

$$L = \int_{-1}^{1} \frac{g(t) - g(0) - tg'(0)}{t^2}\, dt + g(0)\int_{-1}^{1} \frac{dt}{t^2} + g'(0)\int_{-1}^{1} \frac{dt}{t}. \tag{4.7}$$

By assuming that $g(t)$ possesses a continuous second derivative in the neighborhood of $t = 0$, by applying the Gauss–Legendre quadrature rule to the approximation of the first integral in the right-hand side of (4.7) and by taking into account that

$$\int_{-1}^{1} \frac{dt}{t^2} = -2, \quad \int_{-1}^{1} \frac{dt}{t} = 0, \tag{4.8}$$

we obtain the quadrature rule

$$\int_{-1}^{1} \frac{g(t)}{t^2}\, dt = \sum_{i=1}^{n} \mu_i\, \frac{g(t_i)}{t_i^2} - \left(2 + \sum_{i=1}^{n} \frac{\mu_i}{t_i^2}\right) g(0) + E_n(g). \tag{4.9}$$

The same quadrature rule can be obtained by using the classical theory of Gaussian quadrature rules for the polynomials $tP_{2m}(t) = tP_n(t)$ $(n = 2m)$. Such a polynomial is orthogonal to all polynomials of degree up to n with respect to the weight function

$$w(t) = 1/t^2 \tag{4.10}$$

as is clear from the previous developments based on (4.3).

Then for the nodes t_i, which are the roots of $P_{2m}(t)$ $(i = 1(1)n)$, we find directly [4]

$$A_i = \frac{1}{t_i P_n'(t_i)} \int_{-1}^{1} \frac{1}{t} \frac{P_n(t_i)}{t - t_i}\, dt = \frac{\mu_i}{t_i^2}, \quad i = 1(1)n. \tag{4.11}$$

Similarly, for the node $t_{n+1} = 0$ we have

$$A_{n+1} = \frac{1}{P_n(0)} \int_{-1}^{1} \frac{P_n(t)}{t^2}\, dt. \tag{4.12}$$

This quadrature rule,

$$\int_{-1}^{1} \frac{g(t)}{t^2}\, dt = \sum_{i=1}^{n+1} A_i g(t_i) + E_n(g), \tag{4.13}$$

is seen, because of (4.11), (4.12) and the fact that (4.9) is exact for $g(t) = P_n(t)$, to be identical with the quadrature rule (4.9). These quadrature rules are a special case of a more general quadrature rule for finite-part integrals with a double-pole singularity considered in [7, 17].

REFERENCES

[1] C h a w l a, M. M., and R a m a k r i s h n a n, T. R., *Modified Gauss-Jacobi quadrature formulas for the numerical evaluation of Cauchy type singular integrals*, BIT **14**, 14–21, (1974).

[2] E l l i o t t, D., *On the convergence of Hunter's quadrature rule for Cauchy principal value integrals*, BIT **19**, 457–462, (1979).

[3] E l l i o t t, D. and P a g e t, D. F., *Gauss type quadrature rules for Cauchy principal value integrals.* Math. Comp. **33,** 301–309, (1979).

[4] H i l d e b r a n d, F. B., *Introduction to Numerical Analysis,* Tata McGraw-Hill, New Delhi 1956.

[5] H u n t e r, D. B., *Some Gauss-type formulae for the evaluation of Cauchy principal values of integrals,* Numer. Math. **19**, 419–424, (1972).

[6] I o a k i m i d i s, N. I., *General Methods for the Solution of Crack Problems in the Theory of Plane Elasticity,* Doctoral thesis, The National Technical University of Athens, Athens 1976.

[7] I o a k i m i d i s, N. I., *On the numerical evaluation of derivatives of Cauchy principal value integrals,* Computing **27,** 81–88, (1981).

[8] I o a k i m i d i s, N. I., *One more approach to the computation of Cauchy principal value integrals*, J. Comput. Appl. Math. **7**, 289–291, (1981).

[9] I o a k i m i d i s, N. I., *Further convergence results for two quadrature rules for Cauchy type principal value integrals*, Aplik. Mat. **27**, 457–466, (1982).

[10] I o a k i m i d i s, N. I., *On Kutt's Gaussian quadrature rule for finite-part integrals* (submitted for publication).

[11] I o a k i m i d i s, N. I., *On the numerical evaluation of a class of finite-part integrals*, Z. angew. Math. Mech. **63**, 572–574, (1983).

[12] I o a k i m i d i s, N. I. and T h e o c a r i s, P. S., *On the numerical evaluation of Cauchy principal value integrals*, Rev. Roumaine Sci. Tech. Sér. Méc. Appl. **22**, 803–818, (1977).

[13] K u t t, H. R., *The numerical evaluation of principal value integrals by finite-part integration,* Numer. Math. **24**, 205–210, (1975).

[14] K u t t, H. R., *Quadrature Formulae for Finite-Part Integrals*, CSIR Special Report WISK 178, National Research Institute for Mathematical Sciences (CSIR), P. O. Box 395, Pretoria 0001, 1975.

[15] K u t t, H. R., *On the Numerical Evaluation of Finite-Part Integrals Involving an Algebraic Singularity*, CSIR Special Report WISK 179, National Research Institute for Mathematical Sciences (CSIR), P. O. Box 395, Pretoria 0001, 1975.

[16] N i n h a m, B. W., *Generalised functions and divergent integrals,* Numer. Math. **8**, 444–457, (1966).

[17] P a g e t, D. F., *The numerical evaluation of Hadamard finite-part integrals*, Numer. Math. **236**, 447–453, (1981).

[18] P a g e t, D. F., *A quadrature rule for finite-part integrals,* BIT **21**, 212–220, (1981).

[19] P i e s s e n s, R., *Numerical evaluation of Cauchy principal values of integrals*, BIT **10**, 476–480, (1970).

[20] S t r u b l e, G. W., *Orthogonal polynomials: variable-signed weight functions*, Numer. Math. **5**, 88–94, (1963).

Received 30.VII.1982.

Chair of mathematics B
School of engineering
University of Patras
P. O. Box 120 Patras, Greece