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A CONVEX DUAL FOR THE GENERAL PROBLEM OF GEOMETRIC PROGRAMMING

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1. Let us consider the general geometric programming problem

$$(PG) \qquad \inf\{p_0(x) : p_k(x) \leq 1,\ k = 1, \ldots, r;\ x > 0\},$$

where p_k $(k = 0, 1, \ldots, r)$, are posynomials, i.e. for all $x = (x_1, \ldots, x_n) > 0$ we have

$$p_k(x) = \sum_{i \in I_k} u_i(x) = \sum_{i \in I_k} c_i x_1^{a_{i1}} x_2^{a_{i2}} \ldots x_n^{a_{in}},$$

with the coefficients $c_i > 0$ $(i \in I_k,\ k = 0, 1, \ldots, r)$ and the exponents $a_{ij} \in \mathbf{R}$ $(j = 1, \ldots, n;\ i \in I_k,\ k = 0, 1, \ldots, r)$. Here I_k $(k = 0, 1, \ldots, r)$ is the set of the indices i corresponding to the terms of the posynomial p_k, hence $I_h \cap I_s = \emptyset$ if $h \neq s$ and $\bigcup\{I_k : k = 0, 1, \ldots, r\} = \{1, \ldots, m\}$, m being the number of all terms of the posynomials p_k, $k = 0, 1, \ldots, r$. Denote as usually:

$$I_0 = \{1, 2, \ldots, m_0\},$$
$$I_1 = \{m_0 + 1, m_0 + 2, \ldots, m_1\},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$I_r = \{m_{r-1} + 1, m_{r-1} + 2, \ldots, m_r = m\}.$$

The dual of the standard geometric programming problem (PG) ([3], [4]) is the problem

$$(PGD1) \qquad \sup\left\{v_1(y) : y \geq 0;\ \sum_{i=1}^{m} a_{ij} y_i = 0,\ j = 1, \ldots, n;\ \sum_{i \in I_0} y_i = 1\right\},$$

where

$$v_1(y) = \prod_{i=1}^{m} \left(\frac{c_i}{y_i}\right)^{y_i} \prod_{k=1}^{r} (\lambda_k(y))^{\lambda_k(y)},$$

$$\lambda_k(y) = \sum_{i \in I_k} y_i, \qquad k = 1, \ldots, r,$$

for all $y = (y_1, \ldots, y_m) \geq 0$. The restrictions of the dual problem $(PGD1)$ are linear and the objective function of the problem $(PGD1)$ is logarithmic-concave.

In this paper we will associate to a problem (PG) an other dual problem in which the objective function is concave and the restrictions are linear.

We recall (see [1]) that two optimization problems

(P) $$\min \{f(x) : x \in S\}$$

and

(D) $$\max \{f^*(y) : y \in S^*\}$$

are said to be dual, if, under certain conditions that will be specified, the following properties are satisfied:

I. For all $x \in S$ and $y \in S^*$ we have $f(x) \geq f^*(y)$.

II. If the problem (P) has an optimal solution x^0, then the problem (D) has an optimal solution y^0 and $f(x^0) = f^*(y^0)$.

III. If the problem (D) has an optimal solution y^0, then the problem (P) has an optimal solution x^0 and $f^*(y^0) = f(x^0)$.

The results in this paper are directly or indirectly based on the following lemma.

LEMMA 1. *Let m be a natural number, $u_1, \ldots, u_m$ a system of real positive numbers, $y_1, \ldots, y_m$ a system of real nonnegative numbers. Then the inequality*

$$\sum_{i=1}^{m} u_i \geq \ln \prod_{i=1}^{m} \left(\frac{eu_i}{y_i}\right)^{y_i} \tag{1}$$

holds. In (1) equality holds if and only if $u_i = y_i$ for all $i = 1, \ldots, m$.

The *proof* is given in [2].

For what follows we define $(1/y)^y = 1$ if $y = 0$.

Let $y_1 \geq 0, \ldots, y_m \geq 0$ and $\lambda(y) = \sum_{i=1}^{m} y_i$. Then the following lemma is true.

LEMMA 2. *Let $u_1 > 0, \ldots, u_m > 0$, $y_1 \geq 0, \ldots, y_m \geq 0$. Then*

$$\sum_{i=1}^{m} u_i \geq \ln \left[\lambda(y) \prod_{i=1}^{m} \left(\frac{eu_i}{y_i}\right)^{\frac{y_i}{\lambda(y)}}\right], \tag{2}$$

the equality being valid if and only if $u_i = (y_i)/\lambda(y)$ for all $i = 1, \ldots, m$.

Proof. Apply Lemma 1 to the systems of real numbers $u_1, \ldots, u_m$ and $(y_1)/\lambda(y), \ldots, (y_m)/\lambda(y)$.

2. Let us consider the general problem of geometric programming (PG). As dual, we associate the problem:

$(PGD2)$ $$\sup \left\{v(y) : y \geq 0 ;\ \sum_{i=1}^{m} a_{ij} y_i = 0,\ j = 1, \ldots, n\right\},$$

where

$$v(y) = \ln \left[\prod_{i=1}^{m} \left(\frac{ec_i}{y_i}\right)^{y_i} \prod_{k=1}^{r} (\lambda_k(y))^{\lambda_k(y)}\right] - \sum_{k=1}^{r} \lambda_k(y),$$

with

$$\lambda_k(y) = \sum_{i \in I_k} y_i,\ k = 1, \ldots, r.$$

LEMMA 3. *The dual function*

$$v(y) = \ln \left[\prod_{i=1}^{m} \left(\frac{ec_i}{y_i}\right)^{y_i} \prod_{k=1}^{r} (\lambda_k(y))^{\lambda_k(y)}\right] - \sum_{k=1}^{r} \lambda_k(y),$$

for all $y = (y_1, \ldots, y_m) \geq 0$, is concave.

Proof. The dual function v can be written as follows:

$$v(y) = \sum_{i=1}^{m_0} y_i \ln \frac{ec_i}{y_i} + \sum_{k=1}^{r} f_k(y),$$

where

$$f_k(y) = \sum_{i=m_{k-1}+1}^{m_k} y_i \ln c_i - \sum_{i=m_{k-1}+1}^{m_k} y_i \ln y_i + \left(\sum_{i=m_{k-1}+1}^{m_k} y_i\right) \ln \left(\sum_{i=m_{k-1}+1}^{m_k} y_i\right),$$

for all $y = (y_1, \ldots, y_m) \geq 0$, $k = 1, \ldots, r$.

The function

$$\sum_{i=1}^{m_0} y_i \ln \frac{ec_i}{y_i}, \text{ for all } y_1, \ldots, y_{m_0} \geq 0$$

is concave, since it is a sum of m_0 concave functions.

We investigate the concavity of the functions f_k, $k = 1, \ldots, r$.

We have

$$\frac{\partial^2 f_k}{\partial y_i\, \partial y_j}(y) = 1 \Bigg/ \left(\sum_{i=m_{k-1}+1}^{m_k} y_i\right) - \delta_{ij}/y_i,$$

for all $i, j = m_{k-1} + 1, \ldots, m_k$, $k = 1, \ldots, r$, $y > 0$.

Let $y = (y_1, \ldots, y_m) > 0$, $z = (z_1, \ldots, z_m) \in \mathbf{R}^m$ and $k \in \{1, \ldots, r\}$. From Cauchy-Buneakovski-Schwarz's inequality we get

$$\left[\sum_{i=m_{k-1}+1}^{m_k} z_i\right]^2 \le \left(\sum_{i=m_{k-1}+1}^{m_k} \frac{z_i^2}{y_i}\right)\left(\sum_{i=m_{k-1}+1}^{m_k} y_i\right),$$

or

$$\frac{\left[\sum_{i=m_{k-1}+1}^{m_k} z_i\right]^2}{\sum_{i=m_{k-1}+1}^{m_k} y_i} \le \sum_{i=m_{k-1}+1}^{m_k} \frac{z_i^2}{y_i}.$$

Then

$$z^T \nabla^2 f_k(y) z = \sum_{i=m_{k-1}+1}^{m_k} \sum_{j=m_{k-1}+1}^{m_k} \left[\frac{1}{\sum_{s=m_{k-1}+1}^{m_k} y_s} - \frac{\delta_{ij}}{y_i}\right] z_i z_j = \frac{\sum_{i=m_{k-1}+1}^{m_k} \sum_{j=m_{k-1}+1}^{m_k} z_i z_j}{\sum_{i=m_{k-1}+1}^{m_k} y_i} -$$

$$- \sum_{i=m_{k-1}+1}^{m_k} \frac{z_i^2}{y_i} = \frac{\left[\sum_{i=m_{k-1}+1}^{m_k} z_i\right]^2}{\sum_{i=m_{k-1}+1}^{m_k} y_i} - \sum_{i=m_{k-1}+1}^{m_k} \frac{z_i^2}{y_i} \le 0.$$

Hence for all $k \in \{1, \ldots, r\}$, the function f_k is concave. Since the dual function v is sum of concave functions, it is also concave.

We denote by

$$S = \{x \in \mathbf{R}^n : x > 0 ;\ p_k(x) \le 1,\ k = 1, \ldots, r\}$$

and

$$S^* = \left\{y \in \mathbf{R}^m : y \ge 0 ;\ \sum_{i=1}^{m} a_{ij} y_i = 0,\ j = 1, \ldots, n\right\}.$$

We observe that $0 \in \mathbf{R}^m$ is an element of the set S^*, hence the dual problem $(PGD2)$ is always consistent.

We will show that the problems (PG) and $(PGD2)$ are dual in the above-mentioned sense.

THEOREM 1. *Let $x \in S$ and $y \in S^*$. Then $p_0(x) \ge v(y)$, where equality holds if and only if*

$$(3) \qquad y_i = \begin{cases} u_i(x), & \text{if } i \in I_0 \\ \lambda_k(y)\, u_i(x), & \text{if } i \in I_k,\ k = 1, \ldots, r. \end{cases}$$

Proof. By Lemma 1 it follows that

$$(4) \qquad p_0(x) = \sum_{i \in I_0} u_i(x) \ge \ln \prod_{i \in I_0} \left(\frac{e u_i(x)}{y_i}\right)^{y_i},$$

the equality being valid if and only if

$$u_i(x) = y_i \text{ for all } i \in I_0.$$

By lemma 2 it follows that

$$(5) \qquad 1 \ge p_k(x) = \sum_{i \in I_k} u_i(x) \ge \ln\left[\lambda_k(y) \prod_{i \in I_k} \left(\frac{e u_i(x)}{y_i}\right)^{\frac{y_i}{\lambda_k(y)}}\right]$$

for all $k \in \{1, \ldots, r\}$; the equality is true if and only if

$$u_i(x) = \frac{y_i}{\lambda_k(y)} \text{ for all } i \in I_k,\ k = 1, \ldots, r.$$

Since $\lambda_k(y) \ge 0$, from (5) we obtain

$$(6) \qquad \lambda_k(y) \ge \lambda_k(y)\, p_k(x) = \lambda_k(y) \sum_{i \in I_k} u_k(x) \ge \ln\left[(\lambda_k(y))^{\lambda_k(y)} \prod_{i \in I_k} \left(\frac{e u_i(x)}{y_i}\right)^{y_i}\right],$$

for all $k \in \{1, \ldots, r\}$.

Summing the inequalities (4) and (6) for $k \in \{1, \ldots, r\}$, we obtain

$$(7) \qquad p_0(x) + \sum_{k=1}^{r} \lambda_k(y) \ge \ln\left[\prod_{i=1}^{m} \left(\frac{e u_i(x)}{y_i}\right)^{y_i} \prod_{k=1}^{r} (\lambda_k(y))^{\lambda_k(y)}\right],$$

or, equivalently

$$(8) \qquad p_0(x) \ge \ln\left[\prod_{i=1}^{m} \left(\frac{e u_i(x)}{y_i}\right)^{y_i} \prod_{k=1}^{r} (\lambda_k(y)^{\lambda_k(y)}\right] - \sum_{k=1}^{r} \lambda_k(y).$$

Since $u_i(x) = c_i x_1^{a_{i1}} \ldots x_n^{a_{in}}$ with $c_i > 0$ and $a_{ij} \in \mathbf{R}$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$, inequality (8) becomes

$$(9) \qquad p_0(x) \ge \ln\left[\prod_{i=1}^{m} \left(\frac{e c_i}{y_i}\right)^{y_i} x_1^{\sum_{i=1}^{m} a_{i1} y_i} \ldots x_n^{\sum_{i=1}^{m} a_{in} y_i} \prod_{k=1}^{r} (\lambda_k(y))^{\lambda_k(y)}\right] - \sum_{k=1}^{r} \lambda_k(y).$$

From (9) it follows that $p_0(x) \ge v(y)$. This completes the proof of the first part of the theorem.

Now we assume that $p_0(x) = v(y)$. Then inequality (4) and hence all inequalities (6) must become equalities. By lemma 1, inequality (4) becomes equality if and only if $u_i(x) = y_i$ for all $i \in I_0$. By lemma 2, the inequalities in (6) become equalities if and only if $u_i(x) = (y_i)/\lambda_k(y)$ for all $i \in I_k$, $k = 1, \ldots, r$. Hence the equality $p_0(y) = v(y)$ implies the equalities (3).

Now suppose that $x \in S$ and $y \in S^*$ satisfy equality (3). Then by lemma 1 it follows that (4) becomes equality. By lemma 2 each inequality from the second part of relation (6) becomes equality. Since

$$p_k(x) = \sum_{i \in I_k} u_i(x) = \sum_{i \in I_k} \frac{y_i}{\lambda_k(y)} = \frac{\lambda_k(y)}{\lambda_k(y)} = 1$$

for all $k = 1, \ldots, r$, it follows that each of the two inequalities in (6) becomes equality, hence $p_0(x) = v(y)$. This completes the proof of the weak duality theorem.

COROLLARY 1. *We have*

$$\inf \{p_0(x) : x \in S\} \geq \sup \{v(y) : y \in S^*\}.$$

Proof. Apply Theorem 1.

COROLLARY 2. *If the points $x^0 \in S$ and $y^0 \in S^*$ satisfy the equality $p_0(x^0) = v(y^0)$, then x^0 is an optimal solution of the problem (PG) and y^0 is an optimal solution of the problem (PGD2).*

Proof. By Corrolary 1, we have

$$p_0(x^0) \geq \inf \{p_0(x) : x \in S\} \geq \sup \{v(y) : y \in S^*\} \geq v(y^0).$$

Since $p_0(x^0) = v(y^0)$, we obtain

$$p_0(x^0) = \inf \{p_0(x) : x \in S\},$$
$$v(y^0) = \sup \{v(y) : y \in S^*\},$$

and hence x^0 is an optimal solution of the problem (PG) and y^0 an optimal solution of the problem $(PGD2)$.

THEOREM 2. *If the problem (PG) is superconsistent (i.e., there exists $x^1 = (x_1^1, \ldots, x_n^1) > 0$ such that $p_k(x^1) < 1$ for all $k = 1, \ldots, r$) and if x^0 is an optimal solution of the problem (PG), then the problem (PGD2) has an optimal solution y^0 and*

$$p_0(x^0) = v(y^0).$$

Proof. We make a change of variables in problem (PG) by letting

$$x_j = e^{z_j}, \; j = 1, \ldots, n.$$

The transformed problem (PG) is

$$(PG)_z \qquad \inf \{g_0(z) : g_k(z) \leq 1, \; k = 1, \ldots, r\},$$

where

$$g_k(z) = \sum_{i \in I_k} c_i e^{\sum_{j=1}^n a_{ij} z_j}, \quad k = 0, 1, \ldots, r,$$

with $c_i > 0$, $a_{ij} \in \mathbf{R}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. For each $k \in \{0, \ldots, r\}$ the function g_k is convex because it is a positive linear combination of convex functions (exponential functions); hence the transformed program $(PG)_z$ is convex.

Let x^0 be an optimal solution of the problem (PG). Since the problem (PG) is superconsistent, it follows that the transformed problem $(PG)_z$ is superconsistent, too. The transformed problem $(PG)_z$ has an optimal solution z^0 that satisfies the conditions

$$(10) \qquad z_j^0 = \ln x_j^0, \qquad j = 1, \ldots, n,$$

$$(11) \qquad g_k(z^0) \leq 1, \qquad k = 1, \ldots, r.$$

Then, by Karush-Kuhn-Tucker's theorem, there are multipliers μ_k, $k = 1, \ldots, r$ so that

i) $\mu_k \geq 0$, $k = 1, \ldots, r$,

ii) $\mu_k [g_k(z^0) - 1] = 0$, $k = 1, \ldots, r$

iii) $\nabla g_0(z^0) + \sum_{k=1}^r \mu_k \nabla(g_k(z^0) - 1) = 0.$

The condition *iii)* is equivalent with

$$(12) \qquad \sum_{i \in I_0} c_i a_{ij} e^{\sum_{j=1}^n a_{ij} z_j^0} + \sum_{k=1}^r \mu_k \left(\sum_{i \in I_k} c_i a_{ij} e^{\sum_{j=1}^n a_{ij} z_j^0} \right) = 0, \qquad j = 1, \ldots, n.$$

Noting

$$(13) \qquad y_i^0 = \begin{cases} c_i e^{\sum_{j=1}^n a_{ij} z_j^0}, & i \in I_0 \\ \mu_k c_i e^{\sum_{j=1}^n a_{ij} z_j^0}, & i \in I_k, \; k = 1, \ldots, r, \end{cases}$$

from *i)* and (12) it follows that $y^0 = (y_1^0, \ldots, y_m^0) \in S^*$. From (13) we obtain

$$\lambda_k(y^0) = \sum_{i \in I_k} y_i^0 = \sum_{i \in I_k} \mu_k c_i e^{\sum_{j=1}^n a_{ij} z_j^0} = \mu_k g_k(z^0), \quad k = 1, \ldots, r,$$

and from *ii)* we conclude that $\mu_k g(z^0) = \mu_k$, hence $\lambda_k(y^0) = \mu_k$ for all

$k = 1, \ldots, r$. Then the equalities (13) can be rewritten as

$$(14)\qquad y_i^0 = \begin{cases} c_i e^{\sum_{j=1}^{n} a_{ij} z_j^0}, & i \in I_0 \\ \lambda_k(y^0)\, c_i e^{\sum_{j=1}^{n} a_{ij} z_j^0}, & i \in I_k, \quad k = 1, \ldots, r. \end{cases}$$

After using (10) to write relation (14) in terms of z_j, we obtain

$$y_i^0 = \begin{cases} u_i(x^0), & i \in I_0 \\ \lambda_k(y^0) u_i(x^0), & i \in I_k, \quad k = 1, \ldots, r. \end{cases}$$

Then by Theorem 1 we have $p_0(x^0) = v(y^0)$ and hence by Corollary 2 it follows that y^0 is an optimal solution of the problem $(PGD2)$.

The following theorem is a criterion for the existence of an optimal solution of the problem (PG).

THEOREM 3. *If the problem $(PGD2)$ has a feasible solution y^* with all components strict positive and if the problem (PG) is consistent, then the problem (PG) has an optimal solution x^0.*

The *proof* is analogous to the proof of Theorem 8.2 from [4] p. 119.

Next we show how one can obtain an optimal solution for the problem (PG), if an optimal solution of the problem $(PGD2)$ is known.

THEOREM 4. *If the problem $(PGD2)$ has an optimal solution y^0, then any optimal solution x^0 of the problem (PG) satisfies the system*

$$(15)\qquad u_i(x^0) = \begin{cases} y_i^0 v(y^0), & \text{if } i \in I_0 \\ \dfrac{y_i^0}{\lambda_k(y^0)}, & \text{if } i \in I_k,\ k \in \{\{1, \ldots, r\} : \lambda_k(y^0) > 0\}. \end{cases}$$

Proof. Let x^0 be an optimal solution of the problem (PG) and y^0 an optimal solution of the problem $(PGD2)$. Then, by Theorem 2 we have

$$(16)\qquad p_0(x^0) = v(y^0),$$

but by Theorem 1, equality (16) is true iff x^0 and y^0 satisfy equations (3). Hence the equalities (15) are true.

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