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A METHOD OF GENERATING KANTOROVICH-TYPE OPERATORS

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1. Introduction

For a real-valued bounded function F defined on $I = [0, 1]$ the Bernstein operators are given by

$$\mathbf{B}_n F(x) = \sum_{k=0}^{n} F\left(\frac{k}{n}\right) p_{n,k}(x) \qquad (x \in I,\ n \in \mathbf{N})$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}.$$

Supposing that f is Lebesgue integrable on $[0,1]$, let F be the indefinite integral of f:

$$F(x) = \int_0^x f(t)dt.$$

The Kantorovich operators (see Lorentz [2]) are defined as

$$\mathbf{K}_n f(x) = \frac{d}{dx} \mathbf{B}_{n+1} F(x)$$

and they are used to approximate the function f in integral metrics. By a simple calculation $\mathbf{K}_n f(x)$ can be expressed as

$$\mathbf{K}_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt.$$

In the last few years many authors obtained interesting results concerning Kantorovich operators like: estimation of the L^1-norm of $f - \mathbf{K}_n f$ for f a function with bounded variation [1] and characterization of the functions f such that $||f - \mathbf{K}_n f||_{L^p} \le Cn^{-\alpha}$[4], [7].

This paper presents a method which, starting from a net A_n of operators defined on a space of continuous functions, attach a net $\tilde{A}_n$ defined on a space of integrable functions such that some convergence properties are transposed.

2. Main Results

Let X be a σ-locally compact space (i.e. a topological space which is σ-compact and locally compact), μ a regular Borel measure on X.

We need the following generalization of a result of W. Orlicz given for $X = [a, b]$ and μ the Lebesgue measure (see [2]).

THEOREM 1. *Suppose that Φ is a real or complex measurable function on the product space $X \times X$, $M > 0$ and*

$$\int_X |\Phi(x, t)|\, d\mu(t) \le M \qquad \text{a.e. } [\mu],\ x \in X$$

$$\int_X |\Phi(x, t)|\, d\mu(x) \le M \qquad \text{a.e. } [\mu],\ t \in X.$$

Then for a fixed $p \ge 1$ the operator $S : L^p(\mu) \to L^p(\mu)$

$$Sf(x) = \int_X \Phi(x, t) f(t)\, d\mu(t),\ f \in L^p(\mu)$$

is well-defined and $||Sf||_p \le M\, ||f||_p$.

Proof. X is a σ-finite measure space and by Fubini's theorem we have

$$\int_X \int_X |\Phi(x, t)| \cdot |f(t)|^p\, d\mu(t)\, d\mu(x) \le M\, ||f||_p^p$$

and therefore

$$\int_X |\Phi(x, t)| \cdot |f(t)|^p\, d\mu(t) < \infty \ \text{ a.e. } [\mu]\ x \in X.$$

Using Hölder's inequality, we obtain

$$\int_X |\Phi(x, t)| \cdot |f(t)|\, d\mu(t) = \int_X |\Phi(x, t)|^{1/p} \cdot |f(t)| \cdot |\Phi(x, t)|^{1/q}\, d\mu(t) \le$$

$$\le \left(\int_X |\Phi(x, t)| \cdot |f(t)|^p\, d\mu(t)\right)^{1/p} \left(\int_X |\Phi(x, t)|\, d\mu(t)\right)^{1/q} \le$$

$$\le M^{1/q} \left(\int_X |\Phi(x, t)| \cdot |f(t)|^p\right)^{1/p} < \infty$$

$(1/p + 1/q = 1$; if $p = 1$ we do not need the last chain of inequalities — we may take $M^{1/q} = 1)$.

Hence the integral defining Sf exists a.e. $[\mu]$.

$$\int_X |Sf(x)|^p\, d\mu(x) \le M^{p/q} \cdot M ||f||_p^p = M^p ||f||_p^p$$

and the proof is complete.

COROLLARY 1. *Suppose that each member of a net of functions Φ_n defined on $X \times X$ satisfies the conditions of the above theorem and, in addition, the corresponding net $S_n f \to f$ strongly for all the elements f of the set $\mathcal{M} \subseteq L^p(\mu)$ which is dense in $L^p(\mu)$. Then $S_n f \to f$, for all f in $L^p(\mu)$.*

Proof. Using the linearity of S_n we have for all $f, g \in L^p(\mu)$

$$||S_n f - f||_p \le ||S_n(f - g)||_p + ||S_n g - g||_p + ||g - f||_p \le$$
$$\le (M + 1)\, ||f - g||_p + ||S_n g - g||_p.$$

For a given $\varepsilon > 0$, choose $g \in \mathcal{M}$ such that $||f - g||_p < \varepsilon$. Then for sufficiently large n we obtain $||S_n f - f|| \le (M + 2)\varepsilon$ and the proof is complete.

Let $B_n \subseteq X$ be a net of compact sets, such that $0 < \mu(B_n) < \infty$ for all n in a directed set D. Suppose that for each n we have a finite partition $B_{n,k}$, $k \in J_n$ a finite set containing $|J_n|$ elements, such that $\mu(B_{n,k}) = \frac{1}{|J_n|}\mu(B_n)$, $k \in J_n$; let $a_{n,k}$ be a fixed point of each $B_{n,k}$. Denote by $C(X)$ the set of all real or complex continuous functions on X and let $p_{n,k} \in C(X)$, $k \in J_n$, $n \in D$.

We associate to these elements the following operator on $C(X)$

$$A_n f(x) = \sum_{k \in J_n} f(a_{n,k})\, p_{n,k}(x).$$

If $f \in L^1_{\text{loc}}(\mu)$ (i.e. f is integrable on each compact set of X) define

$$\tilde{A}_n f(x) = \sum_{k \in J_n} \frac{1}{\mu(B_{n,k})} \left(\int_{B_{n,k}} f d\mu\right) p_{n,k}(x).$$

THEOREM 2. *If there is a positive constant M such that*

(i) $$\frac{|J_n|}{\mu(B_n)}\int_{B_n}|p_{n,k}|d\mu \le M,\ k\in J_n,\quad n\in D$$

(ii) $$\sum_{k\in J_n}|p_{n,k}(x)|\le M,\quad n\in D,\quad x\in X$$

then, for a fixed $p\ge 1$, $\tilde{A}_n$ is a bounded linear operator on $L^p(\mu)$ and $\|\tilde{A}_n f\|_p\le \le M\|f\|_p$, for all f in $L^p(\mu)$.

Proof. We define $\Phi_n(x,t)=\frac{|J_n|}{\mu(B_n)}p_{n,k}(x)$ for t in $B_{n,k}$, k in J_n, and $\Phi_n(x,t)=0$ for t in the complement of B_n. Since

$$\tilde{A}_n f(x)=\int_X \Phi_n(x,t)f(t)\,d\mu(t)$$

we have only to check the conditions of theorem 1.

$$\int_X|\Phi_n(x,t)|\,d\mu(x)=\frac{|J_n|}{\mu(B_n)}\int_{B_n}|p_{n,k}|d\mu\le M$$

$$\int_X|\Phi_n(x,t)|\,d\mu(t)\le\frac{|J_n|}{\mu(B_n)}\sum_{k\in J_n}\int_{B_{n,k}}|p_{n,k}(x)|d\mu(t)=\sum_{k\in J_n}|p_{n,k}(x)|\le M \text{ q.e.d.}$$

With the notation preceding the theorem 2 we obtain

COROLLARY 2. *Suppose that in addition to the hypotheses of the theorem 2, the net $\tilde{A}_n f\to f$ strongly for all elements f of a dense set $\mathcal{M}\subseteq L^p(\mu)$. Then $\tilde{A}_n f\to f$, for all f in $L^p(\mu)$.*

Proof. The proof is similar to that of Corollary 1.

A very important case appears when X is a compact metric space.

THEOREM 3. *Let X be a compact metric space, and* $\lim_{n\in D}\operatorname{diam}(B_{n,k})=0$ *uniformly for k. If the conditions* (i) *and* (ii) *of the theorem 2 are satisfied and $A_n f\to f$ in the sup-norm of $C(X)$ for all f in $C(X)$ then $\tilde{A}_n f\to f$ strongly, for all f in $L^p(\mu)$.*

Proof. We first show that $\|\tilde{A}_n f-A_n f\|_{L^\infty(X)}\to 0$ for all f in $C(X)$. For arbitrary $\varepsilon>0$ and f in $C(X)$ there is $\delta>0$ such that for all x, x' we have $|f(x)-f(x')|<\varepsilon$ provided that $d(x,x')<\delta$. Since diam $(B_{n,k})<\delta$ for sufficiently large n we obtain for x in X

$$|\tilde{A}_n f(x)-A_n f(x)|\le\sum_{k\in J_n}\frac{1}{\mu(B_{n,k})}\int_{B_{n,k}}|f(t)-f(a_{n,k})|d\mu(t)\cdot|p_{n,k}(x)|\le M.$$

Since $\mu(X)<\infty$ it follows that $\|\tilde{A}_n f-A_n f\|_{L^p(X)}\to 0$, hence $\|\tilde{A}_n f-f\|_{L^p(X)}\to 0$ for all f in $C(X)$. To complete the proof, it remains only to apply Corollary 2 for $\mathcal{M}=C(X)$.

3. Some Applications

In the presence of a net of operators $\tilde{A}_n$ satisfying the conditions of Corollary 2, we formulate the following criterion of compactness.

THEOREM 4. *A bounded set $\mathcal{M}\subset L^p(X)$ is relatively compact iff* $\lim_{n\in D}\|\tilde{A}_n f-f\|_p=0$, *uniformly for all $f\in\mathcal{M}$.*

Proof. It is a direct consequence of the general theorem of Mazur and Phillips (see [6]) since $\tilde{A}_n$ are of finite rank, hence compact operators. q.e.d.

Example 1. $X=[0,1]$, $n\in D=\mathbf{N}$, $J_n=\{0,1,\ldots,n\}$

$$B_n=[0,1],\ B_{n,k}=\left[\frac{k}{n+1},\frac{k+1}{n+1}\right],\ a_{n,k}=\frac{k}{n}.$$

Then $A_n=\mathbf{B}_n$ (Bernstein operator)

$\tilde{A}_n=\mathbf{K}_n$ (Kantorovich operator)

The conditions of theorem 3 are quite easy to check.

Example 2. $X=[0,1]^m$, $n=(n_1,n_2,\ldots,n_m)\in D=\mathbf{N}^m$, $J_n=\{(k_1,k_2,\ldots,k_m)\in\mathbf{N}^m:0\le k_i\le n_i,\ i=1,2,\ldots,m\}$.

$$B_n=[0,1]^m,\ B_{n,k}=\prod_{i=1}^m\left[\frac{k_i}{n_i+1},\frac{k_i+1}{n_i+1}\right],\ a_{n,k}=\left(\frac{k_1}{n_1},\ldots,\frac{k_m}{n_m}\right)\in B_{n,k}$$

$$p_{n,k}(x)=\binom{n_1}{k_1}\cdots\binom{n_m}{k_m}x_1^{k_1}(1-x_1)^{n_1-k_1}\cdots x_m^{k_m}(1-x_m)^{n_m-k_m}.$$

A_n represents the m-dimensional Bernstein operator. It is easy to check the conditions of theorem 3 such that for the corresponding m-dimensional Kantorovich operators $\tilde{A}_n$ we obtain

$$\lim_{n\to\infty}\|\tilde{A}_n f-f\|_{L^p}=0$$

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