MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 12, N° 2, 1983, pp. 141–149

APPROXIMATION OF NON-NEGATIVE CONVEX FUNCTIONS BY POLYNOMIALS WITH POSITIVE COEFFICIENTS IN THE HAUSDORFF METRIC

by

V. AT. KOSTOVA
(Russe)

In the paper an estimate is obtained for the approximation of the non-negative convex function by polynomials with positive coefficients in the Hausdorff metric. It is proved that the estimate is exact with respect to the order $(1/\sqrt{n})$.

We shall use the following notations:
R_Δ — the set of real functions defined on the interval Δ; R_Δ^M — the set of functions $f \in R_\Delta$, $\max |f(x)| \leqslant M$, $x \in \Delta$, $M > 0$; K_Δ^M — the set of convex functions $f \in R_\Delta^M$, i.e. $K_\Delta^M = \{f;\ f \in R_\Delta^M,\ f(\alpha x_1 + (1-\alpha) x_2) \leqslant \leqslant \alpha f(x_1) + (1-\alpha) f(x_2),\ x_1, x_2 \in \Delta,\ \alpha \in [0, 1]\}$; C_Δ — the set of continuous functions $f \in R_\Delta^M$; H_n — the set of polynomials P_n of degree $\leqslant n$; H_n^* — the set of $P_n \in H_n$ of the type:

$$P_n(x) = \sum_{i+j \leqslant n} a_{ij}\, x^i (1-x)^j,\quad a_{ij} \geqslant 0;$$

$\tau(\Delta; f, P_n)$ — the Hausdorff distance between $f \in R_\Delta^M$ and $P_n \in H_n$[7]; $E_{n,\tau}(f) = \inf \{\tau(\Delta; f, P_n);\ P_n \in H_n\}$ — the best approximation or $f \in \in R_\Delta^M$ with $P_n \in H_n$ in the Hausdorff metric; $E_{n,\tau}^*(f) = \inf \{\tau(\Delta; f, P_n\};\ P_n \in H_n^*\}$ — the best approximation of $f \in R_\Delta^M$ with $P_n \in H_n^*$ in the Hausdorff metric.

It is known [8], that for every function $f \in R_\Delta^M$

(1) $$E_{n,\tau}(f) = O(\ln n/n)$$

is valid, where $O(1)$ is a constant depending only on the interval Δ and M. V. Popov [6] proved that for $f \in K_\Delta^M$

(2) $$E_{n,\tau}(f) \leqslant C \ln (e + M) n^{-1}$$

is valid, where C is an absolute constant.

The aim of our examination is to find the best Hausdorff approximation of non-negative functions $f \in K_\Delta^M$ with polynomials $P_n \in H_n^*$. In [1] V. Veselinov proved

THEOREM 1. *Let $f \in R_{[0,1]}^M$, $f(x) \geqslant 0$ for $x \in [0, 1]$. Then*

(3) $$E_{n,\tau}^* = O(\sqrt{\ln n/n})$$

holds, where the constant $O(1)$ depends only on M.

We prove

THEOREM 2. *Let $f \in K_{[0,1]}^M$, $f(x) \geqslant 0$ for $x \in [0, 1]$. Then*

(4) $$E_{n,\tau}^*(f) = O(1/\sqrt{n})$$

is valid, where the constant $O(1)$ depends only on M.

In order to prove Theorem 2 we need some definitions and lemmas. In accordance with [6] we define the one sided Hausdorff distance from the function f to the continuous function g:

$$h(\Delta; f, g) = \max_{x\in\Delta} \min_{\xi\in\Delta} \max [\,|x - \xi|,\ |f(x) - g(\xi)|\,].$$

From the definition of Hausdorff distance [7] we have

$$\tau(\Delta; f, g) = \max \{h(\Delta; f, g), h(\Delta; g, f)\}$$

The following two lemmas are proved in [6].

LEMMA 1. [6] *Let $f, g \in C_\Delta$, $\Delta = [a, b]$, g be monotone in $[a, b]$. Then*

$$h(\Delta; g, f) \leqslant \max \{h(\Delta; f, g), p, q\},$$

where

$$p = \nu(g, f; a), \qquad q = \nu(g, f; b);$$

$$\nu(g, f; x) = \min_{\xi\in\Delta} \max \{|x - \xi|;\ |g(x) - f(\xi)|\}.$$

We'll need its modification, i.e.

COROLLARY 1. *Let $f, g \in C_{[0,1]}$, f, g, be monotone and $f(0) = g(0)$, $f(1) = g(1)$. Then the following equality is true*

$$\tau(\Delta; f, g) = h(\Delta; g, f) = h(\Delta; f, g).$$

LEMMA 2. [6] *Let g_i, $i = 1, 2, \ldots, m$ be monotone increasing continuous functions in the interval $[a, b]$ and f_i, $i = 1, 2, \ldots, m$ are functions such that $h(\Delta; f_i, g_i) \leqslant \delta_i$, $i = 1, 2, \ldots, m$. If $\mu_i \geqslant 0$, $i = 1, 2, \ldots m$ then*

$$h\left(\Delta; \sum_{i=1}^m \mu_i f_i, \sum_{i=1}^m \mu_i g_i\right) \leqslant \max \left\{\sum_{i=1}^m \mu_i \delta_i, \max_i \delta_i\right\}.$$

We'll need its modification, i.e.

COROLLARY 2. *Let $f_i, g_i \in C_{[0,1]}$, $i = 1, 2, \ldots, m$ be monotone increasing functions such that $f_i(0) = g_i(0)$, $f_i(1) = g_i(1)$, $\tau(\Delta; f_i, g) \leqq \delta_i$, $i = 1, 2, \ldots, m$. Then if $\mu_i \geqslant 0$, $i = 1, 2, \ldots, m$ it follows that*

$$\tau\left(\Delta; \sum_{i=1}^m \mu_i f_i; \sum_{i=1}^m \mu_i g_i\right) \leqq \max \left\{\sum_{i=1}^m \mu_i \delta_i, \max_i \delta_i\right\}.$$

Now we'll prove the following three lemmas:

LEMMA 3. *Let $g(\lambda; x)$ be a monotone increasing function defined on $[0, 1]$*

$$g(\lambda; x) = \max \{0, \mu(x - \lambda)/(1 - \lambda)\},$$

where $M > 0$, $\lambda \in (0, 1)$. Then there exists an absolute constant C_0^*, such that

$$\tau([0, 1]; B_n(g(\lambda)), g(\lambda)) \leqslant C_0^* M/\sqrt{n},$$

where

$$B_n(f; x) = \sum_{\nu=0}^n f\left(\frac{\nu}{n}\right) p_{n,\nu}(x),\ p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1 - x)^{n-\nu}$$

is the Bernstein polynomial for $f \in C_{[0,1]}$.

Proof. The function $g(\lambda) \in K_\Delta^M$. Therefore following [4] we have

(5) $$g(\lambda; x) \leqslant B_n(g(\lambda); x).$$

From the definition of the Bernstein polynomial it follows (See [9])

(6) $$B_{k-1}(f; x) - B_k(f; x) = \sum_{\nu=1}^{k-1} S(f; k, \nu) p_{k,\nu}(x),$$

where

$$S(f; k, \nu) = \frac{k-\nu}{k} f\left(\frac{\nu}{k-1}\right) - f\left(\frac{\nu}{k}\right) + \frac{\nu}{k} f\left(\frac{\nu-1}{k-1}\right),\ 1 \leqslant \nu \leqslant k - 1.$$

We express

$$B_{k-1}(f; x) - B_k(f; x) = S_1(f; x) + S_2(f; x),$$

where
$$S_1(f;\ x) = \sum_{|\nu/k - x| \leqslant 2\sqrt{x(1-x)\ln k/k}} S(f;\ k,\ \nu) p_{k,\nu}(x);$$
$$S_2(f;\ x) = \sum_{|\nu/k - x| > 2\sqrt{x(1-x)\ln k/k}} S(f;\ k,\ \nu) p_{k,\nu}(x).$$

First we estimate $S_2(f;\ x)$. From the definition of $S(f;\ k,\ \nu)$ it follows that $S(g(\lambda);\ k,\ \nu) \neq 0$ only for $\lambda - \frac{\lambda}{k} < \frac{\nu}{k} < \lambda - \frac{1-\lambda}{k}$. Then the following relations hold:

(7)
$$\left\{S(g(\lambda);\ k,\ \nu);\ 0 < \frac{\nu}{k} < 1\right\} \leqslant$$
$$\leqslant \max\left\{S(g(\lambda);\ k,\ \nu);\ \lambda - \frac{\lambda}{k} < \frac{\nu}{k} < \lambda - \frac{1-\lambda}{k}\right\}$$
$$\leqslant \max\left\{\left[\frac{M(k-\nu)}{(1-\lambda)k}\left(\frac{\nu}{k-1} - \lambda\right);\ \lambda - \frac{\lambda}{k} < \frac{\nu}{k} \leqslant \lambda\right], \left[\frac{M(k-\nu)}{(1-\lambda)k}\left(\frac{\nu}{k-1} - \lambda\right) - \right.\right.$$
$$\left.\left. - \frac{M}{1-\lambda}\left(\frac{\nu}{k} - \lambda\right);\ \lambda \leqslant \frac{\nu}{k} < \lambda + \frac{1-\lambda}{k}\right]\right\} \leqslant$$
$$\leqslant \max\left\{\left[\frac{M}{1-\lambda}\left(1 + \frac{1}{k} - \lambda\right)\left(\frac{\nu}{k} + \frac{\nu}{k(k-1)} - \lambda\right);\ \lambda - \frac{\lambda}{k} < \frac{\nu}{k} \leqslant \lambda\right],\right.$$
$$\left.\left[\frac{M}{1-\lambda}\left(\frac{\nu}{k-1} - \frac{\nu^2}{k(k-1)} + \lambda\frac{\nu}{k} - \frac{\nu}{k}\right);\ \lambda \leqslant \frac{\nu}{k} < \lambda + \frac{1-\lambda}{k}\right]\right\} \leqslant$$
$$\leqslant \max\left\{\left[\frac{2M\nu}{k(k-1)},\ \lambda - \frac{\lambda}{k} < \frac{\nu}{k} \leqslant \lambda\right], \left[\frac{M\nu(1 - \lambda - \nu + \lambda k)}{(1-\lambda)(k-1)k};\ \lambda \leqslant \frac{\nu}{k} < \lambda + \frac{1-\lambda}{k}\right]\right\} \leqslant$$
$$\leqslant \max\left\{\left[\frac{2M\lambda}{k-1};\ \lambda - \frac{\lambda}{k} < \frac{\nu}{k} \leqslant \lambda\right], \left[\frac{M\nu}{k(k-1)};\ \lambda \leqslant \frac{\nu}{k} < \lambda + \frac{1-\lambda}{k}\right]\right\}$$
$$\leqslant \left\{\frac{2M\lambda}{k-1};\ \lambda - \frac{\lambda}{k} < \frac{\nu}{k} < \lambda + \frac{1-\lambda}{k}\right\}.$$

According to [4] we have

(8)
$$\sum_{|\nu/k - x| > 2\sqrt{x(1-x)\ln n/n}} p_{k,\nu}(x) \leqslant 2n^{-1}.$$

In view of (7) and (8) we get
$$S_2(g(\lambda);\ x) \leqslant 4M\lambda[(k-1)k]^{-1}.$$

Futher we find an estimate for $S_1(g(\lambda);\ x)$. From the definition of $S(f;\ k,\ \nu)$ and (7) it follows that
$$S_1(g(\lambda);\ x) \leqslant \sum_{|\nu/k - \lambda| \leqslant 2\sqrt{\lambda(1-\lambda)\ln k/k}} S(g(\lambda);\ k,\ \nu) p_{k,\nu}(\lambda) =$$
$$= \sum_{|\nu/k - \lambda| \leqslant 1/k} S(g(\lambda);\ k,\ \nu) p_{k,\nu}(\lambda) = \sum_{\lambda - \lambda/k < \nu/k < \lambda + (1-\lambda)/k} S(g(\lambda);\ k,\ \nu) p_{k,\nu}(\lambda) \leqslant$$
$$\leqslant \frac{2M\lambda}{k-1} \sum_{|\nu/k - \lambda| \leqslant 1/k} p_{k,\nu}(\lambda).$$

It is known [3], that for $n^{-4/5} < x < 1 - n^{-4/5}$ and $|\nu/n - x| \leqslant 2[x(1 - x)\ln n/n]^{1/2}$ the following is valid:
$$p_{n,\nu}(x) = \frac{1 + \theta_1(n,\ \nu,\ x)}{\sqrt{2\pi}\, n^{1/2}[x(1-x)]^{1/2}} \exp\left\{-\frac{n(\nu/n - x)^2}{2x(1-x)}\right\},$$
where $|\theta_1(n,\ \nu,\ x)| \leqslant \theta_2(n)$, $\theta_2(n) > 0$, $\lim\limits_{n\to\infty} \theta_2(n) = 0$. Therefore a constant C_3 exists such that

(9)
$$P_{k,\nu}(x) \leqslant C_3[kx(1-x)]^{-1/2}.$$

Now we can estimate:
$$B_{k-1}(g(\lambda);\ x) - B_k(g(\lambda);\ x) \leqslant \frac{2C_3M\lambda}{(k-1)[k\lambda(1-\lambda)]^{1/2}} + \frac{4M\lambda}{(k-1)k} \leqslant$$
$$\leqslant \frac{C_2M[\lambda(1-\lambda)]^{1/2}}{(1-\lambda)(k-1)\sqrt{k}} + \frac{4M\lambda(1-\lambda)}{(1-\lambda)(k-1)k} \leqslant \frac{C_1M[\lambda(1-\lambda)]^{1/2}}{(1-\lambda)(k-1)\sqrt{k}}.$$

The sequence of Bernstein polynomials for a function $f \in C_{[0,1]}$ converges to f [4]. Hence it is true:

(10)
$$0 \leqslant B_n(g(\lambda);\ x) - \max\{0,\ M(x-\lambda)/(1-\lambda)\}$$
$$= \sum_{k=n+1}^{\infty} [B_{k-1}(g(\lambda);\ x) - B_k(g(\lambda);\ x)] \leqslant \frac{C_1M[\lambda(1-\lambda)]^{1/2}}{1-\lambda} \sum_{k=n+1}^{\infty} \frac{1}{(k-1)\sqrt{k}} =$$
$$\leqslant \frac{2C_0M[\lambda(1-\lambda)]^{1/2}}{(1-\lambda)\sqrt{n}} \leqslant \frac{C_0M}{(1-\lambda)\sqrt{n}}.$$

In case of $x \in [\lambda - C_0/\sqrt{n},\ 1]$ we obtain from (5) and (10)

(11)
$$\max\left[0,\ \frac{M}{1-\lambda}(x-\lambda)\right] \leqslant B_n(g(\lambda);\ x) \leqslant \frac{M}{1-\lambda}\left(x - \lambda + \frac{C_0}{\sqrt{n}}\right).$$

For $x \in [0,\ \lambda - C_0/\sqrt{n}]$ we have:
$$0 \leqslant B_n(g(\lambda);\ x] = \sum_{k=n+1}^{\infty} [B_{k-1}(g(\lambda);\ x) - B_k(g(\lambda);\ x)] \leqslant$$
$$\leqslant \sum_{k=n+1}^{\infty} \left\{\frac{4M\lambda}{(k-1)k} + \frac{4M\lambda}{(k-1)\sqrt{k}} \frac{\exp[-C_0^2[2x(1-x)]^{-1}]}{[x(1-x)]^{1/2}}\right\} \leqslant \frac{MC}{\sqrt{n}}.$$

Applying the definition of the Hausdorff distance and (1.1) we get
$$\tau([0,\ 1];\ B_n(g(\lambda)),\ g(\lambda)) \leqslant C_0^* Mn^{-1/2}.$$

LEMMA 4. *Let $f \in K_{[0,1]}^M$, f be a monotone increasing function, $0 \leqslant \leqslant f(x) \leqslant M$. Then for every $\varepsilon > 0$ there exists a function*
$$g(x) = \sum_{i=0}^{n(\varepsilon)-1} \mu_i\, g(\lambda_i;\ x)$$

with

$$g(\lambda_i;\ x) = \max\{0,\ M(x-\lambda_i)/(1-\lambda_i)\},$$

$$\lambda_i \in (0,\ 1),\ M_i \geqslant 0,\ i = 0,\ 1,\ 2,\ \ldots,\ n-1,\ \sum_{i=0}^{n-1}\mu_i = 1,$$

such that

$$\tau([0,\ 1];\ f,\ g) < \varepsilon$$

Proof. Suppose n is sufficiently large and $1/n < \varepsilon$. We split the interval in n uniform parts and define the functions:

$$g_1(x) = nf(1/n)x, \qquad x \leqslant 1/n;$$

$$g_2(x) = \begin{cases} n\left[f\left(\frac{2}{n}\right) - 2f\left(\frac{1}{n}\right)\right]\left(x - \frac{1}{n}\right), & x > 1/n; \\ 0, & x \leqslant 1/n; \end{cases}$$

$$g_i(x) = \begin{cases} n\left[f\left(\frac{i}{n}\right) - 2f\left(\frac{i-1}{n}\right) + f\left(\frac{i-2}{n}\right)\right]\left(x - \frac{i-1}{n}\right), & x > \frac{i-1}{n}; \\ 0, & x \leqslant \frac{i-1}{n}. \end{cases}$$

for $i = 3,\ 4,\ \ldots\ n.$

One can verify that for the function $g(x) = \sum_{i=1}^{n} g_i(x)$ the following is valid;

$$g(i/n) \leqslant f(i/n),\ i = 1,\ 2,\ \ldots,\ n. \tag{12}$$

According to the definition of the Hausdorff distance and (12) we have

$$\tau([0,\ 1];\ f,\ g) < \varepsilon.$$

The function f is convex and has the property

$$f(x) \leqslant g(x) \quad \text{for} \quad x \in [0,\ 1].$$

After some transformations it is easy to see that for g one has

$$g(x) = \sum_{i=1}^{n} \mu_{i-1}\, g\left(\frac{i-1}{n};\ x\right)$$

where

$$g\left(\frac{i-1}{n};\ x\right) = \begin{cases} \frac{nM}{n-i+1}\left(x - \frac{i-1}{n}\right), & x > \frac{i-1}{n}; \\ 0 & x \leqslant \frac{i-1}{n}, \end{cases} \qquad i = 1,\ 2,\ \ldots,\ n.$$

$$\mu_0 = nf(1/n)/M$$

$$\mu_{i-1} = (n-i+1)[f(i/n) - 2f((i-1)/n) + f((i-2)/n)]/M,$$

$$i = 2,\ 3,\ \ldots,\ n.$$

The restriction $\mu_i \geqslant 0,\ i = 0,\ 1,\ 2,\ \ldots\ n-1$ holds because the function f is convex.

LEMMA 5. *Let f be a monotone convex function in the interval* $[0,\ 1]$, $0 \leqslant f(x) \leqslant M$ *for* $x \in [0,\ 1]$, $f(x) = 0$ *for* $x \in [0,\ 2\delta_n]$, *where* $\delta_n = C_0^* M n^{-1/2}$, C_0^* *is the constant from Lemma* 3. *Then*

$$E^*_{n,\tau}(f) \leqslant 2C_0^* M n^{-1/2}$$

holds.

Proof. For every $\varepsilon > 0$ there exists a linear combination

$$g(x) = \sum_{i=0}^{n-1} \mu_i\, g(\lambda_i;\ x)$$

with the property

$$\tau([0,\ 1];\ f,\ g) < \varepsilon, \tag{13}$$

where $\mu_i \geqslant 0,\ i = 0,\ 1,\ \ldots\ n-1,\ \sum_{i=0}^{n-1}\mu_i = 1;\ g(\lambda_i),\ i = 0,\ 1,\ \ldots,\ n-1$ are the convex functions from Lemma 3. Now we shall define a polynomial with positive coefficients of the type

$$P_n(x) = \sum_{i=0}^{n-1} \mu_i\, B_n(g(\lambda_i);\ x).$$

Following (11) this polynomial has the following properties:

(14) 1) $P_n(x) \leqslant C_0^* M n^{-1/2},\ x \in [0,\ \delta_n];\ P_n(0) = 0;\ P_n(1) = M.$

2) P_n is a convex function;

3) $0 \leqslant \sum_{i=0}^{n-1} \mu_i[B_n(g(\lambda_i);\ x) - g(\lambda_i;\ x)] \leqslant P_n(x) - f(x),\ x \in [0,\ 1].$

Due to Corollary 2 and Lemma 3 we have

$$\tau([0,\ 1];\ P_n,\ g) \leqslant C_0^* M n^{-1/2} \tag{15}$$

From (13) and (15) it follows that

$$\tau([0,\ 1];\ P_n,\ f) \leqslant \tau([0,\ 1];\ P_n,\ g) + \tau([0,\ 1];\ g,\ f) \leqslant 2C_0^* M n^{-1/2}$$

Lemma 5 is proved. Now we can prove Theorem 2.

Let $f \in K^M_{[0,1]}$. We suppose that $f \in C_{[0,1]}$ and $\min\{f(x),\ x \in [0,\ 1]\} = f(\beta) = a$, where $a \geqslant 0,\ \beta \in [0,\ 1]$. We denote $\delta_n^* = C_0^* M n^{-1/2}$, where C_0^* is the constant from Lemma 3. We set $f^*(x) = f(x) - a$, $x \in [0, 1]$ and we define a continuous function f_n as follows:

$$f_n(x) = \begin{cases} f^*(x) & x \in [0, \beta]; \\ 0 & x \in [\beta, \beta + 2\delta_n]; \\ f^*(x - 2\delta_n) & x \in [\beta + 2\delta_n, 1 - n^{-1}], \end{cases}$$

$f_n(1) = M - a$, $f_n(x)$ is linear for $x \in [1 - n^{-1}, 1]$. It is obvious that $\tau([0, 1]; f^*, f_n) \leqslant 2\delta_n$. Futher we define f_n as a sum of two functions g_n and h_n, where

$$g_n(x) = \begin{cases} 0 & x \in [0, \beta + \delta_n]; \\ f_n(x) & x \in [\beta + \delta_n, 1], \end{cases}$$

$$h_n(x) = \begin{cases} f_n(x) & x \in [0, \beta + \delta_n]; \\ 0 & x \in [\beta + \delta_n, 1]. \end{cases}$$

The functions g_n and $\bar{g}_n(x) = h_n(1 - x)$ satisfy the restrictions of Lemma 5. Therefore there exist polynomials P_{1n} and P_{2n} with positive coefficients such that

$$\tau([0, 1]; g_n, P_{1n}) \leqslant 2\delta_n; \quad \tau([0, 1]; \bar{g}_n, P_{2n}) \leqslant 2\delta_n.$$

From the definition of g_n, $\bar{g}_n$ and (14) it follows that

(16) $$P_{1n}(x) \leqslant C_0^* M n^{-1/2}, \ x \in [0, \beta + \delta_n];$$

$$P_{2n}(1 - x) \leqslant C_0^* M n^{-1/2}, \ x \in [\beta + \delta_n, 1].$$

Then denoting $P_n^*(x) = P_{1n}(x) + P_{2n}(1 - x)$, $x \in [0, 1]$ we obtain from the definitions of g_n, $\bar{g}_n$ and (16) that

$$\tau([0, 1]; P_n^* f_n) \leqslant$$

$$\leqslant \max\{\tau([0, \beta + \delta_n]; P_n^*, f_n), \tau([\beta + \delta_n; 1], P_n^*, f_n)\} \leqslant$$

$$\leqslant \max\{\tau([0, \beta + \delta_n]; P_{2n}, \bar{g}_n) + \delta_n, \tau([\beta + \delta_n, 1]; P_{1n}, g_n) + \delta_n)\} = 3C_0^* M n^{-1/2}.$$

For the function f^* we have

$$\tau([0, 1]; P_n^* f^*) \leqslant \tau([0, 1]; f^*, f_n) +$$

$$+ \tau([0, 1]; f_n, P_n^*) \leqslant 5C_0^* M n^{-1/2}.$$

Substituting $f(x) = f^*(x) + a$ for $x \in [0, 1]$ and applying the definition of the Hausdorff distance we get:

$$\tau([0, 1]; P_n^*(x), f^*(x)) =$$

$$= \max\{\max_{x \in [0,1]} \min_{\xi \in [0,1]} \max[|x - \xi|, |f(x) - a - P_n^*(\xi)|],$$

$$\max_{x \in [0,1]} \min_{\xi \in [0,1]} \max[|x - \xi|, |f(\xi) - a - P_n^*(x)|]\} = \tau([0, 1]; f(x), P_n^*(x) + a).$$

Therefore there exists a polynomial with positive coefficients $P_n(x) = P_n^*(x) + a$ for which

$$\tau([0, 1]; P_n, f) \leqslant 5C_0^* M n^{-1/2}.$$

holds.

The obtained estimate can not be improved for non-negative convex functions. This can be seen from the following

THEOREM 3. [5] *Let* $f(x) = |x - a|^\alpha$, $0 < a < 1$, $0 < \alpha \leqslant 2$. *Then*

$$\max_{x \in [0,1]} || x - a|^\alpha - Q_n^*(x)| \geqslant M n^{-\alpha/2}$$

for sufficienly large n *and more precisely* $Q_n^*(a) \geqslant M n^{-\alpha/2}$, *where* Q_n^* *is the polynomial of the best uniform approximation. The constant* $M > 0$ *depend only on* a.

Really, let us examine the function $\tilde{f}(x) = |x - 1/2|$, $x \in [0, 1]$. In view of Theorem 3 for the function $\tilde{f}$ and its polynomial of the best uniform approximation $Q_n^* \in H_n^*$ one gets:

$$h([0, 1]; \tilde{f}, Q_n^*) \geqslant M n^{-1/2}.$$

Therefore the Hausdorff distance in the point $x = 1/2$ can not be improved.

Theorem 2 is proved.

REFERENCES

[1] Veselinov, V. M., *Best approximation of functions by polynomials mith positive coefficients in the Hausdorff metric,* C. R. Acad. Bulg. Sci., **27,** 599–602, (1974).

[2] Veselynov, V. M., *The exact constants in the theory of approximation by the Bernstein polynomials in the Hausdorff metric,* C. R. Acad. Bulg. Sci., **27,** 1183–1186, (1974),.

[3] Esseen, G. G., *Uber die asymptotisch beste Approximation stetige Functionen mit Hilfe fon Bernstein Polynomen,* Numer. Math., **2,** 206, (1960).

[4] Lorentz, G. G., *Bernstein Polynomials,* Toronto, 1953.

[5] Lorentz, G. G., *The Degree of Approximation by Polynomials with Positive Coefficients,* Math. Annalen **151,** 239–251, 1963.

[6] Popov, V. A., *Approximation of convex functions by algebraic polynomials in Hausdorff's metric.* Serdica, **1,** 386–398, 1975.

[7] Сендов, Бл., *Хаусдорфовые приближения,* БАН, София, 1978.

[8] Сендов Бл., Попов, В. А. *Точная асимптотика наилучшего приолижения алгебраическими и тригонометрическими многочленами в метрике Хаусдорфа.* Мат. сб., **89,** 138–147, 1972.

[9] Струков. Л. И. Тиман, А. Ф., *Математическое ожидание непрерываных функций случайных величин гладкость и дисперсия.,* Сибирский мат. журнал. **18,** 3, 658–664, 1977.

Received 19.II.1983.

TU „Angel Kanchev"
Center of Mathematics
7004 Russe Bulgaria
Violeta At. Kostova