MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 13, Nº 1, 1984, pp. 15 – 22

DUALITY IN MATHEMATICAL PROGRAMMING IN COMPLEX SPACE. CONVERSE THEOREMS

by

DOREL I. DUCA

(Cluj-Napoca)

1. Introduction

Consider the pair of dual problems:

Primal problem

$$(P) \qquad \begin{cases} \operatorname{Re} f(z) \to \min \\ z \in M \\ g(z) \in S, \end{cases}$$

and

Dual problem

$$(D) \qquad \begin{cases} \operatorname{Re}\,[f(z) - \langle g(z), u\rangle] \to \max \\ (z, u) \in M \times S^* \\ \overline{\nabla_z f(z)} + \nabla_{\bar z} f(z) - \overline{\nabla_z g(z) u} - \nabla_{\bar z} g(z)\bar u = 0, \end{cases}$$

where M is an open nonempty set in $\mathbf{C}^n$, S is a closed convex cone in $\mathbf{C}^m$ and $f : M \to \mathbf{C}$ and $g : M \to \mathbf{C}^m$ are differentiable functions on M.

In this paper some converse duality theorems are given.

2. Notation and Preliminaries

Let $\mathbf{C}^n(\mathbf{R}^n)$ denote the n-dimensional complex (real) vector space with Hermitian (Euclidean) norm $\|\cdot\|$, $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x = (x_j),\ x_j \geqq 0,\ j = 1, \ldots, n\}$ the nonnegative orthant of $\mathbf{R}^n$, and $\mathbf{C}^{m\times n}$ the set of $m \times n$ complex matrices. If A is a matrix or a vector, the $\bar A$, A^T, A^H denote its complex conjugate, transpose and conjugate transpose respectively. For $z = (z_j)$, $w = (w_j) \in \mathbf{C}^n$: $\langle z, w\rangle = w^H z$ denotes the inner product of z and w and $\operatorname{Re} z = (\operatorname{Re} z_j) \in \mathbf{R}^n$ denotes the real part of z.

The nonempty set S in $\mathbf{C}^m$ is a polyhedral cone if it is a finite intersection of closed half-spaces in $\mathbf{C}^m$, each containing 0 in its boundary, i.e. there exists a natural number q and q points $u^1, \ldots, u^q$ in $\mathbf{C}^m$ such that

$$S = \bigcap \{H(u^k) : k \in \{1, \ldots, q\}\},$$

where

$$H(u^k) = \{v \in \mathbf{C}^m : \operatorname{Re} \langle v, u^k\rangle \geqq 0\} \text{ for all } k \in \{1, \ldots, q\}.$$

If $S = \bigcap \{H(u^k) : k \in \{1, \ldots, q\}\}$ is a polyhedral cone in $\mathbf{C}^m$ and $v \in S$, then $S(v)$ is defined to be the intersection of those closed half-spaces $H(u^k)$, $k \in \{1, \ldots, q\}$ which include v in their boundaries, i.e.

$$S(v) = \bigcap \{H(u^k) : k \in E\}, \text{ where } E = \{k \in \{1, \ldots, q\} : \operatorname{Re} \langle v, u^k\rangle = 0\}.$$

The polar S^* of a nonempty set S in $\mathbf{C}^m$ is defined by

$$S^* = \{u \in \mathbf{C}^m : v \in S \Rightarrow \operatorname{Re} \langle v, u\rangle \geqq 0\}.$$

We shall make use of the following [2]: If S and T are closed convex cones in $\mathbf{C}^m$, then $(S \times T)^* = S^* \times T^*$, $(S^*)^* = S$, $(S \cap T)^* = \operatorname{cl}(S^* + T^*)$, where cl denotes closure.

We shall also need the following results:

LEMMA 1 [10]. *Let S be a polyhedral cone in $\mathbf{C}^m$ and let $v \in S$. Then $u \in (S(v))^*$ iff $[u \in S^*$ and $\operatorname{Re} \langle u, v\rangle = 0]$.*

LEMMA 2 [2]. *Let $A \in \mathbf{C}^{m\times n}$, $b \in \mathbf{C}^m$ and $S \subseteq \mathbf{C}^m$ be a polyhedral cone. Then the following are equivalent:*

(i) $Az = b$, $z \in S$ is consistent

(ii) $A^H u \in S^* \Rightarrow \operatorname{Re} \langle u, b\rangle \geqq 0$. (Farkas'lemma)

Let M be an open set in $\mathbf{C}^n$ and let $z^0 \in M$. A function $f : M \to \mathbf{C}$ is said to be differentiable at z^0 if there exist $2n$ complex numbers $A_1(z^0), \ldots, A_n(z^0)$, $B_1(z^0), \ldots, B_n(z^0)$ and a function $h(\cdot\,; z^0) : M \to \mathbf{C}$ continuous at z^0 and vanishing at this point

$$\lim_{z\to z^0} h(z; z^0) = h(z^0; z^0) = 0$$

such that

$$f(z) - f(z^0) = \sum_{j=1}^{n} A_j(z^0)(z_j - z_j^0) + \sum_{j=1}^{n} B_j(z^0)(\bar z_j - \bar z_j^0) + \|z - z^0\|\, h(z; z^0) \text{ for all } z \in M.$$

If for $z = x + iy \in M$ $(x, y \in \mathbf{R}^n)$ we have $f(z) = u(x, y) + iv(x, y)$, then the function f is differentiable at $z^0 = x^0 + iy^0 \in M$ if and only if the functions u and v are differentiable at $(x^0, y^0) \in \mathbf{R}^{2n}$. If we consider the formal differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left[\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right] \text{ and } \frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left[\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right]$$

for all $j \in \{1, \ldots, n\}$, we obtain that

$$A_j(z^0) = \frac{\partial f}{\partial z_j}(z^0), \quad B_j(z^0) = \frac{\partial f}{\partial \bar z_j}(z^0) \text{ for all } j \in \{1, \ldots, n\}.$$

If $f : M \to \mathbf{C}$ is differentiable at $z^0 \in M$ then

$$\nabla_z f(z^0) = \left(\frac{\partial f}{\partial z_1}(z^0), \ldots, \frac{\partial f}{\partial z_n}(z^0)\right)^T \in \mathbf{C}^n,$$

$$\nabla_{\bar z} f(z^0) = \left(\frac{\partial f}{\partial \bar z_1}(z^0), \ldots, \frac{\partial f}{\partial \bar z_n}(z^0)\right)^T \in \mathbf{C}^n.$$

The function $g = (g_k) : M \to \mathbf{C}^m$ is said to be differentiable at $z^0 \in M$ if for any $k \in \{1, \ldots, m\}$ the function g_k is differentiable at z^0. If $g = (g_k) : M \to \mathbf{C}^m$ is differentiable at $z^0 \in M$, then

$$\nabla_z g(z^0) = (\nabla_z g_1(z^0) \ldots \nabla_z g_m(z^0)) \in \mathbf{C}^{n\times m},$$

$$\nabla_{\bar z} g(z^0) = (\nabla_{\bar z} g_1(z^0) \ldots \nabla_{\bar z} g_m(z^0)) \in \mathbf{C}^{n\times m}.$$

The function $g = (g_k) : M \to \mathbf{C}^m$ is said to be differentiable on M if it is differentiable at any $z \in M$. If $g : M \to \mathbf{C}^m$ is a differentiable function at $z^0 \in M$, then

$$(1) \qquad \nabla_z \bar g(z^0) = \overline{\nabla_{\bar z} g(z^0)} \quad \text{and} \quad \nabla_{\bar z} \bar g(z^0) = \overline{\nabla_z g(z^0)}.$$

Let M be an open nonempty set in $\mathbf{C}^n$, let $z^0 \in M$, let S be a closed convex cone in $\mathbf{C}^m$, and let $f : M \to \mathbf{C}^m$ be a differentiable function at z^0. The function f is said to be:

a) *convex with respect to S at z^0* if for any $z \in M$,

$$f(z) - f(z^0) - [\nabla_z f(z^0)]^T (z - z^0) - [\nabla_{\bar z} f(z^0)]^T (\bar z - \bar z^0) \in S;$$

b) *pseudoconvex with respect to S at z^0* if

$$\left.\begin{array}{l} z \in M \\ [\nabla_z f(z^0)]^T (z - z^0) + [\nabla_{\bar z} f(z^0)]^T (\bar z - \bar z^0) \in S \end{array}\right\} \Rightarrow f(z) - f(z^0) \in S;$$

c) *quasiconvex with respect to S at z^0* if

$$\left.\begin{array}{l} z \in M \\ f(z^0) - f(z) \in S \end{array}\right\} \Rightarrow [\nabla_z f(z^0)]^T (z^0 - z) + [\nabla_{\bar z} f(z^0)]^T (\bar z^0 - \bar z) \in S;$$

d) *concave (pseudoconcave, quasiconcave) with respect to S at z^0* if $-g$ is convex (pseudoconvex, quasiconvex respectively) with respect to S at z^0;

e) *convex (concave, pseudoconvex, pseudoconcave, quasiconvex, quasiconcave) with respect to* S *on* M if g is differentiable on M, M is convex and g is convex (concave, pseudoconvex, pseudoconcave, quasiconvex, quasiconcave respectively) with respect to S at any $z^0 \in M$.

When reffering to the objective function of a programming problem in complex space, convexity of real part is of interest. Let M be an open nonempty set in $\mathbb{C}^n$, let T be a closed convex cone in $\mathbf{R}^m$, and let $f: M \to \mathbb{C}^m$ be a differentiable function at $z^0 \in M$. The function f is said to be:

a) *with convex (concave, pseudoconvex, pseudoconcave, quasiconvex, quasiconcave) real part with respect to* T *at* z^0 if f is convex (concave, pseudoconvex, pseudoconcave, quasiconvex, quasiconcave respectively) with respect to $CT = \{v \in \mathbb{C}^m : \operatorname{Re} v \in T\}$ at z^0;

b) *with convex (concave, pseudoconvex, pseudoconcave, quasiconvex, quasiconcave) real part with respect to* T *on* M if f is convex (concave, pseudoconvex, pseudoconcave, quasiconvex, quasiconcave respectively) with respect to $CT = \{v \in \mathbb{C}^m : \operatorname{Re} v \in T\}$ on M.

From Theorem 4 and Corollary 2 of [5] it follows

THEOREM 1. *Let M be a nonempty open set in $\mathbb{C}^n$, let $z^0 \in M$, let S be a polyhedral cone in $\mathbb{C}^m$ and let $f: M \to \mathbb{C}$ be a differentiable function at z^0. Let $A, B \in \mathbb{C}^{m\times n}$ let $b \in \mathbb{C}^m$ and let $g: \mathbb{C}^n \to \mathbb{C}^m$ defined by the formula $g(z) = Az + B\bar{z} + b$ for all $z \in \mathbb{C}^n$. If z^0 is a local solution of the problem*

$$\begin{cases} \operatorname{Re} f(z) \to \min \\ z \in M \\ g(z) \in S, \end{cases}$$

then there exists $v \in S^*$ such that

$$\overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0) - \overline{\nabla_z g(z^0) v} - \nabla_{\bar{z}} g(z^0)\, \bar{v} = 0$$

$$\operatorname{Re} \langle g(z^0), v\rangle = 0.$$

3. Results

THEOREM 2. *Let M be a nonempty open set in $\mathbb{C}^n$, let S be a polyhedral cone in $\mathbb{C}^m$, and let $f: M \to \mathbb{C}$ and $g: M \to \mathbb{C}^m$ be differentiable functions on M. Let (z^0, u^0) be a solution of Dual problem (D). Assume that f has pseudoconvex real part with respect to $\mathbf{R}_+$ at z^0 and g is quasiconcave with respect to $S(g(z^0))$ at z^0. If there exists an open set $U \subseteq \mathbb{C}^m$ containing u^0 and a function $h: U \to M$ differentiable on U such that*

$$h(u^0) = z^0 \tag{2}$$

$$\overline{\nabla_z f(h(u))} + \nabla_{\bar{z}} f(h(u)) - \overline{\nabla_z g(h(u)) u} - \nabla_{\bar{z}} g(h(u))\, \bar{u} = 0 \tag{3}$$

for all $u \in U$, then z^0 is a solution of Primal problem (P) and $\operatorname{Re} f(z^0) = \operatorname{Re} F(z^0, u^0)$, where $F(z, u) = f(z) - \langle g(z), u\rangle$ for all $(z, u) \in M \times \mathbb{C}^m$.

Proof. We define the function $\varphi: U \to \mathbb{C}$ by formula

$$\varphi(u) = \langle g(h(u)), u\rangle - f(h(u)), \text{ for all } u \in U.$$

From (3) it follows that, for each $u \in (U \cap S^*)$, the point $(h(u), u) \in M \times S^*$ is a feasible solution of Dual problem (D). Since $(z^0, u^0) = (h(u^0), u^0)$ is a solution of Dual problem (D) we have that u^0 is a solution of the problem

Minimize $\operatorname{Re} \varphi(u)$ *subject to* $u \in U$, $\psi(u) \in S^*$,

where $\psi: \mathbb{C}^m \to \mathbb{C}^m$ is defined by $\psi(u) = u$ for all $u \in \mathbb{C}^m$. Then by Theorem 1, it follows that there exists a point $v \in (S^*)^*$ such that

$$\overline{\nabla_u \varphi(u^0)} + \nabla_{\bar{u}} \varphi(u^0) - \overline{\nabla_u \psi(u^0) v} - \nabla_{\bar{u}} \psi(u^0)\, \bar{v} = 0, \tag{4}$$

$$\operatorname{Re} \langle \psi(u^0), v\rangle = 0. \tag{5}$$

Since

$$\begin{aligned} \nabla_u \varphi(u) = {} & [\nabla_u h(u)]\, \nabla_z g(h(u))\, \bar{u} + [\nabla_u \bar{h}(u)] \nabla_{\bar{z}} g(h(u))\, \bar{u} - \\ & - [\nabla_u h(u)]\, \nabla_z f(h(u)) - [\nabla_u \bar{h}(u)]\, \nabla_{\bar{z}} f(h(u)), \\ \nabla_{\bar{u}} \varphi(u) = {} & [\nabla_{\bar{u}} h(u)] \nabla_z g(h(u))\, \bar{u} + [\nabla_{\bar{u}} \bar{h}(u)]\, \nabla_{\bar{z}} g(h(u))\, \bar{u} - \\ & - [\nabla_{\bar{u}} h(u)] \nabla_z f(h(u)) - [\nabla_{\bar{u}} \bar{h}(u)]\, \nabla_{\bar{z}} f(h(u)) + g(h(u)), \\ \nabla_u \psi(u) = {} & I,\ \nabla_{\bar{u}} \psi(u) = 0 \text{ for all } u \in U, \end{aligned}$$

from (2) we have

$$\begin{aligned} \nabla_u \varphi(u^0) = {} & [\nabla_u h(u^0)] \nabla_z g(z^0)\, \bar{u}^0 + [\nabla_u \bar{h}(u^0)] \nabla_{\bar{z}} g(z^0)\, \bar{u}^0 - \\ & - [\nabla_u h(u^0)]\, \nabla_z f(z^0) - [\nabla_u \bar{h}(u^0)]\, \nabla_{\bar{z}} f(z^0), \\ \nabla_{\bar{u}} \varphi(u^0) = {} & [\nabla_{\bar{u}} h(u^0)]\, \nabla_z g(z^0)\, \bar{u}^0 + [\nabla_{\bar{u}} \bar{h}(u^0)] \nabla_{\bar{z}} g(z^0) \bar{u}^0 - \\ & - [\nabla_{\bar{u}} h(u^0)]\, \nabla_z f(z^0) - [\nabla_{\bar{u}} \bar{h}(u^0)] \nabla_{\bar{z}} f(z^0) + g(z^0), \\ \nabla_u \psi(u^0) = {} & I,\ \ \nabla_{\bar{u}} \psi(u^0) = 0, \text{ where } I \text{ is the identity map.} \end{aligned} \tag{6}$$

From (1), (2), (6) and (4) we deduce $v = g(z^0)$. Since S is a polyhedral cone we have $(S^*)^* = S$, and hence $g(z^0) = v \in S$. Therefore z^0 is a feasible solution of Primal problem (P). Since $\psi(u^0) = u^0$ and $v = g(z^0)$, from (5) we deduce

$$\operatorname{Re} \langle u^0, g(z^0)\rangle = 0. \tag{7}$$

By Lemma 1, $u^0 \in (S(g(z^0)))^*$, because $u^0 \in S^*$ and $\operatorname{Re} \langle u^0, g(z^0)\rangle = 0$.

Let now z be a feasible solution of Primal problem (P); then

$$g(z) \in S \subseteq S(g(z^0)). \tag{8}$$

Obviously

$$-g(z^0) \in S(g(z^0)). \tag{9}$$

Since $S(g(z^0))$ is a polyhedral cone, from (8) and (9) we obtain

$$g(z) - g(z^0) \in S(g(z^0)).$$

By the quasiconcavity of g with respect to $S(g(z^0))$ at z^0

$$g(z) - g(z^0) \in S(g(z^0)) \text{ implies}$$

$$[\nabla_z g(z^0)]^T (z - z^0) + [\nabla_{\bar{z}} g(z^0)]^T (\bar{z} - \bar{z}^0) \in S(g(z^0)).$$

Since $u^0 \in (S(g(z^0)))^*$ we have

$$\mathrm{Re}\, \langle [\nabla_z g(z^0)]^T (z - z^0) + [\nabla_{\bar{z}} g(z^0)]^T (\bar{z} - \bar{z}^0), u^0 \rangle \geqq 0,$$

or, equivalently

(10) $$\mathrm{Re}\, \langle \overline{\nabla_z g(z^0) u^0 + \nabla_{\bar{z}} g(z^0) \bar{u}^0}, z - z^0 \rangle \geqq 0.$$

Now

$$\mathrm{Re}\, \langle \overline{\nabla_z f(z^0) + \nabla_{\bar{z}} f(z^0)}, z - z^0 \rangle = (\text{by (2) and (3)})$$

$$= \mathrm{Re}\, \langle \overline{\nabla_z g(z^0) u^0 + \nabla_{\bar{z}} g(z^0) \bar{u}^0}, z - z^0 \rangle \geqq (\text{by (10)})$$

$$\geqq 0,$$

which by pseudoconvexity of real part of f with respect to $\mathbf{R}_+$ at z^0 gives $\mathrm{Re} f(z) \geqq \mathrm{Re} f(z^0)$. Thus z^0 is a solution of Primal problem (P). Now, it follows easily that $\mathrm{Re}\, F(z^0, u^0) = \mathrm{Re}\, [f(z^0) - \langle g(z^0), u^0 \rangle] = \mathrm{Re} f(z^0)$, because (7) holds. This completes the proof.

THEOREM 3. *Let M be an open nonempty set in $\mathbf{C}^n$, let S be a polyhedral cone in $\mathbf{C}^m$ and let $f: M \to \mathbf{C}$ and $g: M \to \mathbf{C}^m$ be differentiable functions on M. Let $z^0 \in M$ be a feasible solution of Primal problem (P), let f be with quasiconcave real part with respect to $\mathbf{R}_+$ at z^0 and let g be pseudoconvex with respect to S at z^0. If Dual problem (D) has no feasible solution, then z^0 cannot be a local solution of Primal problem (P) (hence, not a solution of Primal problem (P)).*

Proof. Because Dual problem (D) has no feasible solution, we have that the system

(11) $$\begin{cases} \overline{\nabla_z g(z^0)} u + \nabla_{\bar{z}} g(z^0) \bar{u} = \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0) \\ u \in S^*, \end{cases}$$

has no solution $u \in \mathbf{C}^m$. System (11) can be written as

$$\begin{cases} [\overline{\nabla_z g(z^0)} \quad \nabla_{\bar{z}} g(z^0)] \begin{bmatrix} u \\ v \end{bmatrix} = \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0) \\ \begin{bmatrix} u \\ v \end{bmatrix} \in (S^* \times \overline{S^*}) \cap Q, \end{cases}$$

where $Q = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbf{C}^{2m} : v = \bar{u} \right\}$. Then, by Farkas'lemma (Lemma 2), the system

$$\begin{cases} \begin{bmatrix} [\nabla_z g(z^0)]^T \\ [\nabla_{\bar{z}} g(z^0)]^H \end{bmatrix} w \in ((S^* \times \overline{S^*}) \cap Q)^* \\ \mathrm{Re}\, \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0), w \rangle < 0, \end{cases}$$

has a solution $w = w^0 \in \mathbf{C}^n$. Since $((S^* \times \overline{S^*}) \cap Q)^* = S \times \bar{S} + Q^* = S \times \bar{S} + \{(t, p) \in C^{2m} : p = -\bar{t}\}$, it follows that there exists $(w^0, s^0, r^0, t^0) \in \mathbf{C}^n \times S \times S \times \mathbf{C}^m$ such that

(12) $$[\nabla_z g(z^0)]^T w^0 = s^0 + t^0$$

(13) $$[\nabla_{\bar{z}} g(z^0)]^H w^0 = \bar{r}^0 - \bar{t}^0$$

(14) $$\mathrm{Re}\, \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0), w^0 \rangle < 0.$$

Conjugating (13) and adding to (12), gives

(15) $$[\nabla_z g(z^0)]^T w^0 + [\nabla_{\bar{z}} g(z^0)]^T \bar{w}^0 = s^0 + r^0 \in S,$$

because $s^0, r^0 \in S$ and S is polyhedral cone.

Assume that z^0 is a local solution of Primal problem (P). Since M is open, $z^0 \in M$, and z^0 is a local solution of Primal problem (P), it follows that there exists $r_0 \in \mathbf{R}$, $r_0 > 0$ such that

$$B(z^0; r_0) = \{z \in \mathbf{C}^n : ||z - z^0|| < r_0\} \subseteq M$$

and

(16) $$\mathrm{Re} f(z^0) \leqq \mathrm{Re} f(z) \text{ for all } z \in X \cap B(z^0; r_0),$$

where $X = \{z \in M : g(z) \in S\}$. Let $r_1 = \min\{r_0, r_0/||w^0||\}$ and r be a real number in $]0, r_1[$. Then

(17) $$z^0 + rw^0 \in B(z^0; r_0) \subseteq M.$$

From (14) we have

$$\mathrm{Re}\, \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0), rw^0 \rangle < 0.$$

Since S is cone, from (15) it follows that

$$[\nabla_z g(z^0)]^T (rw^0) + [\nabla_{\bar{z}} g(z^0)]^T \overline{(rw^0)} \in S.$$

By the quasiconcavity of real part of f with respect to $\mathbf{R}_+$ at z^0

$$\mathrm{Re}\, \langle \overline{\nabla_z f(z^0)} + \nabla_{\bar{z}} f(z^0), rw^0 \rangle < 0$$

implies

(18) $$\mathrm{Re} f(z^0 + rw^0) < \mathrm{Re} f(z^0),$$

and by the pseudoconvexity of g with respect to S at z^0

$$[\nabla_z g(z^0)]^T (rw^0) + [\nabla_{\bar{z}} g(z^0)]^T (\overline{rw^0}) \in S$$

implies

(19) $$g(z^0 + rw^0) - g(z^0) \in S.$$

Since S is polyhedral cone, and $g(z^0) \in S$, from (19) we have

(20) $$g(z^0 + rw^0) \in S.$$

From (17), (18) and (20) it follows that $z^0 + rw^0 \in X \cap B(z^0 ; r_0)$ and $\operatorname{Re} f(z^0 + rw^0) < \operatorname{Re} f(z^0)$, which contradicts (16). Hence z^0 cannot be a local solution of Primal problem (P) (hence, not a solution of Primal problem (P)).

REFERENCES

[1] Abrams, R. A., *Nonlinear Programming in Complex Space: Sufficient Conditions and Duality*, J. Math. Anal. Appl., **38**, 3, 619–632 (1972).

[2] Ben–Israel, A., *Linear Equations and Inequalities on Finite Dimensional, Real or Complex, Vector Space: A Unified Theory*, J. Math. Anal. Appl., **27**, 367–389 (1969).

[3] Craven, B. D. and Mond, B., *Converse and Symmetric Duality in Complex Nonlinear Programming*, J. Math. Anal. Appl., **37**, 617–626 (1972).

[4] Duca, D. I., *Constraint Qualifications in Nonlinear Programming in Complex Space*, Studia Univ. Babeş-Bolyai, Math., **23**, 61–65 (1978).

[5] Duca, D. I., *Necessary Optimality Criteria in Nonlinear Programming in Complex Space with Differentiability*, Mathematica-L'Analyse numérique et la théorie de l'approximation, **9**, 2, 163–179 (1980).

[6] Duca, D. I., *On Sufficient Optimality Conditions in Nonlinear Programming in Complex Space*, Mathematica, **22**(45), 2, 263–267 (1980).

[7] Duca, D. I., *Programare matematică în domeniul complex*, Teză de doctorat, Universitatea din Cluj-Napoca, Cluj-Napoca, 1981.

[8] Levinson, N., *Linear Programming in Complex Space*, J. Math. Anal. Appl., **14**, 44–62 (1966).

[9] Mahajan, D. G. and Vartak, M. N., *Duality for Generalized Problems in Complex Space*, Bull. Austral. Math. Soc., **17**, 11–22 (1976).

[10] Mond, B. and Craven, B. D., *Sufficient Optimality Conditions for Complex Programming with Quasi-Concave Constraints*, Math. Operationsforsch. Statist., Ser. Optimization, **8**, 4, 445–453 (1977).

[11] Parida, J., *On Converse Duality in Complex Nonlinear Programming*, Bull. Austral. Math. Soc., **13**, 421–427 (1975).

Received 12.XII.1983.

University of Cluj-Napoca
Faculty of Mathematics
R–3400, Cluj-Napoca
ROMANIA