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SOME GENERALIZATION OF CERTAIN O. HADŽIĆ CONTRACTION TYPE THEOREMS

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Abstract

At first in this paper we give fixed point theorems for mappings with a contractive iterate at the point. Theorems 2.1 and 2.2 give us possibility to get fixed point theorems in some generalized metric spaces. For example we can obtain fixed point theorems of I.A. Rus [13]—[15] type in (X, d), where $d: X^2 \to R_+^n$, $n \in N$ (see Remark 2.3). Next we prove O. Hadžić [5] and W. Netes [12] type theorems about fixed points of contractive mappings in locally convex and sequentially complete spaces. We use here the comparative method of Wazewski type and therefore our theorems are slighty general and stronger than the above cited results of O. Hadžić and W. Netes.

1. Auxiliary notes

Let $(G; +, 0, \leqq)$ be the algebraic system where $+$ and $\leqq$ denote a commutative binary relation on G and binary reflexive, antysymmetric and transitive relation on G, respectively.

Assume that the partial order has the additional property:

$(\leqq.\ 1)$ $g_1 \leqq g_2$ imply $g_1 + g \leqq g_2 + g$ for $g_1, g_2, g \in G$.

Take $G_+ = \{g \in G : g \geqq 0\}$ we will consider the algebraic system $(G_+; +, 0, \leqq, \downarrow)$, where $\downarrow$ denotes the limit operator on the set $S(G_+)$ of all non — increasing sequences with values from G_+. Let the limit —

operator have the additional properties:

($\downarrow$.1) $g_n \downarrow g,\ g_n' \downarrow g \Rightarrow g_n + g_n' \downarrow g + g,$

($\downarrow$.2) $g_n \downarrow g,\ g_n' \downarrow g',\ g_n \leqq g_n'$ for $n \in \mathbf{N} \Rightarrow g \leqq g',$

($\downarrow$.3) $(g_n) = (g) \Rightarrow g_n \downarrow g,$

($\downarrow$.4) $g_n \downarrow g \Rightarrow g_{n+1} \downarrow g.$

In the sequel $S_0(G_+)$ will denote the subclass of all sequences of $S(G_+)$ convergent to null.

Assume that $(G_+ ; \leqq)$ has the additional property

$$(\leqq.2)\ g_1 + g_2 \leqq g \Rightarrow g_1,\ g_2 \leqq g \text{ for } g_1, g_2, g \in G_+.$$

We say that a sequence $(g_n) \in G_+^{\mathbf{N}}$ is (o) — convergent to 0 and that 0 is the (o) — limit of this sequence, if there exists a sequence $(g_n') \in S_0(G_+)$ such that $g_n \leqq g_n'$ for $n \in \mathbf{N}$. We denote this fact by $0 = (o) - \lim_{n \in \mathbf{N}} g_n$ or $g_n \xrightarrow{o} 0$.

In this paper we consider also the second algebraic system $(P_+ ; +, 0, \leqq, \downarrow)$, which has the same properties as the fi first one but in addition $(P_+ ; \leqq)$ is a σ — conditionally complete lattice, i.e. every countable and upper bounded subset included in P_+ has the least upper bound.

Let now X be some nonempty set. The pair (X, ρ) ((X, d), respectively) is called M_0 — space if $\rho : X^2 \to G_+$ ($d : X^2 \to P_+$, respectively). We will use the following additional conditions

(M.1) $\rho(x, x) = 0$ for $x \in X$,

(M.2) $\rho(x, y) = 0$ implies $x = y$ for $x, y \in X$,

(M.3) $\rho(x, y) \leqq \rho(x, z) + \rho(z, y)$ for $x, y, z \in X$,

(M.4) $\rho(x, y) = \rho(y, x)$ for $x, y \in X$.

If M_0 — space has properties $(M . i_1) - (M . i_s)$, $s = 1, 2, 3, 4$, then (X, ρ) is $M_{i_1, \ldots, i_s}$ — space, $i_1 < i_2 < \ldots < i_s$, $s = 1, \ldots, 4$. The pair (X, ρ) is a generalized metric space, if (X, ρ) is $M_{1,2,3,4}$ — space.

In M_4—space we may introduce some class of convergent sequences. The sequence $(x_n) \in X^{\mathbf{N}}$ is convergent to $x \in X$ iff there exists a sequence $(q_n) \in S_0(G_+)$ and a positive integer n_0 such that $\rho(x_n, x) \leqq q_n$ for $n \geqq n_0$; we write then $x_n \to x$.

Let $T : X \to X$ and $w : X \to \mathbf{N}$ be given mappings. The sequence $(x_n)_{n \in N_0}$, $x_0 \in X$, $x_{n+1} = T^{w(x_n)} x_n$, $n = 0, 1, \ldots$ we call an w — orbit of T starting from x_0 or (T, w, x_0) — orbit.

Let (X, ρ) be M_4 — space. Then (T, w, x_0) — orbit is said a Cauchy w — orbit of T or a Cauchy (T, w, x_0) — orbit, if there exists a sequence $(q_n) \in S_0(G_+)$ such that for $n \in N_0$, $m \in \mathbf{N}$, $\rho(x_n, x_{n+m}) \leqq q_n$. We say that (X, ρ) is (T, w, x_0) — orbitally complete M_4 — space, if a Cauchy (T, w, x_0)—orbit is convergent to $x \in X$. Obviously, (X, ρ)

is (T, w)—orbitally complete if (X, ρ) is (T, w, x_0)—orbitally complete for any $x_0 \in X$.

The mapping $F : X \to X$ is (T, w, x_0)—orbitally continuous if it has the property: if (T, w, x_0)—orbit is convergent to some $x \in X$ then $Fx_n \to Fx$ and F is (T, w)—orbitally continuous if F is (T, w, x_0)—orbitally continuous for any $x_0 \in X$.

Remark. 1.1. It is well known that if (X, ρ) is a metric space and $T : X \to X$ is a Banach contraction, then T is for example uniformly continuous on X. On the other hand the above property do not occur for generalized contractions. In general the generalized contractions are at the most orbitally continuous.

Example 1.1. The function $T_1 : \mathbf{R}_+ \to \mathbf{R}_+$, $T_1 x = ax$ for irrational $x \geqq 0$ and $T_1 x = 0$ for rational $x \geqq 0$, $0 \leqq a < \frac{1}{5}$, is a generalized contraction: $|T_1 x - T_1 y| \leqq a(|x - y| + |x - T_1 x| + |y - T_1 y| + |x - T_1 y| + |y - T_1 x|)$, $x, y \in \mathbf{R}_+$, $x = 0$ is the unique fixed point of T_1 which is incontinuous on $[0, \infty)$ but T_1 is $(T_1, 1)$ —orbitally continuous on X.

The above definitions of (T, w)—orbitally completeness and (T, w)—orbitally continuity of F are slight modifications of the well — known corresponding definitions considered, e.g. in papers of L. Ćiric [3] and [4].

2. Fixed point theorems for mappings with a contractive iterate at the point

At first we will formulate the fixed point theorem for the Sehgal-type selfmappings on a non—void set on which two generalized metrics are given.

Let X be a nonempty set and let $\rho : X^2 \to G_+$, $d : X^2 \to P_+$ $T : X \to X$ and $w : X \to \mathbf{N}$ be given mappings. Assume that (X, d) is $M_{3,4}$—space and for $x, y \in X$ the inequality holds

$$(2.1)\quad d(T^{w(x)}x, T^{w(x)}y) \leqq a(d(x, y), d(x, T^{w(x)}x), d(y, T^{w(x)}y), d(x, T^{w(x)}y), d(y, T^{w(x)}x)),$$

where the function $a : P_+^5 \to P_+$ is non—decreasing and has the additional properties:

a) A is upper semicontinuous, $A(r) := a(r, r, 2r, r, 2r)$, $r \in P_+$,

b) $A_n(r_0) \xrightarrow{o} 0$ for some $r_0 \geqq 0$, where $A_n := A^n$, $n = 0, 1, \ldots$.

Suppose that

(2.2) there exist $x_0 \in X$ and $s \in \mathbf{N}$ that

a) $d(x_0, T^k x_0) \leqq r_0$ for $k \in \mathbf{N}$,

b) (X, ρ) is (T, w, x_0)—orbitally complete $M_{2,3,4}$—space,

c) T^s is (T, w, x_0)—orbitally continuous with respect to ρ.

(2.3) for (T, w, x_0)—orbit, (x_n), $\rho(x_n, T^k x_n) \leqq b(d(x_n, T^k x_n))$, $n \in \mathbf{N}_0$, $k \in \mathbf{N}$, where $b: P_+ \to G_+$ is non — decreasing and upper semicontinuous function such that $b(0) = 0$.

THEOREM 2.1. *If the assumptions* (2.1)—(2.3) *are fulfilled then there exists a fixed point $\bar{x}$ of T^s in X. Moreover if s is a divisor of m, $m = w(\bar{x})$ and (X, d) is M_2—space then $\bar{x}$ is a unique fixed point of T^s in $B_d(\bar{x}, r_0) = \{x \in X : d(\bar{x}, y) \leqq r_0\}$ and if $A_n(r) \xrightarrow{0} 0$ for each $r \in P_+$ then $\bar{x}$ is a unique fixed point of T in X.*

Proof. We denote $p_{n,1} = d(x_n, T^1 x_n)$, $x_n = T^{w(x_{n-1})} x_{n-1}$, $n = 1, 2, \ldots, p_n = \sup\{p_{n,1} : 1 \geqq 1\}$ and $\bar{p}_n = \sup\{p_k : k \geqq n\}$, $n \in \mathbf{N}$. Then $p_{n+1} \leqq \leqq A(p_n) \leqq \ldots \leqq A_{n+1}(r_0)$ (see the proof of Theorem 2.1 ([11]) and $\bar{p}_n \xrightarrow{0} 0$. Thus (x_n) is a Cauchy (T, w, x_0)—orbit in (X, d). From (2.3), (x_n) is a Cauchy (T, w, x_0)—orbit in (X, ρ). Thus from (2.2) c) there exists $\bar{x} \in X$ that $\rho(x_n, \bar{x}) \xrightarrow{0} 0$ as $n \to \infty$. On the other hand from (2.3), $\rho(T^s x_n, x_n) \xrightarrow{0} 0$ as $n \to \infty$. Obviously in $M_{2,3,4}$ —space (X, ρ) must be $\bar{x} = T^s\bar{x}$. If for some $1 \in \mathbf{N}$, $m = 1 \cdot s$, where $m = w(\bar{x})$, then also $\bar{x} = T^m\bar{x}$. If (X, d) is $M_{2,3,4}$—space and $d(\bar{x}, \bar{y}) \leqq r_0$ for $\bar{y} = T^m\bar{y}$, then $d(\bar{x}, \bar{y}) \leqq A(d(\bar{x}, \bar{y})) \leqq \leqq \ldots \leqq A_n(r_0)$, $n \in \mathbf{N}$, and thus $\bar{x} = \bar{y}$. If $\bar{x} = T^s\bar{x}$ is a unique fixed point of T^s in X then $\bar{x}$ is a unique fixed point of T in X.

Remark 2.1. In the case $G_+ = P_+ = \mathbf{R}_+$ we obtain the metrical version of Theorem 2.1. In particular the well—known result of Maia [2] is included in Theorem 2.1. For this purpose it is sufficient to take (X, ρ) and (X, d) as metric spaces, (X, ρ)-a complete $w(x) = 1$ for $x \in X$ and T — continuous with respect to ρ, $a(r_1, \ldots, r_5) = k \cdot r_1$, $r_i \in \mathbf{R}_+$, $i = 1, \ldots, 5$, $0 \leqq k < 1$ and $b = id_{\mathbf{R}_+}$.

THEOREM 2.2. *Let (X, d) be $M_{2,3,4}$ —space, $d: X^2 \to P_+$ and let $T: X \to X$ and $w: X \to \mathbf{N}$ be such that the assumptions* (2.1)—(2.2) *holds with $d = \rho$. Then there exists $\bar{x} \in B(x_0, r_0)$ such that $\bar{x} = T^s\bar{x}$. If s is a divisor of m, $m = w(\bar{x})$, then $\bar{x}$ is a unique fixed point of T in $B(\bar{x}, r_0) \cup \cup B(x_0, r_0)$. Moreover, if $A_n(2r_0) \xrightarrow{0} 0$ as $n \to \infty$ then also $T^n x_0 \to \bar{x}$ as $n \to \infty$ and $\bar{x}$ is a unique fixed point of T in $B(\bar{x}, 2r_0) \cup B(x_0, 2r_0)$.*

Proof. We have as in Theorem 2.1, $x_n \to \bar{x}$, $\bar{x} \in X$, where (x_n) is (T, w, x_0)—orbit. We get $d(\bar{x}, x_0) \leqq d(x_n, \bar{x}) + d(x_n, x_0) \leqq r_n + r_0$ for $n \in \mathbf{N}$, where $(r_n) \in S_0(P_+)$ and thus $d(\bar{x}, x_0) \leqq r_0$ i.e. $\bar{x} \in B(x_0, r_0)$. Obviously, $\bar{x} = T^s\bar{x}$ and if s is a divisor of m, then $\bar{x}$ is a unique fixed point of T^s in $B(\bar{x}, r_0)$ (see the proof of Theorem 2.1). If $s = 1$, then for $\bar{y} \in B(x_0, r_0)$, $\bar{y} = T\bar{y}$, $d(\bar{x}, \bar{y}) \leqq d(\bar{x}, x_n) + d(x_n, \bar{y}) \leqq A_n(r_0) + A_n(r_0)$ and consequently $\bar{x} = \bar{y}$ and $\bar{x}$ is a unique fixed point of T in $B(x_0, r_0) \cup \cup B(\bar{x}, r_0)$. In the case $A_n(2r_0) \xrightarrow{0} 0$ as $n \to \infty$ we have for $k \geqq m = w(\bar{x})$, $\bar{x} = T\bar{x}$, $k = 1 \cdot m + q$, $0 \leqq q < m$, $m = w(\bar{x})$, $t_n \leqq A(t_{n-m})$, where $t_n = = \sup\{t_k : k \geqq n\}$, $n = m, m+1, \ldots, t_k = d(T^k x_0, \bar{x})$, $k \in \mathbf{N}$. From the estimation $t_k \leqq A_1(\sup\{d(T^k x_0, \bar{x}) : 0 \leqq k < m\})$ we get $t_n \xrightarrow{0} 0$, where $n = = 1 \cdot m + q$, $0 \leqq q < m$ and (A_1) is the sequence as in (2.1) *b)*.

Remark 2.2. In the paper [11] we give some conditions on the function a which guarante the boundedness of the 1—orbit of T starting from $x_0 \in X$ and the (T, w, x_0)—orbitally continuity of T^m, where $m = w(\bar{x})$, $\bar{x} = \lim_{n \to \infty} T^{w(x_n)} x_n$.

a) Let the assumptions of Theorem 2.2 be fulfilled without (2.2) c) and $A_n(r) \xrightarrow{0} 0$ for any $r \in P_+$. Then there exists a unique fixed point of T in X. It is enough to prove (see the proof of Theorem 2.2 of [11]) that T^m is (T, w, x_0) —orbitally continuous, where $m = w(\bar{x})$, $\bar{x} = \lim T^{w(x_n)} x_n$. We have $r_{n+1} \leqq A(\bar{u}_n + \bar{w}_n + \bar{r}_n)$, $n \in \mathbf{N}$, where $\bar{r}_n = \sup\{d(T^m x_k, T^m\bar{x}) : k \geqq n\}$, $\bar{w}_n = \sup\{w_n : k \geqq n\}$, $w_k = d(x_k, T^m x_k)$, $\bar{u}_n \sup\{d(x_k, x_{k+1}) : k \geqq n\}$, $k, n \in \mathbf{N}$. Thus $\bar{x}$ is a unique fixed point of T^m in X and $\bar{x}$ is a unique fixed point of T in X.

b) Let (X, d) be $M_{2,3,4}$ —space, $d: X^2 \to \mathbf{R}_+$. Let the inequality (2.1) of Theorem 2.2 holds with $a(r_1, \ldots, r_5) = k_1 \cdot r_1 + \ldots + k_5 \cdot r_5$ for k_i, $r_i \in \mathbf{R}_+$, $i = 1, \ldots, 5$, Suppose that

(2.4) $$k_1 + k_2 + k_3 + k_4 + k_5 < 1 \text{ and } k_1 + 2k_3 + k_4 + k_5 < 1$$

and there exists $x_0 \in X$ such that (X, d) is (T, w, x_0)—orbitally complete. Then there exists a unique fixed point $\bar{x}$ of T in X and $T^n x_0 \to \to \bar{x}$ as $n \to \infty$ (see Remark 2.2 of [11]). For some less restrictive conditions on the comparative function a in the case $d: X^2 \to \mathbf{R}_+$ see J. Matkowski [9].

Remark 2.3. Let (X, d) be $M_{2,3,4}$ —space with $d: X^2 \to \mathbf{R}^n_+$, $n \in \mathbf{N}$. Suppose that the inequality (2.1) of Theorem 2.2 holds with $a(r_1, \ldots, r_5) = K_1 \cdot r_1 + K_2 \cdot r_2 + \ldots + K_5 \cdot r_5$, $r_i \in \mathbf{R}^n_+$, $i = 1, \ldots, 5$, where K_i, $i = 1, \ldots, 5$ are non —negative $n \times n$ — matrices such that

(2.5) *a)* there exist products

$$L_1 = (I - K_3 - K_4)^{-1}(I + K_2 + K_5)$$

$$L_2 = (I - K_3 - K_4)^{-1}(I + K_3 + K_5)$$

b) $r(K_3 + K_4) < 1$, $r(L_2) < 1$ and $r(L) < 1$,

where $L = K_1 + K_2 + K_3 + K_4 + K_5$ and $r(M)$ denotes the spectra radius of M, and there exists $x_0 \in X$ such that (X, d) is (T, w, x_0)—orbitally complete. Then there exists a unique fixed point $\bar{x}$ of T in X and $T^n x_0 \to \bar{x}$ as $n \to \infty$. (see Remark 2.2 of [11] and Remark 3.4 of (10]). For fixed point theorems for multivalued contractions and common fixed point theorems for contractions in such generalized metric space see papers of I.A. Rus [13]—[15].

3. Some generalization of O. Hadžič and W. Netes theorems

Here we consider the case of Hadžić-Netes type conditions putting on the class of operators fulfilling the generalized contraction conditions.

Let (X, d) be $M_{2,3,4}$—space defined by the family of semimetrics $d = (d_j)_{j\in J}$. Let $T: X \to X$ and $w: X \to \mathbf{N}$ be given mappings such that

(3.1) $$d_j(T^{w(x)}x, T^{w(x)}y) \leqq a_j(d(x, y), d(x, T^{w(x)}x), d(y, T^{w(x)}y), d(x, T^{w(x)}y), d(y, T^{w(x)}x)),$$

where $a_j: (\mathbf{R}_+^J)^5 \to \mathbf{R}_+$ is non—decreasing and

a) A_j is upper semicontinuous, $A_j(r) = a_j(r, r, 2r, r, 2r)$, $r \in \mathbf{R}_+^J$, $j \in J$,

b) $A_n^j(r_0) \to 0$ as $n \to \infty$ for some $r_0 \in \mathbf{R}_+^J$, where $A_0^j = id_{\mathbf{R}_+^J}$ and $A_{n+1}^j = A_j \circ A_n^j$, $n = 0, 1, \ldots,$

(3.2) there exist $x_0 \in X$ and $s \in \mathbf{N}$ that

a) $d_j(x_0, T^k x_0) \leqq r_0(j)$ for $k \in \mathbf{N}$ and $j \in J$,

b) (X, d) is (T, w, x_0)—orbitally complete

c) T^s is (T, w, x_0)—orbitally continuous.

THEOREM 3.1. *If the assumptions* (3.1)—(3.2) *are fulfilled then there exists* $\bar{x} \in B(x_0, r_0) = \{x \in X: p_j(x, x_0) \leqq r_0(j),\ j \in J\}$ *such that* $\bar{x} = T^s\bar{x}$. *If s is a divisor of m, $m = w(\bar{x})$, then $\bar{x}$ is a unique fixed point of T^s in $B(x_0, r_0)$ and if $s = 1$ then $\bar{x}$ is a unique fixed point of T in $B(x_0, r_0) \cup B(\bar{x}, r_0)$. Moreover, if $A_n(2r_0) \overset{0}{\to} 0$ as $n \to \infty$ then $T^n x_0 \to \bar{x}$ and $\bar{x}$ is a unique fixed point of T in $B(x_0, 2r_0) \cup B(\bar{x}, 2r_0)$.*

We may omit the proof that all assumptions of Theorem 2.2 are fulfilled.

Example 3.1. Let the function a_j, $j \in J$ of Theorem 3.1 have the form

(3.3) $$a_j(r_1, \ldots, r_5) = \sum_{i=1}^{5} k_i(j)\, r_{i, f_i(j)},\quad k_i(j) \geqq 0,$$

$i = 1, \ldots, 5$, $j \in J$, where $f_i: J \to J$, $r_i = (r_{i,j})_{j\in J}$, $i = 1, \ldots, 5$.

In this case we assume that the series

$$(1)\ \sum_{n=2}^{\infty} A_{n-1}(j, x_0) \quad \text{and} \quad (2)\ \sum_{n=2}^{\infty} \bar{A}_{n-1}(j, x_0)$$

are convergent, where

$$A_1(j, x_0) = \bar{A}_1(j, x_0) = \max_{w \leqq w(x_0)} d_j(T^w x_0, x_0),$$

$$A_n(j \cdot x_0) = \sum_{i_1=1}^{5} \ldots \sum_{i_{n-1}=1}^{5} \prod_{l=0}^{n-1} k_{i_{l+1}}(F_l(j, i_0, \ldots, i_1)) \cdot m(x_0, n)$$

$$\bar{A}_n(j, x_0) = \sum_{i_1=1}^{5} \ldots \sum_{i_{n-1}=1}^{5} \prod_{l=0}^{n-1} \bar{k}_{i_{l+1}}(F_l(j, i_0, \ldots, i_1)) \cdot m(x_0, n),\quad n = 2, 3, \ldots,$$

where

$$F_{l+1}(j, i_0, \ldots, i_{l+1}) = f_{i_l} \circ f_{i_{l-1}} \circ \ldots \circ f_{i_0})(j)$$

for

$$l = 0, 1, \ldots, f_{i_0} = id_J,\ \bar{k}_i = k_i \text{ for } i \in \{1, 4, 5\},\ \bar{k}_2 = 0,\ \bar{k}_3 = 2k_3$$

and

$$m(x_0, n) = \max_{w \leqq w(x_0)} \{d_{f^n(j)}(T^w x_0, x_0)\},\quad n \in \mathbf{N}.$$

Remark 3.1. If in Example 3.1 the series (1)—(2) are convergent then the comparative function $A = (A_j)_{j\in J}$ fulfils the assumptions of Theorem 3.1 (see also Remark 2.2 *b*)).

Remark 3.2. If in Theorem 3.1, (X, d) is a locally convex and sequentially complete Hausdorff space with the topology defined by the family of seminorms $d = (d_j)_{j\in J}$, $a_j(r_1, \ldots, r_5) = k_1 \cdot j\, r_1, f_1(j)$ and T is continuous then we obtain the stronger version of Theorem of [5]. In [5], O. Hadžić assume that $\sum_{n=1}^{\infty} n \cdot A_n < \infty$ $\left(\text{we assume only that } \sum_{n=1}^{\infty} A_n < \infty\right)$ and She prove the uniqueness of the fixed point of T in the set $B(x_0, S)$, $S = \sum_{n=1}^{\infty} A_n$ (we get the uniqueness in $B(x_0, S) \cup B(\bar{x}, S)$). It is easy to see that without the assumption that T is continuous we may to prove in this case that T^m, $m = w(\bar{x})$, is (T, w, x_0)—orbitally continuous and thus $\bar{x}$ is a unique fixed point of T^m in $B(\bar{x}, S)$.

Example 3.2. Let (X, d), T, w and x_0 be such as in Example 3.1. and the comparative function have the form (3.3). Suppose that

a) $$\forall_{j\in J} \exists_{n(j)\in N} \exists_{k_0(j)\in(0,1)} \left(n \geqq n(j) \Rightarrow \sum_{i_1=1}^{5} \ldots \sum_{i_{n-1}=1}^{5} \prod_{l=0}^{n-1} k_{i_{l+1}}(F_l(j, \ldots, i_1)) \leqq k_0(j)\right)$$

and

b) $$\forall_{j\in J} \exists_{c(j)} \exists_{p(j)\in(0,k_0^{-1}(j))} \forall_{n\in N_0} (k \leqq w(x_0) \Rightarrow d_{F_n}(j, i_0, \ldots, i_n)(x_0, T^k x_0) \leqq \leqq c(j) \cdot p^n(j)),$$

where $F_l(j, i_0, \ldots, i_l)$ is defined as in Example 3.1, $l = 0, 1, \ldots$ and $\bar{k}_i = k_i$ for $i \in \{1, 2, 4, 5\}$, $\bar{k}_3 = 2k_3$.

In this case

$$\bar{A}_n(j, x_0) = c(j)p^{n-1}(j) \sum_{i_1=1}^{5} \ldots \sum_{i_{n-1}=1}^{5} \prod_{l=0}^{n-1} \bar{k}_{i_{l+1}}(F_l(j, i_0, \ldots, i_1))$$

and

$$\sum_{n=1}^{\infty} \bar{A}_n(j, x_0) < \infty.$$

Remark 3.3. From Example 3.2 we get the generalization of theorem of W. Netes [12]. For this purpose it is sufficient to place (X, d) as the locally convex and sequentially complete Hausdorff space with the topology defined by the family of seminorms $d = (d_j)_{j \in J}$, T — continuous and $a_j(r_1, \ldots, r_5) = k(j)\, r_{1, f_1(j)}$, $j \in J$, $r_i \in \mathbf{R}_+^J$, $i = 1, \ldots, 5$.

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