MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 12, N° 2, 1983, pp. 151 — 155

A NOTE ON POPOVICIU'S INEQUALITY FOR BILINEAR FORMS

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1. Popoviciu ([4]) gave necessary and sufficient conditions for a matrix $X = (x_{ij})$ in order that the inequality $\sum_{i,j=1}^{n} x_{ij} a_i b_j \geq 0$ holds for all (ev. nonnegative), nondecreasing n-tuplets $a = (a_1 \ldots, a_n)$ and $b = (b_1, \ldots \ldots, b_n)$. Here we give a simple proof of his results and, as an application give a refinement of a rearrangement-inequality due to Hardy, Littlewood and Polya.

2. We first recall Abels inequality as it is stated and proved in [3]. „Let $a_1, \ldots, a_n$ be a sequence of real numbers. Let $0 \leq b_1 \leq b_2 \leq \ldots \ldots \leq b_n$. Then, if $m := \min\limits_{1 \leq k \leq n} \sum_{i=k}^{n} a_i$, $M := \max\limits_{1 \leq k \leq n} \sum_{i=k}^{n} a_i$, so $mb_n \leq a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \leq Mb_n$"
A simple consequence of this is, with the notation $\mathfrak{M}_+ = \{x \in \mathbf{R}_n^+ : x_1 \leq x_2 \leq \ldots \leq x_n\}$, given by the

COROLLARY 1. *For any set $a_1, a_2 \ldots, a_n$ of real numbers the following statements are equivalent:*

(*i*) $\bigwedge\limits_{1 \leq k \leq n} \sum_{i=k}^{n} a_i \geq 0$

(*ii*) $\bigwedge\limits_{b \in \mathfrak{M}_+} \sum_{i=1}^{n} a_i b_i \geq 0$

Proof: (*i*) $\Rightarrow$ (*ii*) follows directly from Abel's inequality with $m = 0$. (*ii*) $\Rightarrow$ (*i*) follows if one successively chooses the vectors

$b^{(k)} := (0, 0, 0, \ldots, 0, 1, 1, \ldots, 1) \in \mathbf{R}^n_+$ (first 1 at the k-th place) as vectors b from $\mathfrak{M}_+$.

As an immediate consequence of this Corollary we arrive at the result due to Popoviciu (stated in [3], and proved by seemingly much more complicated methods in [4]).

THEOREM 1. *The following two statements are equivalent:*

$$(i) \quad \bigwedge_{a,b \in \mathfrak{M}_+} F(a, b) := \sum_{i,j=1}^{n} x_{ij}\, a_i b_j \geqslant 0$$

$$(ii) \quad \bigwedge_{1 \leqslant r,\ s \leqslant n} \ \sum_{i=r}^{n} \sum_{j=s}^{n} x_{ij} \geqslant 0$$

Proof: We have the following equivalents: $F(a, b) \geqslant 0$ holds for all $a,b \in \mathfrak{M}_+$ exactly if $\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} x_{ij}\, b_j\right) \geqslant 0$ for all $a,b \in \mathfrak{M}_+$. By Corollary 1 this is the case exactly if for all $1 \leqslant r \leqslant n$ and for all $b \in \mathfrak{M}_+$ one has $\sum_{i=r}^{n} \sum_{j=1}^{n} x_{ij}\, b_j \geqslant 0$, or-rewritten – if $\sum_{j=1}^{n} b_j \left(\sum_{i=r}^{n} x_{ij}\right) \geqslant 0$. Once again, applying Corollary 1 $\sum_{j=1}^{n} b_j \left(\sum_{j=r}^{n} x_{ij}\right) \geqslant 0$ for all $b \in \mathfrak{M}_+$ is the case exactly if for all $1 \leqslant s \leqslant n$ $\sum_{j=s}^{n} \sum_{i=r}^{n} x_{ij} \geqslant 0$. This implies the desired result.

If we do not impose the condition of positivity to the increasing sequences, we arrive at the notation $\mathfrak{M} := \{x \in \mathbf{R}^n : x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n\}$. From Theorem 1 we easily derive another result due to Popoviciu.

THEOREM 2. *The following statements are equivalent:*

$$(i) \quad \bigwedge_{a,b \in \mathfrak{M}} F(a,b) = \sum_{i,j=1}^{n} x_{ij}\, a_i b_j \geqslant 0$$

$$(ii) \quad \bigwedge_{1 \leqslant r,\ s \leqslant n} \ \sum_{i=r}^{n} \sum_{j=s}^{n} x_{ij} \geqslant 0 \ \text{and} \ \bigwedge_{1 \leqslant r \leqslant n} \ \sum_{i=r}^{n} \sum_{j=1}^{n} x_{ij} = 0, \ \bigwedge_{1 \leqslant s \leqslant n} \ \sum_{i=1}^{n} \sum_{j=s}^{n} x_{ij} = 0.$$

Proof: $(i) \Rightarrow (ii)$ The first condition of (ii) follows by Theorem 1 already, since $\mathfrak{M}_+ \in \mathfrak{M}$. Now choose

$$(a, b) = ((0, 0, \ldots, 0, 1, 1, \ldots, 1), (1, 1, \ldots, 1)) \in \mathfrak{M} \times \mathfrak{M}$$

and then

$$(a, b) = ((0, 0, \ldots, 0, 1, 1, \ldots, 1), (-1, -1, \ldots, -1)) \in \mathfrak{M} \times \mathfrak{M}$$

where the first 1 of a occurs at the r-th place. These tuplets (a, b) plugged into (i) yield the inequalities $0 \leqslant \sum_{i,j=1}^{n} x_{ij}\, a_i b_j = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} x_{ij} = \sum_{i=r}^{n} \sum_{j=1}^{n} x_{ij}$

and $0 \leqslant -\sum_{i=r}^{n} \sum_{j=1}^{n} x_{ij}$. From this, the second condition of (ii) follows; the third follows similarly.

$(ii) \Rightarrow (i)$ Let $-\alpha := \min\{0, a_1, a_2, \ldots, a_n\}$, $-\beta := \min\{0, b_1, b_2, \ldots, \ldots, b_n\}$, for given $a, b \in \mathfrak{M}$. Then $a + (\alpha, \alpha, \ldots, \alpha) \in \mathfrak{M}_+$ and $b + (\beta, \beta, \ldots, \beta) \in \mathfrak{M}_+$. By the first of the conditions (ii) and by Theorem 1

$$0 \leqslant F(a + (\alpha, \alpha, \ldots, \alpha), b + (\beta, \beta, \ldots, \beta)) = \sum_{i,j=1}^{n} x_{ij}(a_i + \alpha)(b_j + \beta) =$$

$$= \sum_{i,j=1}^{n} x_{ij}\, a_i b_j + \alpha\left(\sum_{i,j=1}^{n} x_{ij}\, b_j\right) + \beta\left(\sum_{i,j=1}^{n} x_{ij}\, a_i\right) + \alpha\beta\left(\sum_{i,j=1}^{n} x_{ij}\right)$$

Now, successively putting $r = n, n-1, n-2, \ldots, 1$ in the second of the conditions (ii) one sees that it is equivalent to:

$$\bigwedge_{1 \leqslant r \leqslant n} \ \sum_{j=1}^{n} x_{rj} = 0.$$

Hence $\sum_{i,j=1}^{n} x_{ij}\, a_i = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} x_{ij} = 0$. Similarly $\sum_{i,j=1}^{n} x_{ij}\, b_j = 0$. Finally putting $r = 1$ in the second condition of (ii), we have $\sum_{i,j=1}^{n} x_{ij} = 0$. Hence from (1) we yield (i).

Remark 1. It is not sufficient for (i) to demand the first two conditions of (ii) only. This is seen by the example

$$X = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = (x_{ij}) \quad a = (-1, -1) \quad b = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Here

$$F(a, b) = a \times b = (-1, -1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2 < 0$$

Remark 2. Let S_n be the group of permutations on $\{1, 2, \ldots, n\}$, let $\varepsilon \in S_n$ and consider the matrix $P_\varepsilon = (p_{ij})$ $1 \leqslant i, j \leqslant n$ where, by definition $p_{ij} = \delta_{\varepsilon(i)j}$. We calculate

$$\sum_{i,j=1}^{n} p_{ij}\, a_i b_j = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \delta_{\varepsilon(i)j} b_j = \sum_{i=1}^{n} a_i b_{\varepsilon(i)}.$$

Defining the inner product $\langle a, Mb \rangle := \sum_{i,j=1}^{n} m_{ij}\, a_i b_j$ for arbitrary matrices $M = (m_{ij})$, we get: There holds an inequality $\sum_{i=1}^{n} a_i b_{\sigma(i)} \leqslant \sum_{i=1}^{n} a_i b_{\pi(i)}$ for permutations $\sigma, \pi \in S_n$ and arbitrary $a, b \in \mathfrak{M}_+$, if and only if $\langle a, P_\sigma b\rangle \leqslant \langle a, P_\pi b\rangle$ for all $a, b \in \mathfrak{M}_+$, i.e. if and only if the matrix $X := P_\pi - P_\sigma$ has the property $0 \leqslant \langle a, Xb\rangle$ for all $a, b \in \mathfrak{M}_+$, or, by Theorem 1, if and only if $\bigwedge_{1\leqslant r,\, s\leqslant n} \sum_{i=r}^{n}\sum_{j=s}^{n} x_{ij} = \sum_{i=r}^{n}\sum_{j=s}^{n} (\delta_{\pi(i)j} - \delta_{\sigma(i)j}) \geqslant 0$. Again, for an arbitrary $\varepsilon \in S_n$

$$\sum_{i=r}^{n}\sum_{j=s}^{n} \delta_{\varepsilon(i)j} = \sum_{i=r}^{n} |\{j \geqslant s : j = \varepsilon(i)\}| = |\{i \geqslant r : \varepsilon(i) \geqslant s\}|.$$

Hence we have the

COROLLARY *Given two permutations $\sigma, \pi \in S_n$, the. following two conditions are equivalent:*

(i) $\bigwedge_{a,b\in\mathfrak{M}_+} \sum_{i=1}^{n} a_i b_{\pi(i)} \leqslant \sum_{i=1}^{n} a_i b_{\sigma(i)}$

(ii) $\bigwedge_{1\leqslant r,\, s\leqslant n} |\{i \geqslant r : \pi(i) \geqslant s\}| \leqslant |\{i \geqslant r : \sigma(i) \geqslant s\}|$

Condition (ii) is nothing else, than another formulation of a condition given in [2] for the same problem. However in [2] deeper investigations in connection with (i) are carried through. For (i), which — by the way — is a thorough refinement of a rearrangement-inequality given already in [1], p. 261, it is shown for example that $\sum_{i=1}^{n} a_i(b_{\sigma(i)} - b_{\pi(i)}) \geqslant 0$ can be written as an expression that consists of definite summands that obviously are positive.

Remark 3. After (!) this paper had been typed already, the author learned, that meanwhile there appeared other interesting papers on Popoviciu's inequality. The reader should consult especially Pečarić's work ([5]) and the literature given there. Integral — analogues and extensions, as well as nice applications are given there. His method of proof differs from ours, in that it uses an integral-identity rather than Abel's inequality. In Theorem 8 he also notes an extension of (our) Theorem 1 to multilinear forms. It could be proved along the same lines, we used, by repeated application of Abel's inequality.

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Received 29.III.1983.

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