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A CLASS OF GENERALIZED SZÁSZ OPERATORS AND THEIR CONVERGENCE PROPERTIES

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1. Introduction

Szász [10] has shown that for a finite function $f(t)$ defined on $[0, \infty)$ and having a suitable rate of growth as $t \to \infty$, the operators

$$(1) \qquad S_u(f, x) = e^{-ux} \sum_{k=0}^{\infty} f\left(\frac{k}{u}\right) \frac{(ux)^k}{k!}, \qquad u > 0, \ x \geq 0$$

converge, as $u \to \infty$, to $f(x)$ at each point $t = x \geq 0$ where $f(t)$ is continuous. Jakimovski and Leviatan [5] extended the Szász operator (1) to a class of operators $J_u(f, x)$ (see below), generated by the Appel polynomials, and established their convergence as $u \to \infty$ to real and analytic functions f.

The purpose of this note is to introduce a broad family of operators $P_u(f, x; A, G)$, which includes the operators $J_u(f, x)$ of [5], establish their convergence properties and obtain estimates on the rate of convergence. In addition, we provide an application of the new class of operators to the problem of signal reconstruction from quantized noisy data (Masry and Cambanis [8]). The convergence properties of the new class are stated in Section 2, the application is considered in Section 3, and the proofs are collected in Section 4.

2. The Operators $P_u(f, x; A, G)$ and Their Convergence Properties

Let

$$A(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad G(z) = \sum_{n=1}^{\infty} \gamma_n z^n \tag{2}$$

be two analytic functions in the disk $|z| < R$, $R > 1$, and suppose $A(1) \neq 0$, $G(1) \neq 0$. Define the polynomials $p_n(x) \equiv p_n(x; A, G)$ by

$$A(z)\, e^{xG(z)} = \sum_{n=0}^{\infty} p_n(x) z^n, \qquad x \geq 0. \tag{3a}$$

These polynomials are called, variously, sets of "A-type zero" (Sheffer [9]), "generalized Appel polynomials" (Erdélyi [4, Vol. 3: Chap. 19] and "Sheffer type" (Boas and Buck [1, Section 10]). To each function $f(t)$ on $[0, \infty)$ associate the operators

$$\begin{aligned} P_u(f, x) &\equiv P_u(f, x; A, G) \\ &= \frac{e^{-uxG(1)}}{A(1)} \sum_{k=0}^{\infty} (f(k/u)\, p_k(ux), \qquad u > 0,\ x \geq 0. \end{aligned} \tag{3b}$$

The family of operators $P_u(f, x; A, G)$ is therefore generated by the two functions $A(z)$ and $G(z)$ of (2). In the special case where $G(z) \equiv z$, $P_u(f, x)$ reduces to the operator $J_u(f, x)$ studied by Jakimovski and Leviatan [5]. The Szász operator $S_u(f, x)$ corresponds to $A(z) \equiv 1$, $G(z) \equiv z$.

It is seen from (3a) and (3b) that if $f(t)$ has an exponential growth, $|f(t)| \leq M \exp(\mu t)$ on $[0, \infty)$ for some constants $M > 0$, $\mu \geq 0$, then $P_u(f, x)$ exists for $x \geq 0$ and $u > \mu/\ln R$.

A sufficient condition for the operators $P_u(f, x)$ to be positive in $[0, \infty)$ is that the coefficients $\{a_n\}$ and $\{\gamma_n\}$ in (2) are nonnegative. This can be seen from (2) and (3a) from which we obtain the following explicit expression for the Sheffer polynomials $p_n(x)$:

$$p_n(x) = \sum_{j=0}^{n} [\Sigma a_{k_0} \gamma_{k_1} \ldots \gamma_{k_j}] \frac{x^j}{j!}, \qquad x \geq 0,\ n = 0, 1, \ldots \tag{4}$$

where the inner sum extends over all sets of integers $\{k_i\}_{i=0}^{j}$ satisfying $k_0 \geq 0$, $k_i \geq 1$, $i = 1, \ldots, j$ such that $\Sigma_{i=0}^{j} k_i = n$. Henceforth we shall assume that the sequences $\{a_n\}$, $\{\gamma_n\}$ are nonnegative so that $P_u(f, x)$ is a linear positive operator of the interpolation type.

Our first objective is to find appropriate conditions on the analytic functions $A(z)$ and $G(z)$ of (2) which assure the point-wise convergence of $P_u(f, x)$ to $f(x)$, as $u \to \infty$, at points of continuity of f. For functions f

which are *bounded* on $[0, \infty)$ such convergence is assured by Korovkin's theorem [7, *p.* 6] provided $P_u(e_j, x) \to e_j(x)$, as $u \to \infty$, for $e_j(x) = x^j$, $j = 0, 1, 2$. We have

PROPOSITION. *$P_u(e_j, x) \to e_j(x)$, $j = 0, 1, 2$, as $u \to \infty$, if and only if $G'(1) = 1$.*

Note that since $G'(1) < \infty$ by (2), the condition $G'(1) = 1$ is merely a normalization for $G(z)$. The convergence of $P_u(f, x)$ to $f(x)$ is given in

THEOREM 1. *Let $G'(1) = 1$ and suppose that $|f(t)| \leq M \exp(\mu t)$ on $[0, \infty)$ for some constants $M > 0$ and $\mu \geq 0$. If $f(t)$ is continuous at $t = x_0$, then as $u \to \infty$, $P_u(f, x)$ converges uniformly at $x = x_0$ to $f(x_0)$.*

The notion of uniform convergence at a point x_0 is due to Szász [10] (given $\varepsilon > 0$ there exists a $\delta \equiv \delta(\varepsilon)$ and $u_0 \equiv u_0(\varepsilon)$ such that $|P_u(f, x) - f(x_0)| < \varepsilon$ for $|x - x_0| < \delta$ *and* $u > u_0$). As uniform convergence at each point of a compact set D implies uniform convergence over the set D, we have

COROLLARY. *If, in addition, $f(t)$ is continuous on the finite interval $[a, b]$, $0 \leq a < b < \infty$, then, as $u \to \infty$, $P_u(f, x)$ converges to $f(x)$ uniformly over $[a, b]$.*

When $G(z) \equiv z$, Theorem 1 was proved in [5].

Under stronger conditions on f we obtain the uniform convergence of $P_u(f, x)$ to $f(x)$ on $[0, \infty)$.

THEOREM 2. *Let $G'(1) = 1$ and suppose that $f(t)$ is continuous on $[0, \infty)$ and $\lim_{t \to \infty} f(t)$ exists. Then $P_u(f, x)$ converges to $f(x)$, uniformly on $[0, \infty)$, as $u \to \infty$.*

When $G(z) \equiv z$, Theorem 2 was proved in [5].

Next we obtain explicit bounds on the degree of approximation of f, on the compact interval $[0, b]$, by the family of operators $P_u(f, x; A, G)$ of (3) with the aim of exhibiting the dependence of these bounds on the functions A, G generating the family. In doing so we shall utilize the general theory on the degree of approximation of continuous functions by a sequence of linear positive operators (see, for example, Devore [2, *pp.* 28–29] for functions defined on compact intervals and Ditzian [3] for functions defined on $[0, \infty)$ or $(-\infty, \infty)$).

THEOREM 3. *Let $G'(1) = 1$.*

(a) Suppose $f(t)$ is uniformly continuous on $[0, \infty)$. Then

$$|P_u(f, x) - f(x)| \leq 2\omega(f, \alpha_u(b)) \tag{5}$$

uniformly in x over $[0, b]$, $0 < b < \infty$, where $\omega(f, \delta)$ is the modulus of continuity of f on $[0, \infty)$ and

$$\alpha_u^2(b) = \frac{b[1 + G''(1)]}{u} + \frac{A'(1) + A''(1)}{A(1)u^2} \tag{6}$$

(b) Suppose $|f(t)| \leq M \exp(\mu t)$ on $[0, \infty)$ for some constants $M > 0$, $\mu \geq 0$. Let $I = [0, b]$, $0 < b < \infty$, and $I_1 = [0, b - \eta]$ for some

$0 < \eta < b$. *If $f(t)$ is continuous on I, then*

(7) $$|P_u(f,\ x) - f(x)| \le 2\omega_I(f,\ \alpha_u(b)) + K_{f,G}\left(\frac{1}{u}\right) + O(1/u^2)$$

uniformly in x over I_1, where $\omega_I(f, \delta)$ is the modulus of continuity of f on I, $\alpha_u^2(b)$ as in (6), the constant $K_{f,G}$ is given by

(8) $$K_{f,G} = \frac{b}{\mu^2}\,[Me^{\mu b} + \max_{t \in I_1} |f(t)|][1 + G''(1)]$$

and $O(1/u^2)$ is uniform in x over I_1.

Note that Part (*a*) of Theorem 3 imposes a severe restriction on the growth of f at infinity, but provides a simple bound in terms of the modulus of continuity of f on $[0, \infty)$. On the other hand, the more complex bound in Part (*b*) given in terms of the modulus of continuity of f over $[0, b]$ plus "correction" terms, is derived under weak assumptions on the growth of f at infinity. We also remark that in obtaining the bound (7) of Part (*b*), we sought to determine the dependence of the first two significant terms of the bound on the function f and on the family of operators $P_u(f,\ x\,;\ A,\ G)$; hence our explicit determination of the constant $K_{f,G}$. Of course, asymptotically as $u \to \infty$, the first term $2\omega_I(f,\ \alpha_u(b))$ is dominant.

We finally remark that among the family of operators $P_u(f,\ x\,; A,\ G)$ undre consideration, the Szász operator $S_u(f,\ x)$, corresponding to $A(z) \equiv 1$, $G(z) \equiv z$, gives the smallest values of $\alpha_u^2(b)$ and $K_{f,G}$ (cf. (6), (8)) and therefore the smallest bound of the form (5), (7). For the Szász operator $S_u(f,\ x)$ bounds of the form (5), (7) were first given by Ditzian [3].

3. An Application

It is well-known that a continuous function $f(t)$ on $(-\infty, \infty)$ cannot be reconstructed from its sign, $\operatorname{sgn}[f(t)]$, $-\infty < t < \infty$; this is due to the fact that a signum operation retains only the zero crossings information about $f(t)$ and the latter does not determine $f(t)$ even when $f(t)$ is analytic (see, for example, Titchmarsh [11]). The addition of "contamination" to $f(t)$ prior to the signum operation improves the situation as shown by Masry and Cambanis [8]. Here we employ the class of operators $P_u(f,\ x\,;\ A,\ G)$ considered in Section 2 for the reconstruction scheme as follows.

Let $\{X_k\}$ be a sequence of independent identically distributed random variables with distribution $F(x)$. Add $\{X_k\}$ to samples $\{f(k/u)\}$ of $f(t)$ prior to the signum operation to obtain the data set

(9) $$Z_k = \operatorname{sgn}\,[f(k/u) + X_k], \qquad k = 0,\ \pm 1,\ \ldots$$

where $u > 0$. For the sake of simplicity of analysis we shall assume that (see [8])

(10*i*) *i*) $f(t)$ is uniformly continuous on $[0, \infty)$ such that $|f(t)| \le M$ for all $t \ge 0$ where M is known and finite

(10*ii*) *ii*) The distribution $F(x)$ of the "noise" $\{X_k\}$ is uniform over $[-M, M]$.

Given the data set $\{Z_k\}_{k=0}^{\infty}$, estimate $f(t)$ on $[0, \infty)$ by

(11) $$\hat{f}_u(t) = M \sum_{k=0}^{\infty} h_k(ut)\, Z_k, \qquad t \ge 0,\ u > 0$$

where

(12) $$h_k(x) = \frac{e^{-xG(1)}}{A(1)}\, p_k(x), \qquad x \ge 0,\ k = 0,\ 1,\ \ldots$$

is the kernel of the operator $P_u(f,\ x)$ considered in Section 2. We show that as $u \to \infty$, $\hat{f}_u(t)$ converges in quadratic mean and with probability one to $f(t)$. Note that no such convergence is possible in the absence of the "noise" $\{X_k\}$.

THEOREM 4. *Under assumption (10) on f and $\{X_k\}$ and the assumption $G'(1) = 1$ on the generating function $G(z)$ of (2) we have for every $t \ge 0$ and $u > 0$*

$$E[f_u(t) - f(t)]. \le 4\omega^2(f \cdot \alpha_u(t)) + M^2 \exp\,(-2u\gamma_{n_0} t)\, I_0(2u\gamma_{n_0} t)$$

where "E" is the expectation operator, $\omega(f, \delta)$ is the modulus of continuity of f on $[0, \infty)$,

(13) $$\alpha_u^2(t) = \frac{[1 + G''(1)]t}{u} + \frac{A'(1) + A''(1)}{A(1)u^2},$$

$I_0(x)$ is the modified Bessel function of the first kind of order zero and γ_{n_0} is the first nonzero coefficient of the generating function $G(z)$.

Since $I_0(x) = (e^x/\sqrt{2\pi x})(1 + O(1/x))$ as $x \to \infty$, (Erdélyi [4, *Vol.* 2: *p.* 86]), Theorem 4 implies that $\hat{f}_u(t)$ converges to $f(t)$ in quadratic mean as $u \to \infty$ for every $t > 0$. In particular, for $f \in \text{Lip}\ \alpha$, $0 < \alpha \le 1$, on $[0, \infty)$ we have

(14) $$E[\hat{f}_u(t) - f(t)]^2 = O(u^{-\min(\alpha,\ 1/2)})$$

uniformly in t on compact subsets of $(0, \infty)$.

The convergence in quadratic mean of Theorem 4 can be strengthened to probability one convergence as in

THEOREM *Let $f(t)$ be* Lip α, $0 < \alpha \le 1$, *on* $[0, \infty)$ *and assume* 10(*ii*) *and* $G'(1) = 1$ *to hold. Then for each fixed* $t \in (0, \infty)$ *we have with probability one*

$$n^{\theta}|\hat{f}_n(t) - f(t)| \to 0 \ \ as\ n \leftarrow \infty.$$

for all θ satisfying $0 < \theta < 1/2 \min\,(\alpha,\ 1/2)$.

Theorem 5 implies the convergence of $\hat{f}_n(t)$ to $f(t)$ corresponding to almost every realization of the "noise" sequence $\{X_k\}$.

4. Proofs.

We need the following lemmas.

LEMMA 1. *Let the Sheffer polynomials $p_n(x)$ be given by* (2), (3*a*). *For $|z| < R$, $u > 0$, and $x \geq 0$ we have*

$$S \equiv e^{-uxG(z)} \sum_{k=0}^{\infty} (k - ux)^2 p_k(ux) z^k = u^2 K_u(x, z)$$

where

$$K_k(x, z) = x^2 A(z) [1 - zG'(z)]^2$$
$$+ \frac{xz}{u} \{2A'(z)[zG'(z) - 1] + A(z)[G'(z) + zG''(z)]\} \tag{15}$$
$$+ \frac{z}{u^2} [zA''(z) + A'(z)].$$

Proof. Expanding $(k - ux)^2$ in S and using (3*a*) we obtain

$$S = z^2 \frac{d^2}{dz^2} [A(z) e^{uxG(z)}] + (1 - 2ux) z \frac{d}{dz} [A(z) e^{uxG(z)}] + (ux)^2 A(z) e^{uxG(z)}$$

and the result follows by differentiation and collection of terms. □

LEMMA 2. *Under the assumptions of Lemma* 1 *and $G'(1) = 1$ we have for every $\mu \geq 0$, $x \geq 0$ that*

$$K_k(x, e^{\mu/u}) = \frac{xA(1)}{u} [1 + G''(1)] + O(1/u^2)$$

where the $O(1/u^2)$ term is uniform in x over compact intervals.

Proof. Let $z = e^{\mu/u}$ in (15) and write

$$K_u(x, e^{\mu/u}) = S_1 + S_2 + S_3$$

for the 3 terms on the right-hand side of (15). From (2) we have for $u > \mu/\ln R$ that

$$\left.\begin{array}{l} A^{(n)}(e^{\mu/u}) = A^{(n)}(1) + O(1/u) \\ G^{(n)}(e^{\mu/u}) = G^{(n)}(1) + O(1/u). \end{array}\right| , \quad n = 0, 1, \ldots$$

Then

$$S_1 = x^2 [A(1) + O(1/\mu)][1 - G'(1) + O(1/u)]^2$$
$$= x^2 O(1/u^2)$$

since $G'(1) = 1$. In a similar fashion

$$S_2 = \frac{x}{u} \{2A'(1)[G'(1) - 1] + A(1)[G'(1) + G''(1)] + O(1/u)\}$$
$$= \frac{x}{u} A(1)[1 + G''(1)] + xO(1/u^2).$$

Finally $S_3 = IO(1/u^2)$ and the result follows. □

Proof of Proposition. It is clear from (3*a*) that $P_u(e_0, x) = e_0$. Now

$$P_u(e_1, x) = \frac{e^{-uxG(1)}}{uA'(1)} \sum_{k=0}^{\infty} k p_k(ux)$$

and using (3*a*) in the manner of the proof of Lemma 1, we obtain

$$P_u(e_1, x) = G'(1) x + \frac{A'(1)}{A(1)u}.$$

Similary

$$P_u(e_2, x) = \frac{e^{-uxG(1)}}{u^2 A(1)} \sum_{k=0}^{\infty} k^2 p_k(ux) = \frac{A'(1) + A''(1)}{u^2 A(1)} +$$
$$+ \frac{x[2G'(1)A'(1) + A(1)G''(1) + A(1)G'(1)]}{uA(1)} + x^2 [G'(1)]^2.$$

Hence $P_u(e_j, x) \to e_j(x)$ for $j = 0, 1, 2$, as $u \to \infty$, if and only if $G'(1) = 1$. □

Proof of Theorem 1. Under the assumption $|f(t)| \leq M \exp(\mu t)$, $P_u(f, x)$ exists for $x \geq 0$ and $u > \mu/\ln R$. For any $\delta > 0$ and $x \geq 0$

$$|P_u(f, x) - f(x)| \leq \frac{e^{-uxG(1)}}{A(1)} \left\{ \sum_{\left|\frac{k}{u} - x\right| < \delta} + \sum_{\left|\frac{k}{u} - x\right| \geq \delta} \right\} \left| f\left(\frac{k}{u}\right) - f(x) \right| p_k(ux)$$
$$\equiv S_1 + S_2, \tag{16}$$

We estimate S_1 as follows. Let x_0 be a point of continuity of f. Given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that for each t satisfying $|t - x| < \delta$ we have $|t - x_0| \leq |t - x| + |x - x_0| \leq 2\delta$ for all $|x - x_0| < \delta$ and $|f(t) - f(x)| \leq |f(t) - f(x_0)| + |f(x) - f(x_0)| < \varepsilon$. Hence, uniformly in x satisfying $|x - x_0| < \delta$, we have

$$S_1 \leq \frac{\varepsilon e^{-uxG(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(ux) = \varepsilon. \tag{17}$$

For estimating S_2, let $u > \mu/\ln R$ and note that for x satisfying $|x - x_0| < \delta$

$$\left| f\left(\frac{k}{u}\right) - f(x) \right| \leq M[(e^{\mu/u})^k + e^{\mu(x_0+\delta)}] \leq M[1 + e^{\mu(x_0+\delta)}](e^{\mu/u})^k$$

so that

$$S_2 \le \frac{M}{A(1)}\,[1 + e^{\mu(x_0+\delta)}]\, e^{-uxG(1)} \sum_{\left|\frac{k}{u}-x\right| \ge \delta} (e^{\mu/u})^k\, p_k(ux)$$

$$\le \frac{M}{A(1)\,\delta^2}\,[1 + e^{\mu(x_0+\delta)}]\, e^{-uxG(1)} \sum_{k=0}^{\infty} (e^{\mu/u})^k \left(\frac{k}{u} - x\right)^2 p_k(ux)$$

$$\le \frac{M}{A(1)\,\delta^2}\,[1 + e^{\mu(x_0+\delta)}]\, e^{u(x_0+\delta)[G(e^{\mu/u})-G(1)]}\, K_u(x_0 + \delta, e^{\mu/u})$$

where the last inequality follows by Lemma 1. As $\exp\{u(x_0 + \delta)[G(e^{\mu/u}) - G(1)]\} \to \exp\{\mu(x_0 + \delta)G'(1)\}$ as $u \to \infty$ and $K_u(x_0 + \delta, e^{\mu/u}) \le \text{const.}/u$ by Lemma 2, we have, uniformly in x satisfying $|x - x_0| < \delta$, that

$$S_2 = O(1/u) < \varepsilon$$

for sufficiently large u. The result now follows from (16) and (17). □

Proof of Theorem 2. Define, as in [5] the function

$$\Phi(x) = \begin{cases} f\left(\ln \frac{1}{x}\right), & 0 < x \le 1 \\ \lim\limits_{t\to\infty} f(t), & x = 0. \end{cases}$$

$\Phi(x)$ is continuous on $[0, 1]$. Hence, for every $\varepsilon > 0$ there exists a polynomial $q_N(x) = \sum_{l=0}^{N} c_l x^l$ such that $|\Phi(x) - q_N(x)| < \varepsilon$ for $0 \le x \le 1$. Hence for $Q_N(t) = q_N(e^{-t})$ we have $|f(t) - Q_N(t)| < \varepsilon$ for $0 \le t < \infty$. Now

(18) $$|P_u(f, x) - f(x)| \le |P_u(f, x) - P_u(Q_N, x)| + |P_u(Q_N, x) - Q_N(x)| + |Q_N(x) - f(x)| \le 2\varepsilon + |P_u(Q_N, x) - Q_N(x)|$$

since $|P_u(f, x) - P_u(Q_N, x)| \le P_u(1, x)\sup_{t\ge 0}|f(t) - Q_N(t)| < \varepsilon$. Hence it suffices to consider the uniform convergence of $P_u(Q_N, x)$ to $Q_N(x)$. By linearity of the operator $P_u(f, x)$ and (3a) we have

$$P_u(Q_N, x) = \sum_{l=0}^{N} c_l \frac{A(e^{-l/u})}{A(1)}\, e^{-ux[G(1)-G(e^{-l/u})]}.$$

With $\Delta_u \equiv A(e^{-l/u})/A(1)$ and $\beta_u \equiv \frac{u}{l}\,[G(1) - G(e^{-l/u})]$ we have

(19) $$P_u(Q_N, x) - Q_N(x) = \sum_{l=0}^{N} c_l\, [\Delta_u e^{-\beta_u l x} - e^{-lx}].$$

Note that by (2), $0 < \Delta_u \le 1$ and $\Delta_u \to 1$ as $u \to \infty$; also,

$$0 \le \beta_u \le \sum_{n=1}^{\infty} n\gamma_n = G'(1) = 1 \text{ and } \beta_u \to G'(1) = 1 \text{ as } u \to \infty.$$

Hence

$$|\Delta_u e^{-\beta_u l x} - e^{-lx}| \le \Delta_u|e^{-\beta_u l x} - e^{-lx}| + (1 - \Delta_u)\, e^{-lx}$$
$$\le \left(\frac{1}{\beta_u} - 1\right) + (1 - \Delta_u) \to 0 \text{ as } u \to \infty,$$

so that by (19), $|P_u(Q_N, x) - Q_N(x)| \to 0$ as $u \to \infty$ uniformly in x over $[0, \infty)$ and the result follows from (18). □

Proof of Theorem 3. (*a*) The argument here is straightforward: Since $f(t)$ is uniformly continuous on $[0, \infty)$ we have for $0 \le t, x < \infty$ and for every $\delta > 0$

$$|f(t) - f(x)| \le \left[1 + \frac{(t-x)^2}{\delta^2}\right] \omega(f, \delta)$$

and as in the classical argument for functions defined on a compact interval [2] we have

$$|P_u(f, x) - f(x)| \le 2\,\omega(f, \alpha_u(x)),\quad 0 \le x < \infty$$

where $\alpha_u^2(x) \equiv P_u((t - x)^2, x)$. By (3b) and Lemma 1 we find

(20) $$\alpha_u^2(x) = \frac{K_u(x, 1)}{A(1)} = \frac{x[1 + G''(1)]}{u} + \frac{A'(1) + A''(1)}{A(1)u^2}$$

since $G'(1) = 1$. The result now follows.

(*b*) As f is continuous only over $I = [0, b]$ an argument of Ditzian [3] shows that for $x \in I_1 \subset I$, $t \in [0, \infty)$, and $\delta > 0$

$$|f(t) - f(x)| \le \left[1 + \frac{(t-x)^2}{\delta^2}\right] \omega_I(f, \delta) + (\|f\| + Me^{\nu t})\, \frac{(t-x)^2}{\eta^2}$$

where $\|f\| = \max_{t\in I_1}|f(t)|$. Hence for $x \in I_1$,

$$|P_u(f, x) - f(x)| \le \left[1 + \frac{\alpha_u^2(x)}{\delta^2}\right] \omega_I(f, \delta) + \frac{\|f\|}{\eta^2}\, \alpha_u^2(x) + \frac{M}{\eta^2}\, P_u(e^{\nu t}(t - x)^2, x).$$
(21)

Using Lemma 1 we find

(22) $$P_u(e^{\nu t}(t - x)^2, x) = \frac{e^{ux[G(e^{\nu/u})-G(1)]}}{A(1)}\, K_u(x, e^{\nu/u})$$

for $x \ge 0$. Lemma 2 gives the asymptotic behavior of $K_u(x, e^{\nu/u})$ as $u \to \infty$. We need an estimate for rate of convergence of $\exp\{ux[G(e^{\nu/u}) - G(1)]\}$ to $\exp(\mu x)$ as $u \to 0$: With $\Delta_u = \frac{u}{\mu}[G(e^{\mu/u}) - G(1)]$ we find by (2) that $\Delta_u \ge \sum_{n=1}^{\infty} n\gamma_n = G'(1) = 1$ and that

$$\Delta_u - 1 \le \frac{\mu}{u} \sum_{n=1}^{\infty} n^2 \gamma_n e^{n\mu/u} = O(1/u).$$

Hence

$$|e^{\Delta_u \mu x} - e^{\mu x}| \leq e^{\Delta_u \mu x} \mu x(\Delta_u - 1) = O(1/u)$$

uniformly in x over compact intervals. It then follows by Lemma 2 and (22) that

(23) $$P_u(e^{\mu t}(t-x)^2, x) = \frac{[1+G''(1)]\, x\, e^{\mu x}}{u} + O(1/u^2)$$

where the $O(1/u^2)$ is uniform in x over compact intervals. Now by (20) (23) and the choice $\delta^2 = \alpha_u^2(x)$ in (21) we have for $x \in I_1$

$$|P_u(f, x) - f(x)| \leq 2\omega_t(f, \alpha_u(x)) + \left\{\frac{x}{\eta^2}[Me^{\mu x} + ||f||][1+G''(1)]\left(\frac{1}{u}\right)\right\} + O(1/u^2)$$

and the result follows. □

Proof of Theorem 4. We first note that the series (11), defining the estimate $\hat{f}_u(t)$, converges in quadratic mean and with probability one since the Z_k's are independent, $E[Z_k^2] = 1$ and $\Sigma_{k=0}^{\infty} h_k^2(ux) < \infty$ (as shown below) (see, for example, Kawata [6, Theorem 12.4.2]). Now

(24) $$E[\hat{f}_u(t) - f(t)]^2 = (\text{Bias}[\hat{f}_u(t)])^2 + \text{Var}[\hat{f}_u(t)]$$

where $\text{Bias}[\hat{f}_u(t)] = E[\hat{f}_u(t)] - f(t)$; $\text{Var}[\hat{f}_u(t)] = E\{\hat{f}_u(t) - E[\hat{f}_u(t)]\}^2$, and we obtain bounds on the bias and variance terms. Since

$$E[Z_k] = \frac{1}{2M}\int_{-M}^{M} \text{sgn}[f(k/u) + y]dy = \frac{f(k/u)}{M}$$

we have by (11)

$$E[\hat{f}_u(t)] = \sum_{k=0}^{\infty} f(k/u)\, h_k(ut) = P_u(f, t).$$

It then follows by Theorem 3(i) that

(25) $$|\text{Bias}[\hat{f}_u(t)]| = |P_u(f, t) - f(t)| \leq 2\omega(f, \alpha_u(t))$$

where $\alpha_u^2(t)$ is given in (13). Next $\text{Var}[Z_k] = E[Z_k^2] - (E[Z_k])^2 = 1 - [f(k/u)/M]^2 \leq 1$ for all $k \geq 0$. Hence by (11)

(26) $$\text{Var}[\hat{f}_u(t)] \leq M^2 \sum_{k=0}^{\infty} h_k^2(ut).$$

and we estimate the sum in (26) as follows: By (3a) with $z = e^{i\lambda}$ we have

$$p_k(ux) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\lambda} A(e^{i\lambda})\, e^{uxG(e^{i\lambda})}\, d\lambda$$

It then follows by Parseval's theorem and (12) that

(27) $$\sum_{k=0}^{\infty} h_k^2(ut) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|A(e^{i\lambda})|^2}{A^2(1)} e^{-2ux\text{Re}[G(1)-G(e^{i\lambda})]}\, d\lambda.$$

Since

$$\text{Re}[G(1) - G(e^{i\lambda})] = \sum_{n=1}^{\infty} \gamma_n[1 - \cos n\lambda] \geq \gamma_{n_0}(1 - \cos n_0 \lambda),$$

where γ_{n_0} is the first nonzero coefficient of $G(z)$, and $|A(e^{i\lambda})| \leq A(1)$ we have from (27)

$$\sum_{k=0}^{\infty} h_k^2(ut) \leq (1/2\pi)\exp(-2u\gamma_{n_0} t)\int_{-\pi}^{\pi} \exp(-2u\gamma_{n_0} t\cos n_0\lambda)\, d\lambda =$$
$$= \exp(-2u\gamma_{n_0} t)\, I_0(2u\gamma_{n_0} t)$$

and by (26)

(28) $$\text{Var}[\hat{f}_u(t)] \leq M^2 \exp(-2u\gamma_{n_0} t)\, I_0(2u\gamma_{n_0} t).$$

The result now follows by (24), (25) and (28). □

Proof of Theorem 5. Follows in the manner of the proof of Theorem 4.3(a) in [8].

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