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A CLASS OF JACKSON — TYPE OPERATORS

by

LUCIANA LUPAŞ
(Sibiu)

1. Let $C(K)$, $K = [-1, 1]$, be the normed linear space of real functions defined and continuous on K; this space is considered normed by means of the uniform norm. We shall consider $\mathfrak{P}_n$ the set of all polynomials with real coefficients, of the degree $\leqslant n$, and $P_0, P_1 \ldots, P_n, \ldots$ the sequence of Legendre polynomials defined as

$$P_n(x) = \frac{1}{2^n\, n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n.$$

Likewise,

(1) $$x_{0n} < x_{1n} < \ldots < x_{nn}$$

are the roots of P_{n+1}.

If $L_n : C(K) \to \mathfrak{P}_n$ is the Lagrange interpolation operator with respect to (1), i.e.

(2) $$(L_n f)(x) = \sum_{k=0}^{n} f(x_{kn}) \frac{P_{n+1}(x)}{(x - x_{kn})\, P'_{n+1}(x_{kn})}$$

then $\|L_n\| \geqslant \frac{\ln n}{8\sqrt{\pi}}$, $n = 1, 2, \ldots$ ([3]). This means that there is at least one function f_0, $f_0 \in C(K)$, so that $\lim\limits_{n\to\infty} \|f_0 - L_n f_0\| \neq 0$.

The aim of this paper is to „modify" the operators L_n, $n = 1, 2, \ldots$ so that the new sequence of operators converges pointwise, on the whole space $C(K)$, to the identity operator.

Using the Christoffel-Darboux formula ([5]), that is

$$K_n(x, t) = \frac{n+1}{2} \cdot \frac{P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)}{x - t}$$

where

$$K_n(x, t) = \sum_{j=0}^{n} \frac{2j+1}{2} P_j(x) P_j(t), \tag{3}$$

from (2) we get

$$K_n(x, x_{kn}) = \frac{n+1}{2} \frac{P_{n+1}(x) P_n(x_{kn})}{x - x_{kn}}.$$

On the other hand $(1 - x^2)P'_{n+1}(x) = (n + 1)[P_n(x) - xP_{n+1}(x)]$ and, therefore

$$\frac{P_{n+1}(x)}{(x - x_{kn}) P'_{n+1}(x_{kn})} = \frac{2(1 - x_{kn}^2) K_n(x, x_{kn})}{(n + 1)^2 |P_n(x_{kn})|^2} = \frac{K_n(x, x_{kn})}{K_n(x_{kn}, x_{kn})}.$$

It follows that Lagrange operator $L_n : C(K) \to \mathfrak{P}_n$ can be represented as

$$(L_n f)(x) = \sum_{k=0}^{n} C_{kn}(f) P_k(x) \tag{4}$$

where the functionals $C_{jn} : C(K) \to \mathbf{R}$, $j = 0, 1, \ldots, n$, are defined by

$$C_{jn}(f) = \frac{2j+1}{(n+1)^2} \sum_{k=0}^{n} f(x_{kn}) \frac{(1 - x_{kn})^2 P_j(x_{kn})}{|P_n(x_{kn})|^2}. \tag{5}$$

From (4)

$$C_{kn}(f) = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) (L_n f)(x)\, dx,$$

that is

$$C_{kn}(P_j) = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) P_j(x) dx = \delta_{kj}, \qquad 0 \leqslant j, k \leqslant n. \tag{6}$$

2. Taking into consideration the representation (4) as a starting point, we define the operators $A_n : C(K) \to C(K)$, $n = 1, 2, \ldots$, by

$$(A_n f)(x) = \sum_{k=0}^{n} m_{kn} C_{kn}(f) P_k(x) \tag{7}$$

where the functionals C_{kn}, $k = 0, 1, \ldots, n$, had been defined by (5), while $M = \|m_{kn}\|$, $m_{kn} \in \mathbf{R}$, is a triangular matrix still undetermined. Now we give possibilities of choice of this matrix M so that $\|A_n\| \leqslant C_0$, $n = 1, 2, \ldots$, and moreover

$$\lim_{n \to \infty} \|f - A_n f\| = 0 \quad \text{for every} \quad f \in C(K).$$

In this case the numbers m_{kn} are called „convergence multipliers".

In the following we shall prove that there are matrices M for which

$$\|f - A_n f\| \leqslant C\omega\left(f; \frac{1}{n}\right), \qquad n = 1, 2, \ldots, f \in C(K), \tag{8}$$

where C is a constant independent of n and $\omega(f; .)$ is the modulus of continuity. Since $A_n(C(K)) \subseteq \mathfrak{P}_n$, the inequality (8) implies that among the operators of the type (7) there also are Jackson-type operators, i.e. polynomial operators which furnish us an approximation of the order of best approximation of continuous functions by polynomials.

LEMMA 1. *If $A_n : C(K) \to C(K)$ is defined by (7), then*

$$(A_n f)(x) = \sum_{k=0}^{n} D_n(x, x_{kn}; M) f(x_{kn}) \tag{9}$$

where

$$D_n(x, x_{kn}; M) = \frac{\tilde{K}_n(x, x_{kn}; M)}{K_n(x_{kn}, x_{kn})}$$

with $K_n(x, t)$ given by (3) and

$$\tilde{K}_n(x, t; M) = \sum_{j=0}^{n} m_{kn} \frac{2j+1}{2} P_j(x) P_j(t).$$

Moreover

$$(A_n P_j)(x) = m_{jn} P_j(x), \qquad j = 0, 1, \ldots, n, \tag{10}$$

and using the notation $e_k(t) = t^k$ we have

$$\begin{gathered}(A_n e_0)(x) = m_{0n} e_0(x), \qquad (A_n e_1)(x) = m_{1n} e_1(x), \\ (A_n e_2)(x) = m_{2n} e_2(x) + \frac{m_{0n} - m_{2n}}{3}.\end{gathered} \tag{11}$$

Proof. From (7) and (5)

$$(A_n f)(x) = \sum_{j=0}^{n} m_{jn} \frac{2j+1}{2} P_j(x) \sum_{k=0}^{n} \frac{P_j(x_{kn})}{K_n(x_{kn}, x_{kn})} f(x_{kn}) =$$

$$= \sum_{k=0}^{n} \frac{\tilde{K}_n(x, x_{kn}; M)}{K_n(x_{kn}, x_{kn})} f(x_{kn}) = \sum_{k=0}^{n} D_n(x, x_{kn}; M) f(x_{kn}).$$

At the same time, (6) and (7) imply (10), while the equalities $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ together with (10) prove (11).

Let us denote by $\mathcal{S}$ the set of all sequences $Q = (q_n)_{n=1}^{\infty}$ with the properties:

1) $q_n \in \mathfrak{P}_n$;

2) $q_n \geqslant 0$ on K;

3) $\int_{-1}^{1} q_n(t)\, dt = 1.$

If $q_n \in \mathcal{S}$ and $q_n(x) = \sum_{k=0}^{n} m_{kn} \frac{2k+1}{2} P_k(x)$ then

$$m_{jn} = m_j(q_n) := \int_{-1}^{1} q_n(t) P_j(t) dt,\ j = 0, 1, \ldots, n;\ n = 1, 2, \ldots, \tag{12}$$

and

$$m_0(q_n) = 1.$$

In the following we consider that m_{jn} are defined by (12), i.e. the matrix M is generated by an arbitrary sequence $Q = (q_n)$, $Q \in \mathcal{S}$. In order to underline the above mentioned dependence, we denote

$$(A_n f)(x) = (A_n f;\ Q)(x) := \sum_{k=0}^{n} m_k(q_n)\, C_{kn}(f)\, P_k(x) \tag{13}$$

where $m_k(q_n)$ is given by (12).

LEMMA 2. *If $Q = (q_n)$, $Q \in \mathcal{S}$, then the polynomial operators $A_n : C(K) \to C(K)$, $n = 1, 2, \ldots,$ defined by* (13) *are linear and positive, with the properties*

1) $A_n e_0 = e_0,\ A_n e_1 = m_1(q_n)\, e_1,\ A_n e_2 = m_2(q_n)\, e_2 + \frac{1 - m_2(q_n)}{3}$;

2) $\| A_n \| = 1,\ n = 1, 2, \ldots$;

3) $(A_n \Omega_2)(x) = [m_2(q_n) - 2m_1(q_n) + 1]\, e_2(x) + \frac{1 - m_2(q_n)}{3}$

where

$$\Omega_2(t, x) = (t - x)^2.$$

Proof. Taking into account a result established by T. KOORNWINDER [1], for $(x, y) \in (-1, 1)$ we have

$$P_k(x) P_k(y) = \frac{1}{\pi} \int_{-1}^{1} P_k(t) (|\varphi(x, y, t)|_+)^{-\frac{1}{2}} dt$$

where

$$\varphi(x, y, t) = (1 - x^2)(1 - y)^2 - (t - xy)^2$$

and $|z|_+ = \frac{z + |z|}{2}$. If $M = \| m_k(q_n) \|$, then for $x \in (-1, 1)$ we get

$$D_n(x, x_{kn};\ M) = \frac{1}{\pi K_n(x_{kn}, x_{kn})} \int_{-1}^{1} q_n(t) (|\varphi(x, x_{kn}, t)|_+)^{-\frac{1}{2}} dt \geqslant 0.$$

Also

$$D_n(-1, x_{kn};\ M) = \frac{q_n(-x_{kn})}{K_n(x_{kn}, x_{kn})} \geqslant 0$$

and

$$D_n(1, x_{kn};\ M) = \frac{q_n(x_{kn})}{K_n(x_{kn}, x_{kn})} \geqslant 0.$$

Hence

$D_n(x, x_{kn};\ M) \geqslant 0$ for every $x \in [-1, 1]$, $k = 0, 1, \ldots, n$.

By (9), $f \geqslant 0$ implies $(A_n f;\ Q) \geqslant 0$ on K.

The equalities 1) and 3) are consequences of (11) while the monotony of A_n as well as the fact that $A_n e_0 = e_0$ prove that $\| A_n \| = 1$, $n = 1, 2, \ldots$

THEOREM 3. *The convergence multipliers $m_j = m_j(q_n)$ generated by a sequence $Q = (q_n) \in \mathcal{S}$ verify the inequalities*

$$|m_j| \leqslant 1, \qquad j = 0, 1, \ldots, n;$$

$$|m_k - m_{k+1}| \leqslant (k + 1)(1 - m_1),\ k = 0, 1, \ldots, n - 1;$$

$$1 - m_j \leqslant \frac{j(j+1)}{2} (1 - m_1).$$

Proof. Since $|P_j(t)| \leqslant 1$, $P_j(1) = 1$, $t \in [-1, 1]$, from (10) we get $|m_j| = |(A_n P_j)(1)| \leqslant 1$. Simultaneously

$$P_k(t) - P_{k+1}(t) = (k + 1)(1 - t) R_k^{(1,0)}(t) \tag{14}$$

where by $R_k^{(\alpha,\beta)}$ we denote the Jacobi polynomial of the degree k, normalized by the condition $R_k^{(\alpha,\beta)}(1) = 1$. It is known that for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ we have $\| R_k^{(\alpha,\beta)} \| = 1$; therefore, (14) implies

$$D_k(t) = |P_k(t) - P_{k+1}(t)| \leqslant (k + 1)(1 - e_1(t)).$$

On the other hand

$$|m_k - m_{k+1}| = |(A_n P_k)(1) - (A_n P_{k+1})(1)| \leqslant (A_n D_k)(1) = (k + 1)(1 - m_1).$$

From

$$1-m_j=\sum_{\nu=0}^{j-1}(m_\nu-m_{\nu+1})\leqslant\sum_{\nu=0}^{j-1}|m_\nu-m_{\nu+1}|\leqslant(1-m_1)\sum_{\nu=0}^{j-1}(\nu+1)$$

one finds

$$1-m_j\leqslant\frac{j(j+1)}{2}(1-m_1).$$

Similarly it is proved that $|m_1|<1$.

THEOREM 4. *If* $Q\in\mathcal{S}$, $Q=(q_n)$ *and* $\lim\limits_{n\to\infty}m_1(q_n)=1$, *then for every* f, $f\in C(K)$, *we have.*

$$\lim_{n\to\infty}||f-(A_nf;\,Q)||=0.$$

Moreover

$$||f-(A_nf;\,Q)||\leqslant(1+\sqrt{2})\omega(f;\,\sqrt{1-m_1(q_n)}). \tag{15}$$

Proof. We have $m_0(q_n)=1$, $0\leqslant 1-m_2(q_n)\leqslant 3\,(1-m_1(q_n))$. This means that $\lim\limits_{n\to\infty}m_1(q_n)=1$ implies $\lim\limits_{n\to\infty}m_2(q_n)=1$ and from lemma 2 we conclude with $\lim\limits_{n\to\infty}||h-(A_nh;\,Q)||=0$, for every polynomial h of the degree two. According to T. POPOVICIU ([4]) and P. P. KOROVKIN ([2]) it follows that $\lim\limits_{n\to\infty}||f-(A_nf;\,Q)||=0$ for all continuous functions f, $f\in C(K)$.

By means of lemma 2 and theorem 3 we obtain

$$||(A_n\,\Omega_2;\,Q)||\leqslant 2(1-m_1(q_n)).$$

But for $f\in C(K)$ and $\delta>0$, the inequality

$$||f-(A_nf;\,Q)||\leqslant\left\{1+\frac{1}{\delta}\,||(A_n\,\Omega_2;\,Q)||^{\frac{1}{2}}\right\}\omega(f;\,\delta) \tag{16}$$

is verified. For $\delta=\sqrt{1-m_1(q_n)}$ we give (15).

3. In the following we shall stick to some sequences of linear positive operators, illustrating the generality of the method presented in the previous paragraph. For this, we consider the sequences of polynomials

$$Q_1=(a_n),\qquad Q^*=(b_n^*),\qquad Q_2=(c_{2n})$$

where

$$a_n(x)=\frac{n+1}{2}\left(\frac{1+x}{2}\right)^n$$

$$b_n(x)=\frac{(1-z_1^2)}{2^{1+d}}(1+x)^d\left[\frac{R_s^{(0,d)}(x)}{x-z_1}\right]^2, \tag{17}$$

$z_1=z_1(n)$ being the greatest root of the Jacobi polynomial

$$R_s^{(0,d)}\ \text{and}\ s=s(n)=1+\left[\frac{n}{2}\right],\ d=d(n)=n-2\left[\frac{n}{2}\right].$$

Likewise, we define

$$c_{2n}(x)=\frac{3\left(\frac{n+2}{2}\right)^2}{2(n^2+3n+3)}|R_n^{(2,0)}(x)|^2.$$

According to a formula given by Bateman (see [5]), if

$$\left(\frac{1+x}{2}\right)^n=\sum_{k=0}^n c_{kn}P_k(x)$$

then

$$\sum_{k=0}^n c_{kn}P_k(x)P_k(y)=\left(\frac{x+y}{2}\right)^nP_k\left(\frac{1+xy}{x+y}\right), \tag{18}$$

where

$$c_{kn}=\frac{(n!)^2(2k+1)}{(n-k)!\,(n+k+1)!}.$$

Using the above equalities, we have

$$a_n(x)=\sum_{j=0}^n m_{jn}\frac{2j+1}{2}P_j(x),\quad m_{jn}=m_j(a_n)=\frac{n!(n+1)!}{(n+k+1)!\,(n-k)!}.$$

Also, from (18) with $M=||m_j(a_n)||$ we find

$$\tilde{K}_n(x,t;\,M)=\frac{n+1}{2}\left(\frac{x+t}{2}\right)^nP_n\left(\frac{1+tx}{x+t}\right).$$

In conclusion, from (9)

$$(A_nf;\,Q)(x)=\frac{1}{(n+1)\,2^n}\sum_{k=0}^n\frac{(1-x_{kn})^2(x+x_{kn})\,P_n\left(\frac{1+x_{kn}\cdot x}{x+x_{kn}}\right)}{|P_n(x_{kn})|^2}f(x_{kn}).$$

For these operators $1-m_1(a_n)=\dfrac{2}{n+2}$; therefore

$$\lim_{n\to\infty}||f-(A_nf;\,Q_1)||=0.$$

for every f which belongs to $C(K)$.

Since

$$(A_n\Omega_2;\,Q_1)(x)=\frac{6x^2}{(n+2)(n+3)}+\frac{2(n+1)}{(n+2)(n+3)}$$

we have

$$\| (A_n \Omega_2 ; Q_1) \| \leqslant \frac{2}{n},$$

so that from (16) one obtains

$$\| f - (A_n f ; Q_1) \| < (1 + \sqrt{2})\, \omega\left(f ; \frac{1}{\sqrt{n}}\right), \qquad f \in C(K).$$

Let us consider the sequence $Q^* = (b_n^*)$ defined by (17). We have $m_1(b_n) = z_1(n)$ and

$$m_2(b_n) = 1 - 3(1 - z_1(n)) + \frac{3}{2}(1 - z_1(n))^2 + \frac{3}{2(n+3)}(1 - z_1^2(n)).$$

If $r_1(n)$ denotes the greatest root of the Legendre polynomial P_n, then

$$z_1(n) = r_1(s) \text{ for } n \text{ even}$$

and the equality

$$P_s(x) + P_{s+1}(x) = (1 + x) R_s^{(0,1)}(x)$$

furnishes us

$$z_1(n) < r_1(s + 1) \text{ for } n \text{ odd}.$$

Thus

$$z_1(n) < r_1(s + 1), \quad s = 1 + \left[\frac{n}{2}\right], \quad n = 1, 2, \ldots .$$

But (see [5], Theorem 6.21.1 and pag. 139) $z_1(n) \geqslant r_1(s)$, $n = 2m + 1$, and $r_1(s + 1) < \cos \frac{\pi}{2s + 3}$. This enables us to assert that there exists a sequence v_n, $v_n \in \left(\frac{1}{5}, \pi\sqrt{2}\right)$, such that

$$\sqrt{1 - m_1(b_n^*)} = \frac{v_n}{n}, \qquad n = 2, 3, \ldots .$$

From (15) we conclude that there exists a positive constant C, $C < (2 + \sqrt{2})\pi$ such that

$$\| f - (A_n f ; Q^*) \| \leqslant C\, \omega\left(f ; \frac{1}{n}\right), \qquad f \in C(K).$$

This proves that the polynomial operators $f \to (A_n f ; Q^*)$ are operators of the Jackson type.

It remains to estimate the order of approximation given by the operators A_{2n} generated by the sequence $Q_2 = (c_{2n})$.

Since

$$R_n^{(2,0)}(x) = \sum_{k=0}^{n} (n + k + 2)(n - k + 1) \frac{2k + 1}{2} P_k(x)$$

it is easy to see that

$$m_1(c_{2n}) = \frac{n^2 + 3n}{n^2 + 3n + 3}, \quad (A_{2n}\Omega_2 ; Q_2)(x) = \frac{3(2n + 1) - 3x^2((2n - 3)}{(2n + 3)(n^2 + 3n + 3)}.$$

By means of (16) we conclude with

$$\| f - (A_{2n} f ; Q_2) \| \leqslant (1 + \sqrt{3})\, \omega\left(f ; \frac{1}{n}\right),$$

i.e., $A_{2n} : f \to (A_{2n} f ; Q_2)$, $n = 1, 2, \ldots$, are of Jackson-type.

In a natural way, the following problem of optimal approximation may be considered: to find in $\mathcal{S}$ a sequence $Q_* = (q_n^*)$ of polynomials for which

$$(19) \qquad \Delta(Q_*) = \min_{Q \in \mathcal{S}} \Delta(Q) \qquad \text{where} \qquad \Delta(Q) = \sqrt{1 - m_1(q_n)}.$$

We have shown that there exists a sequence Q_* which satisfies (19), and moreover that $Q_* = Q^* = (b_n^*)$, b_n^* being defined in (17). It may be noted that the linear operators A_n furnishes us some summability methods for Lagrange interpolation (see [6]).

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Facultatea de mecanică
Str. I. Rațiu nr. 7
2400 *SIBIU, R. S. România*