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ON THE EQUATION $f(z) = (\beta z + \alpha)f(a)$ IN THE CLASS OF UNIVALENT FUNCTIONS

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1. Let S be the class of functions $f(z) = z + a_2 z^2 + \ldots$, $f(0) = 0$, $f'(0) = 1$ which are regular and univalent in the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$. Consider the equation

(1) $$f(z) = (\beta z + \alpha)f(a)$$

where α, β and $a(|a| < 1)$ are complex numbers.

If $\alpha = 0$ or $\alpha = 1 - a\beta$, then, for every $f \in S$, this equation has only the solution $z = 0$ respectively $z = a$. We exclude these banal cases and suppose that $\alpha \neq 0$, $\alpha \neq 1 - a\beta$.

In the present paper we put the problem te estimate the moduli of the solution of the equation (1) when the functions f belongs to the class S. Our task is to find the maximum disk with its centre the origin and containing no solution of the above equation.

The particular case $\beta = 0$ is solved by PETRU T. MOCANU [2].

First, we may without loss in generality suppose $0 < a < 1$.

In fact, if z_0 is the extremal solution of the equation (1) and if $f_0(z)$ is the extremal function, then we can consider the function $f_1(z) = e^{-it} f_0(z \cdot e^{it})$ $t = arg(a)$ belonging to S and we have

$$f_1(z_1) = (\beta z_1 + \alpha)f_1(|a|)$$

where

$$z_1 = z_0 e^{-it}, \ |z_1| = |z_0|.$$

2. Let $z_0 = re^{i\theta}$ be a solution of the equation (1) which has an extremal value (minimum, for instance) and let $f_0(z) \in S$ be the extremal function. We note that there exists this extremal function because S is a compact space. We have $f_0(z_0) = (\beta z_0 + \alpha)f(a)$.

Now we consider a variation of the function $f_0(z)$ which is given by Schiffer—Golusin's formula [1]:

$$f_0^*(z) = f_0(z) + \lambda V(z\,;\zeta\,;\psi) + 0(\lambda^2),\ \ \lambda > 0,\ \ |\zeta| < 1$$

where

$$V(z\,;\zeta\,;\psi) = e^{i\psi}\frac{f_0^2(z)}{f_0(z) - f_0(\zeta)} - e^{i\psi} f_0(z)\left[\frac{f_0(\zeta)}{\zeta f_0'(\zeta)}\right]^2 - \tag{2}$$

$$- e^{i\psi}\frac{z f_0'(z)}{z-\zeta}\,\zeta\left[\frac{f_0(\zeta)}{\zeta f_0'(\zeta)}\right] + e^{-i\psi}\frac{z^2 f_0'(z)}{1-\bar\zeta z}\,\bar\zeta\,\overline{\left[\frac{f_0(\zeta)}{\zeta f_0'(\zeta)}\right]^2}.$$

For λ sufficiently small $f_0(z) \in S$.
The equation (1) becomes:

$$f_0(z) + \lambda V(z\,;\zeta\,;\psi) + 0(\lambda^2) = (\beta z + \alpha)[f_0(a) + \tag{3}$$

$$+ \lambda V(a\,;\zeta\,;\psi)] + 0(\lambda^2).$$

This equation has the solution:

$$z_0^* = z_0 + \lambda h + 0(\lambda^2).$$

From (3) we obtain the value of h, by replacing z by z_0, differentiating with respect to λ and making $\lambda = 0$. Namely we have

$$h = \frac{(\beta z_0 + \alpha) V(a\,;\zeta\,;\psi) - V(z_0\,;\zeta\,;\psi)}{f_0'(z_0) - \beta f_0(a)}$$

In the sequel we shall cut out the index 0.
According to the formula (2) we obtain

$$h = \frac{1}{f'(z) - \beta f(a)}\left\{e^{i\psi}\frac{(\beta z + \alpha) f^2(a)}{f(a) - f(\zeta)} - e^{i\psi}(\beta z + \alpha) f(a)\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 - \right.$$

$$- e^{i\psi}\frac{(\beta z + \alpha) a f'(a)}{a - \zeta}\,\zeta\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 + e^{-i\psi}\frac{(\beta z + \alpha) a^2 f'(a)}{1 - \bar\zeta a}\,\bar\zeta\,\overline{\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2} -$$

$$- e^{i\psi}\frac{f^2(z)}{f(z) - f(\zeta)} + e^{i\psi} f(z)\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 + e^{i\psi}\frac{z f'(z)}{z - \zeta}\,\zeta\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 -$$

$$\left. - e^{-i\psi}\frac{z^2 f'(z)}{1 - \bar\zeta z}\,\bar\zeta\,\overline{\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2}\right\}.$$

Now we introduce the notations:

$$\begin{cases} f = f(z) = (\beta z + \alpha) f(a), & \gamma = \beta z + \alpha \\ l = f'(z) - \beta f(a), & m = f'(a) \\ n = f'(z), & w = f(\zeta) \end{cases} \tag{4}$$

The above expression of h becomes:

$$h = \frac{1}{l}\left[e^{i\psi}\frac{f^2}{f - \gamma w} - e^{i\psi}\frac{am\gamma}{a - \zeta}\,\zeta\left(\frac{w}{\zeta w'}\right)^2 + e^{-i\psi}\frac{a^2 m}{1 - \bar\zeta a}\,\bar\zeta\,\overline{\left(\frac{w}{\xi w'}\right)^2} - \right.$$

$$\left. - e^{i\psi}\frac{f^2}{f - w} + e^{i\psi}\frac{nz}{z - \zeta}\,\zeta\left(\frac{w}{\zeta w'}\right)^2 - e^{-i\psi}\frac{nz^2}{1 - \bar\zeta z}\,\bar\zeta\,\overline{\left(\frac{w}{\zeta w'}\right)^2}\right].$$

Since the solution z is extremal, it follows:

$$|z^*|^2 = z^* \cdot \bar z^* = |z|^2 + 2\lambda \mathrm{Re}(\bar z h) + 0(\lambda^2) \geqq |z|^2.$$

Hence it must be verified the inequality

$$\mathrm{Re}(h\bar z) \geq 0.$$

According to the above expression of h, this relation may be written

$$\mathrm{Re}\left\{e^{i\psi}\left[\frac{\bar z(\gamma - 1) f^2 w}{l(f - w)(f - \gamma w)} - \left(\frac{am\bar z\gamma}{l(a - \xi)} - \frac{nz\bar z}{l(z - \zeta)} - \frac{a^2\bar m z\bar\gamma}{l(1 - a\zeta)} + \right.\right.\right.$$

$$\left.\left.\left. + \frac{\bar n z\bar z^2}{l(1 - \bar z\zeta)}\right)\zeta\left(\frac{w}{\zeta w'}\right)^2\right]\right\} \geqq 0,$$

Since ψ is arbitrary, it follows that the extremal function $w = f(\zeta)$ must satisfy the differential equation:

$$\left(\frac{\zeta w'}{w}\right)^2\frac{\bar z(\gamma - 1) f^2 w}{l(f - w)(f - \gamma w)} = \left[\frac{am\bar z\gamma}{l(a - \zeta)} - \frac{nz\bar z}{l(z - \zeta)} - \frac{a^2\bar m z\bar\gamma}{l(1 - a\zeta)} + \frac{\bar n z\bar z^2}{\bar l(1 - \bar z\zeta)}\right]\zeta$$

i.e.

$$\left(\frac{\zeta w'}{w}\right)^2\frac{\bar z(\gamma - 1) f^2 w}{l(f - w)(f - \gamma w)} = \left\{\frac{am\bar l\bar z\gamma - a^3 m l z\bar\gamma - a^2(\bar l m\bar z\gamma - l\bar m z\bar\gamma)\zeta}{l\bar l(a - \zeta)(1 - a\zeta)} - \right. \tag{5}$$

$$\left. - \frac{r^2[\bar l n - r^2 l\bar n + (l\bar n - \bar l n)\bar z\zeta]}{l\bar l(z - \zeta)(1 - \bar z\zeta)}\right\}\zeta.$$

This equation may be written in the form

$$\left(\frac{\zeta w'}{w}\right)^2\frac{\bar z(\gamma - 1) f^2 w}{(f - w)(f - \gamma w)} = \frac{\sum_{k=0}^{4} a_k\zeta^k}{\bar l(z - \zeta)(1 - \bar z\zeta)(a - \zeta)(1 - a\zeta)}$$

where:

$$a_0 = 0,\ \ a_4 = \frac{a\bar z}{l}\left[a(l\bar m z\bar\gamma - \bar l m\bar z\gamma) - r^2(l\bar n - \bar l n)\right].$$

Using the variational formula

$$f^*(z) = f(z) + \lambda\left[f(z) - z f'(z)\frac{1 + e^{i\psi} z}{1 - e^{i\psi} z}\right] + 0(\lambda^2)$$

it may be shown that $a_4 = 0$.

Hence the eqation (5) may be written:

$$\left(\frac{\zeta w'}{w}\right)^2 \frac{\bar z(\gamma-1)f^2 w}{(f-w)(f-w\gamma)} = \left\{\frac{a\bar l m\bar z\gamma - a^3\bar m l z\bar\gamma + ar^2(l\bar n - \bar l n)\zeta}{\bar l(a-\zeta)(1-a\zeta)} - \right.$$

$$\left. - \frac{r^2[(l\bar n - \bar l n)\bar z\zeta + \bar l n - l\bar n r^2]}{\bar l(z-\zeta)(1-\bar z\zeta)}\right\}\zeta$$

or

$$(6) \qquad \left(\frac{\xi w'}{w}\right)^2 \frac{\bar z(\gamma-1)f^2 w}{(f-w)(f-\gamma w)} = \frac{(a_1 + a_2\zeta + a_3\zeta^2)\zeta}{(z-\zeta)(1-\bar z\zeta)(a-\zeta)(1-a\zeta)}$$

where

$$a_1 = \frac{a}{\bar l}\,[r^2\bar l m\gamma - a^2 l\bar m\bar z^2\bar\gamma - r^2(\bar l n - l\bar n r^2)],$$

$$a_2 = \frac{1}{\bar l}\{-a(1+r^2)[\bar l m\bar z\gamma - a^2 l\bar m z\bar\gamma + r^2(l\bar n - \bar l n)] +$$

$$+ r^2(1+a^2)(\bar l n - l\bar n r^2) - ar^2(l\bar n - \bar l n)\bar z\},$$

$$a_3 = \frac{1}{\bar l}\{a\bar z[\bar l m\bar z\gamma - a^2 l\bar m z\bar\gamma + r^2(l\bar n - \bar l n)] + r^2(1+a^2)\bar z(l\bar n - \bar l n) -$$

$$- ar^2(\bar l n - l\bar n r^2)\}.$$

Making $\zeta \to 0$ we obtain

$$(7) \qquad a_1 = ar^2(\gamma - 1).$$

Comparing this expression with the above expression of a_1 we obtain the formula

$$(8) \qquad r^2\bar l m\gamma - a^2 l\bar m\bar z^2\bar\gamma - r^2\bar l n + r^4 l\bar n = \bar l r^2(\gamma - 1).$$

3. It may be shown ([1]), that the extremal function $w = f(\zeta)$ maps the unit disk $|\zeta| < 1$ onto the entire w — plane slit along a finite number of analytic arcs. Let $\zeta = q$, $|q| = 1$, be the point which is mapped into an end — point of a slit. Then the trinomial $a_1 + a_2\zeta + a_3\zeta^2$ has the double solution $\zeta = q$, and the equation (6) may be written:

$$\left(\frac{\zeta w'}{w}\right)^2 \frac{\bar z(\gamma-1)f^2\cdot w}{(f-w)(f-\gamma w)} = \frac{a_1(1-\bar q\zeta)^2\cdot\zeta}{(z-\zeta)(1-\bar z\zeta)(a-\zeta)(1-a\zeta)}$$

or, according to (7),

$$(9) \qquad \left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2 w}{(f-w)(f-\gamma w)} = \frac{az(1-\bar q\zeta)^2\zeta}{(z-\zeta)(1-\bar z\zeta)(a-\zeta)(1-a\zeta)}$$

Making $\zeta \to z$ we obtain the formula

$$(10) \qquad n = \frac{a(\gamma-1)(1-\bar q z)^2}{(1-r^2)(z-a)(1-az)},$$

and similary, marking $\zeta \to a$ we obtain the formula

$$(11) \qquad \gamma m = \frac{z(\gamma-1)(1-a\bar q)^2}{(1-a^2)(z-a)(1-a\bar z)}.$$

From (8), (10), (11) we get, by eliminating n and m:

$$(12) \qquad \begin{cases} \dfrac{(1-r^2)(1-a\bar z)(1-qa)^2 - (1-a^2)(1-az)(1-q\bar z)^2}{(1-r^2)(1-az)(1-\bar q a)^2 - (1-a^2)(1-a\bar z)(1-\bar q\bar z)^2} = \\[2ex] \dfrac{az(a-\bar z)(1-q\bar z)^2}{(1-r^2)z(1-\bar q a)^2 - a(1-a\bar z)(1-\bar q z)^2 - (1-r^2)(z-a)(1-a\bar z)}. \end{cases}$$

Now we introduce the notation

$$(13) \qquad \begin{cases} (1-r^2)(1-az)(1-qa)^2 - (1-a^2)(1-az)(1-q\bar z)^2 = b_1 + \\ + b_2 q + b_3 q^2, \\ (1-r^2)z(1-\bar q a)^2 - a(1-a\bar z)(1-\bar q z)^2 - (1-r^2)(z-a)(1-a\bar z) = \\ = c_1 + c_2\bar q + c_3\bar q^2 \\ az(a-\bar z)(1-q\bar z)^2 = d_1 + d_2 q + d_3 q^2 \end{cases}$$

where:

$$b_1 = (1-r^2)(1-a\bar z) - (1-a^2)(1-az)$$
$$b_2 = -2a(1-r^2)(1-a\bar z) + 2\bar z(1-a^2)(1-az)$$
$$b_3 = a^2(1-r^2)(1-a\bar z) - \bar z^2(1-a^2)(1-az)$$
$$c_1 = z(1-r^2) - a(1-a\bar z) - (1-r^2)(z-a)(1-a\bar z)$$
$$c_2 = 2az(r^2 - a\bar z)$$
$$c_3 = az(a-z)$$
$$d_1 = az(a-\bar z),\ d_2 = -2ar^2(a-\bar z),\ d_3 = ar^2 z(a-\bar z).$$

The equation (12) becomes:

$$(b_1 + b_2 q + b_3 q^2)(c_1 + c_2\bar q + c_3\bar q^2) = (\bar b_1 + \bar b_2\bar q + \bar b_3\bar q^2)(d_1 + d_2 q + d_3 q^2).$$

Because $q\bar q = 1$, this equation may be written in the form

$$(14) \qquad \begin{cases} (c_1 b_3 - \bar b_1 d_3)^2 q^2 + (b_1 c_3 - \bar b_3 d_1)\bar q^2 + (b_2 c_1 + b_3 c_2 - \bar b_1 d_2 - \bar b_2 d_3)q + \\ + (b_1 c_2 + b_2 c_3 - \bar b_2 d_1 - \bar b_3 d_2)\bar q + b_1 c_1 + b_2 c_2 + b_3 c_3 - \bar b_1 d_1 - \\ - \bar b_2 d_2 - \bar b_3 d_3 = 0. \end{cases}$$

Further on we note:

$$q = \rho e^{i\tau},\ |\rho| = 1,\ \rho^2 = 1,$$
$$c_1 b_3 - \bar b_1 d_3 = r_1 e^{i\theta_1},\ b_1 d_3 - \bar b_3 d_1 = r_2 e^{i\theta_2}$$
$$b_2 c_1 + b_3 c_2 - \bar b_1 d_2 - \bar b_2 d_3 = r_3 e^{i\theta_3},$$
$$b_1 c_2 + b_2 c_3 - \bar b_2 d_1 - \bar b_3 d_2 = r_4 e^{i\theta_4},$$
$$b_1 c_1 + b_2 c_2 + b_3 c_3 - \bar b_1 d_1 - \bar b_2 d_2 - \bar b_3 d_3 = r_5 e^{i\theta_5}$$

where ρ, τ, r_k, θ_k, $k = \overline{1,5}$ are real numbers.

Equation (14) becoms in this case

$$\rho^2[r_1e^{i(\theta_1+2\tau)} + r_2e^{i(\theta_2-2\tau)}] + \rho[r_3e^{i(\theta_3+\tau)} + r_4e^{i(\theta_4-\tau)}] + r_5e^{i\theta_5} = 0$$

From the above equation we obtain the equation which is verified by ρ and τ:

$$(15)\qquad \begin{cases} \rho^2[r_1\cos(\theta_1+2\tau) + r_2\cos(\theta_2-2\tau)] + \rho[r_3\cos(\theta_3+\tau) + \\ \qquad + r_4\cos(\theta_4-\tau)] + r_5\cos\theta_5 = 0 \\ \rho^2[r_1\sin(\theta_1+2\tau) + r_2\sin(\theta_2-2\tau)] + \rho[r_3\sin(\theta_3+\tau) + \\ \qquad + r_4\sin(\theta_4-\tau)] + r_5\sin\theta_5 = 0 \end{cases}$$

Because $\rho = \pm 1$ the equation (15) gives two values for q, coresponding to the two extremal values.

4. Separating the variables in the equation (9) we obtain:

$$(16)\qquad \frac{f\,dw}{\sqrt{w(f-w)(f-\gamma w)}} = \frac{\sqrt{az}\,(1-\bar{q}\zeta)d\zeta}{\sqrt{H(\zeta)}},$$

where

$$(17)\qquad H(\zeta) = \zeta(z-\zeta)(1-\bar{z}\zeta)(1-a\zeta)(a-\zeta).$$

Integration of (16) gives us the relation

$$(18)\qquad \int_0^w \frac{f\,dw}{\sqrt{w(f-w)(f-\gamma w)}} = \int_0^\zeta \frac{\sqrt{az}(1-\bar{q}t)}{\sqrt{H(t)}}\,dt'.$$

The paths of integration must correspond by the map $w = f(\zeta)$. Since $f = f(z)$ we have

$$(19)\qquad \int_0^f \frac{f\,dw}{\sqrt{w(f-w)(f-\gamma w)}} = \int_0^z \frac{\sqrt{az}(1-\bar{q}t)dt}{\sqrt{H(t)}}.$$

Marking the substitution $w = fu$, (19) becomes:

$$(20)\qquad \sqrt{\frac{az}{f}}\int_0^z \frac{1-\bar{q}t}{\sqrt{H(t)}}\,dt = \int_0^1 \frac{du}{\sqrt{u(1-u)(1-\gamma u)}}.$$

We write

$$(21)\qquad F(\zeta) = \frac{1}{2}\sqrt{az}\int_0^\zeta \frac{1-\bar{q}t}{\sqrt{H(t)}}\,dt.$$

Next we have

$$\int_0^1 \frac{du}{\sqrt{u(1-u)(1-\gamma u)}} = 2K(\sqrt{\gamma})$$

where

$$(22)\qquad K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2}\int_0^1 \frac{du}{\sqrt{u(1-u)(1-k^2u)}}$$

is the elliptic complete integral of the first kind. We integrate along the straight line segment [0,1].

Then (20) becomes:

$$(23)\qquad \frac{1}{\sqrt{f}}F(z) = K(\sqrt{\gamma}).$$

Hence

$$(24)\qquad f = \left[\frac{F(z)}{K\sqrt{(\gamma)}}\right]^2.$$

Since for $\zeta = z$ and $\zeta = a$ we have $w = f$, respectively, $w = \frac{1}{\gamma}f$, we deduce from (16):

$$\sqrt{\frac{az}{f}}\int_z^a \frac{1-\bar{q}t}{\sqrt{H(t)}}\,dt = \int_1^{\frac{1}{\gamma}} \frac{dw}{\sqrt{w(f-w)(f-\gamma w)}}$$

Making the substitution $w = fu$, we deduce

$$\sqrt{\frac{az}{f}}\int_z^a \frac{1-\bar{q}t}{\sqrt{H(t)}}\,dt = \int_1^{\frac{1}{\gamma}} \frac{du}{\sqrt{u(1-u)(1-\gamma u)}}$$

or

$$(25)\qquad \frac{1}{\sqrt{f}}[F(a) - F(z)] = iK'(\sqrt{\gamma})$$

where

$$(26)\qquad iK'(k) = \int_1^{\frac{1}{k}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2}\int_1^{\frac{1}{k^2}} \frac{du}{\sqrt{u(1-u)(1-k^2u)}}.$$

From (23) and (24) we deduce

$$\frac{F(a)}{F(z)} - 1 = \frac{iK'(\sqrt{\gamma})}{K(\sqrt{\gamma})}. \tag{27}$$

This equation gives us the root of extremal modulus $z = re^{i\theta}$. The formula (24) gives the value of $f = f(z)$.
In order to find the extremal function $w = f(\zeta)$ let us replace in the first integral of (18), w by fv^2. We deduce:

$$\int_0^{\sqrt{\frac{w}{f}}} \frac{dv}{\sqrt{(1-v^2)(1-\gamma v^2)}} = \sqrt{\frac{\gamma}{f}}\, F(\zeta)$$

where $F(\zeta)$ is given by the formula (21).
Then the extremal function will be

$$w = f \operatorname{sn}^2 \frac{1}{\sqrt{f}} F(\sqrt{\gamma}) \tag{28}$$

where $\operatorname{sn}(Z)$ is the Jacobi's elliptic function with the periods $4K(\sqrt{\gamma})$ and $2iK'(\sqrt{\gamma})$. The value of f is given by (24).
Finally we found the following result;

THEOREM 1. *The solution* $z = re^{i\theta}$ *of extremal modulus of the equation* $f(\zeta) = (\beta\zeta + \alpha)f(a)$, $(0 < a < 1,\ \alpha \neq 0,\ \beta \neq 0,\ \alpha \neq 1 - \beta a)$, *where* $f(\zeta) \in S$, *is given by the equation*

$$\frac{F(a)}{F(z)} - 1 = \frac{iK'(\sqrt{\gamma})}{K(\sqrt{\gamma})}$$

where

$$F(\zeta) = \frac{1}{2}\sqrt{az}\int_0^{\zeta} \frac{1 - \bar{q}t}{\sqrt{H(t)}}\, dt,$$

$q = \rho e^{i\tau}$, ρ *and* τ *verify the equation* (15),

$$H(t) = t(z - t)(1 - \bar{z}t)(a - t)(1 - at).$$

The extremal function is given by the formula

$$w = f \operatorname{sn}^2 \frac{1}{\sqrt{f}} F(\zeta)$$

where $\operatorname{sn}(Z)$ *is the Jacobi's elliptic function with the periods* $4K(\sqrt{\gamma})$, $2iK'(\sqrt{\gamma})$ *and* f *is given by*

$$f = \left[\frac{F(z)}{K(\sqrt{\gamma})}\right]^2.$$

5. In the sequel we shall study the limit case $a \to 1$.
Taking $a = 1$ in the formula (12), we obtain $q = 1$.
Next, from (17) we have

$$H(\zeta) = \zeta(z - \zeta)(1 - \bar{z}\zeta)(1 - \zeta)^2.$$

If we take $a = 1$ in the formula (21) we obtain

$$F(\zeta) = \frac{1}{2}\sqrt{z}\int_0^{\zeta} \frac{(1 - \bar{q}t)\, dt}{(1 - t)\sqrt{t(z - t)(1 - \bar{z}t)}}.$$

We have further

$$F(a) = F(1) = \frac{1}{2}\int_0^{\zeta} \frac{(1 - \bar{q}t)\, dt}{(1 - t)\sqrt{t(z - t)(1 - \bar{z}t)}}.$$

For $q \neq 1$, the last integral, evidently, is divergent. Hence we must have necessarily $q = \bar{q} = 1$. This means that in the case $a = 1$, we have to find the minimum of the involved moduli only. If r is this minimum, then the disk $|\zeta| < r$ will be the largest disk with centre the origin and containing no solution of the equation $f(\zeta) = (\beta\zeta + \alpha)f(1)$, when f belongs to the class S. We note that if $f(1)$ is not defined, the equation $f(\zeta) = (\beta\zeta + \alpha)f(1)$ evidently contains no solution in the disk $|\zeta| < r$, so that the above statement holds for the all class S.
In this limit case $(a = 1)$ we have

$$F(\zeta) = \frac{1}{2}\sqrt{z}\int_0^{\zeta} \frac{dt}{\sqrt{t(z - t)(1 - \bar{z}t)}}. \tag{29}$$

Let us make the substitution $\zeta = zt$. We deduce

$$F(\zeta) = \frac{1}{2}\sqrt{z}\int_0^{\frac{\zeta}{z}} \frac{dt}{\sqrt{t(1 - t)(1 - r^2t)}}. \tag{30}$$

Hence

$$F(z) = \frac{1}{2}\sqrt{z}\int_0^{1} \frac{dt}{\sqrt{t(1 - t)(1 - r^2 t)}} = \sqrt{z}\, K(r). \tag{31}$$

Introduction of (31) into (24) yields

$$f = z\left[\frac{K(r)}{K(\sqrt{\gamma})}\right]^2 = re^{i\theta}\left[\frac{K(r)}{K(\sqrt{\gamma})}\right]^2. \tag{32}$$

For $\zeta = 1$, (30) becomes:

$$F(1) = \frac{1}{2}\sqrt{z}\int_0^{\frac{1}{z}} \frac{dt}{\sqrt{t(1-t)(1-r^2t)}}.$$

The equation (30) becomes:

$$\frac{\displaystyle\int_0^{\frac{1}{z}} \frac{dt}{\sqrt{t(1-t)(1-r^2t)}}}{\displaystyle\int_0^{1} \frac{dt}{\sqrt{t(1-t)(1-r^2t)}}} - 1 = \frac{iK'(\sqrt{\gamma})}{K(\sqrt{\gamma})}.$$

This equation may be written also in the two equivalent forms:

$$\frac{1}{2K(r)}\int_1^{\frac{1}{r}e^{-i\theta}} \frac{dt}{\sqrt{t(1-t)(1-r^2t)}} = \frac{iK'(\sqrt{\gamma})}{K(\sqrt{\gamma})} \tag{33}$$

and

$$\frac{1}{2K(r)}\int_0^{\frac{1}{r}e^{-i\theta}} \frac{dt}{\sqrt{t(1-t)(1-r^2t)}} = \frac{K(\sqrt{\gamma}) + iK'(\sqrt{\gamma})}{K(\sqrt{\gamma})}. \tag{34}$$

Finally wo found the following result:

THEOREM 2. *If $z = re^{i\theta}$ is the root of minimum modulus of the equation $f(\zeta) = (\beta\zeta + \alpha)f(1)$, when f belongs to the class S, then r and θ verify the equation (33) or (34). The extremal function will be given by the formula*

$$w = z\left[\frac{K(r)}{K(\sqrt{\gamma})}\right]^2 \operatorname{sn}^2 \frac{1}{2}\frac{K(\sqrt{\gamma})}{K(r)}\int_0^{\zeta} \frac{dt}{\sqrt{t(z-t)(1-\bar{z}t)}}.$$

Let D_f, be the image of the unit disk $|\zeta| < 1$ by $w = f(\zeta)$ and d_f be the image of the disk $|\zeta| < r$ by $w = \dfrac{f(\zeta)}{\beta\zeta + \alpha}$, where $f(\zeta)$ is a function of the class S.

We found the following geometrical meaning: $|\zeta| < r$ is the largest disk whose image d_f lies entierly in the domain D_f, when f belongs to the class S, i.e.:

$$d_f \subset D_f, \quad \text{for} \quad f \in S.$$

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