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ON SOME INEQUALITIES INVOLVING CONVEX SEQUENCES

by

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Several inequalities connected with convex sequences are known. Let us mention those of Nanson [4], Steinig [6] and Ozeki (see [2]). In what follows, we shall use a simple method which allows the substitution of the conditions

(1) $$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n \geq 0, \text{ for } n \geq 1,$$

that characterize convex sequences, by

(2) $$m \leq \Delta^2 a_n \leq M, \text{ for } n \geq 1.$$

The obtained inequalities are not only more general, but, as we shall see on some examples, they strengthen the initial inequalities. The same method was used for functions in [5] and [1].

Before presenting the main results, let us give a representation theorem of sequences that satisfy (2). A more general result is given in [7], but we sketch here the proof which is simple in this particular case.

THEOREM 1. *The real sequence $(a_n)_{n\geq 1}$ satisfies* (2) *if and only if there is a sequence $(b_n)_{n\geq 1}$ which verifies*

(3) $$m \leq b_n \leq M, \text{ for } n > 2$$

such that

(4) $$a_n = \sum_{k=1}^{n} (n - k + 1) b_k, \textit{ for any } n \geq 1.$$

Proof. Any sequence $(a_n)_{n\geq 1}$ may be writen as (4) by taking

$$b_1 = a_1,\ b_n = a_n - \sum_{k=1}^{n-1} (n - k + 1) b_k, \text{ for } n \geq 2.$$

Because (4) implies

$$\Delta^2 a_n = b_{n+2} \tag{5}$$

the conditions (2) and (3) are equivalent.

The method which gives the results what follow is based on a simple remark: if the sequence $(a_n)_{n\geq 1}$ satisfies (2), then the sequences $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$, given by

$$c_n = a_n - m\frac{n^2}{2}, \quad d_n = M\frac{n^2}{2} - a_n \tag{6}$$

are convex. So, we can apply to these the results valid for convex sequences. To complete the proofs one requires only some simple calculations which we omit. As a matter of fact, we content ourselves to present for exemplification only two results: the first obtained from the inequality of Nanson [4], the second from that of Steinig [6].

THEOREM 2. *If the sequence* $(a_n)_{n\geq 1}$ *satisfies* (2), *then for any* $n \geq 1$ *hold*

$$\frac{2n+1}{6} m \leq \frac{a_1 + a_3 + \ldots + a_{2n+1}}{n+1} - \frac{a_2 + a_4 + \ldots + a_{2n}}{n} \leq \frac{2n+1}{6} M \tag{7}$$

and

$$\frac{n(2n+1)}{6} m \leq a_1 - a_2 + a_3 - a_4 + \ldots + a_{2n+1} - \frac{a_1 + a_3 + \ldots + a_{2n+1}}{n+1} \leq \frac{n(2n+1)}{6} M \tag{8}$$

Applications. Let $a_n = a^{n-1}$. Then $\Delta^2 a_n = a^{n-1}(a-1)^2$. If $a > 1$, then $m = (a-1)^2$ and (7) gives us

$$\frac{1 + a^2 + \ldots + a^{2n}}{n+1} - \frac{a + a^3 + \ldots + a^{2n-1}}{n} \geq \frac{2n+1}{6}(a-1)^2 \tag{9}$$

This is an improvement of an inequality of Wilson (see [3]).

From (8) it follows

$$1 - a + a^2 - a^3 + \ldots + a^{2n} - \frac{1 + a^2 + \ldots + a^{2n}}{n+1} \geq \frac{n(2n+1)}{6}(a-1)^2 \tag{10}$$

which is an improvement of an inequality of Steinig [6]. If $0 < a < 1$, then $m = 0$ and $M = (a-1)^2$, so that (7) and (8) give

$$0 \leq \frac{1 + a^2 + \ldots + a^{2n}}{n+1} - \frac{a + a^3 + \ldots + a^{2n-1}}{n} \leq \frac{2n+1}{6}(a-1)^2, \tag{11}$$

respectively

$$0 \leq 1 - a + a^2 - a^3 + \ldots + a^{2n} - \frac{1 + a^2 + \ldots + a^{2n}}{n+1} \leq \frac{n(2n+1)}{6}(a-1)^2. \tag{12}$$

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